Lagrangian description of homogeneous turbulence

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We propose an approach to the determination of the spectral energy density of hydrodynamical turbulence which is based on using the properties of a quasi-two-dimensional wavepacket, taken as the structural element of the turbulence. We show that there exist three characteristic $E(k)$ functional behaviors. For the long-wavelength region we get the law $E(k) \propto k^{2\lambda^{1/2}}$ which takes into account the cluster structure of a vortex. For the scale-invariant and dissipative regions we get, respectively, the laws $E(k) \propto k^{-|\beta|}\ln(k_0/k)$ and $E(k) \propto \ln(k_0/k)$. We confirm the conclusions of the theory by a specially designed experiment.

Turbulence is macroscopic chaos and always has structure. The separate vortices, complex eddy formations, are its structural elements. It is not always convenient to determine the characteristics of the turbulence through a study of the vortices by the usual Eulerian method (Ref. 1, Vol. 1, p. 460). This refers especially to the experimental study of large-scale structures with space-correlation characteristics which depend strongly on the boundary conditions (Ref. 2, p. 13).

We consider in what follows the possibility of a Lagrangian approach based on a direct study of the properties of vortex motion of a liquid with a free surface. It is well known[3] that the equations of motion of a liquid with a free surface are in several cases the same as the equations describing phenomena in compressible plasma media. The results of the present paper can thus be relevant to turbulence of a different physical nature.

1. QUASI-TWO-DIMENSIONAL VORTEX PACKET MODEL

We choose the structure element of the turbulence in the form of a quasi-two-dimensional vortex packet with a core of radius r_0, schematically shown in Fig. 1. Quasi-two-dimensionality means the presence of a perturbation $h(x,y,t)$ of the surface level of the vortex and the neglect of small changes in the averaged characteristics of the motion in the equilibrium thickness h_0. We denote by θ the angle between the direction of the free fall acceleration **g** and the z-axis, along with the rotational velocity of the vortex core $\mathbf{\Omega}_0$ is directed. In the $\theta = 0$ case, for which we have designed an experiment, h is the perturbation of the free surface of the liquid. In the $\theta \neq 0$ cases one must take h to be the perturbation of the surface level of the vortex packet, submerged in its own medium. To take into account the effect of a different orientation of the vortex packet in the turbulent medium on the quantity h we introduce into our discussion the value $\langle h \rangle$ of the component of h along the direction of **g**, averaged over the angle θ, which we calculate in terms of a Gibbs distribution of the potential energy of the particles of the liquid relative to their rotational energy:

$$\langle h_g \rangle = h \frac{\int_0^{\pi/2} \exp(-\alpha\cos\theta)\cos\theta\sin\theta\, d\theta}{\int_0^{\pi/2} \exp(-\alpha\cos\theta)\sin\theta\, d\theta} = \frac{h}{\alpha}\left(1 - \frac{1}{\exp(\alpha)-1}\right), \qquad (1.1)$$

$$\alpha = \frac{2hg}{\Omega^2 r^2},$$

where $\Omega(r)$ is the rotational velocity of the particles of the liquid at a distance r from the center of the vortex. The integration in (1.1) is from 0 to $\pi/2$. This is connected with the fact that only perturbations of the upper half in the thickness h_0 affect the dynamics of the vortex packet. It follows from (1.1) that for $\alpha \gg 1$ we have

$$\langle h_g \rangle = \frac{\Omega^2 r^2}{2g} = \eta_0,$$

and we can thus separate the effect of the centrifugal forces from the nonequilibrium part of the perturbation of the surface level of the vortex:

$$h(x, y, t) = \eta_0 + \eta(x, y, t).$$

The role of the perturbation $\eta(x,y,t)$ will then appear for $\alpha \approx 1$, i.e., in those cases where the drop in pressure is comparable with the effect of the rotation. This is essentially a formulation of the so-called quasi-geostrophic condition of geophysical hydrodynamics (Ref. 4, p. 226), which we shall use in what follows.

The structural element model used is more relevant to two-dimensional turbulence. However, one must take into account that in real turbulence, predominantly in the long-wavelength region, the role of two-dimensional vortex formations will dominate, since due to gyroscopic effects they are more stable than three-dimensional structures. In other words, due to the tendency of vortex tubes to stretch in a

FIG. 1. Sketch of a quasi-two-dimensional vortex packet.

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turbulent medium the motion inside a vortex tube will basically be two-dimensional (Ref. 5, p. 113).

2. INITIAL EQUATIONS

We must determine the velocity **v** and pressure p fields caused by the initial circulation

$$\Gamma_0=2\pi\Omega_0 r_0^2$$

in the (x,y) or (r,φ) plane, in a Cartesian or polar coordinate system, respectively. The vortex motion is characterized by the vorticity curl **v** in each point, and some average of its value [e.g., over the polar angle φ—$\langle \text{curl } \mathbf{v}\rangle = 2\boldsymbol{\Omega}(r)$] must be maintained by external action, which is the condition for stationary turbulence generation. The connection between $\Omega(r)$ and Γ_0 can be established through Kelvin's theorem:

$$\int_S \langle \text{curl } \mathbf{v}\rangle dS=2\int_S \Omega(r)\,dS=\Gamma_0, \qquad (2.1)$$

where S is the singly connected surface subtended by the vortex as a whole with respect to the average dynamic characteristics. The proviso about S is connected with the fact that the Kelvin theorem for vortex motion is applicable inside a loop of the separatrix of the stream function.[6] The single-valued connection used by us in what follows between the pressure and the density generalizes the applicability of this theorem to the case where there is viscosity (Ref. 7, p. 31).

One can take into account a given average vorticity field 2Ω in the equations of motion through chaning to a reference frame consisting of an infinite set of annular layers differentially rotating with angular velocity $\Omega(r)$ in which centrifugal and Coriolis forces act upon the particles of the liquid. We then look for the velocity $\mathbf{v}(x,y,t)$ and pressure $p(x,y,t)$ fields perturbed about the average vorticity and centrifugal force pressure fields. We write the final expressions for the Euler quantities $\mathbf{v}_{\text{Eul}}$ and p_{Eul} in the form

$$\mathbf{v}_{\text{Eul}}=\mathbf{u}+\mathbf{v}+[\boldsymbol{\Omega}\mathbf{r}],\quad p_{\ni}=p_u+p+\rho[\boldsymbol{\Omega}\mathbf{r}]^2/2, \qquad (2.2)$$

where **u** is the translational velocity of the vortex, which is the same as the Lagrangian velocity when the center of the vortex has a rectilinear trajectory and p_u and ρ are the pressure and the density of the medium.

In accordance with the quasi-two-dimensional nature of the problem we start from the equations of the "shallow water" theory (Ref. 7, p. 60). Generalizing the continuity equation to the $\eta=\eta(x,y,t)$ case and taking viscosity into account in the equations of motion we have

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-g\nabla\eta-2[\boldsymbol{\Omega}\mathbf{v}]+\nu\Delta\mathbf{v}+\frac{\nu}{3}\,\text{div}\,\mathbf{v}, \qquad (2.3)$$

$$\text{div}\,\mathbf{v}=-\frac{1}{h_0}\left(\frac{\partial\eta}{\partial t}+(\mathbf{v}\nabla)\eta\right). \qquad (2.4)$$

Taking the curl of (2.3) and using (2.4) we get an equation for the stream function ψ:

$$\frac{\partial\Delta\psi}{\partial t}+J(\Delta\psi,\psi)+\frac{2\Omega(r)}{h_0}\left(\frac{\partial\eta}{\partial t}+J(\psi,\eta)\right)=\nu\Delta^2\psi, \qquad (2.5)$$

where ∇^2 is the Laplacian. In (2.5) we have used the following notation:

$$v_x=\frac{\partial\psi}{\partial y},\quad v_y=-\frac{\partial\psi}{\partial x},\quad J(\psi,\eta)=\frac{\partial\psi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\eta}{\partial x},$$

and also the condition

$$(\mathbf{v}\nabla)\Omega\ll\Omega(\nabla\mathbf{v}). \qquad (2.6)$$

Equation (2.5) contains two unknown functions ψ and η and the problem consists of determining ψ for arbitrary η. This is why the right-hand side of (2.4) is not a trivial expression after we have introduced the stream function, although we have div $\mathbf{v}=0$. Inequality (2.6) means that we are considering long-wavelength perturbations of ψ which are stretched out in r and vary with φ.

When there exist functional relations

$$\eta=\eta(\psi),\quad \Delta\psi=\Delta\psi(\psi)$$

the Jacobian J in (2.5) vanishes. To obtain such simplifications we use linear relations which have a well defined physical meaning. We can uniquely construct from the basic parameters of the problem the simple relationship

$$\eta=-h_0\psi/\psi_0,\quad \psi_0=gr_0/\Omega_0, \qquad (2.7)$$

where the minus sign reflects the formation of depressions on the surface of the vortex the size of which is proportional to the rotational velocity of the annular layer considered. On the other hand, the relation (2.7) is an expression of the quasi-geostrophic motion condition mentioned earlier.

To establish a relationship $\nabla^2\psi=F(\psi)$ we take into account the possibility that different perturbation waves form and interact in the vortex packet considered. The usual harmonic analysis of the linearized system (2.3) and (2.4) [without the term $(\mathbf{v}\nabla)\mathbf{v}$] indicates the possibility that there may occur three kinds of perturbation waves: a transverse vorticity wave, and also longitudinal elastic and entropy waves with the respective spectra

$$\omega_1^2=4\Omega^2\cos^2\alpha+\nu^2k^4,\quad (\boldsymbol{\Omega}\mathbf{k})=\Omega_z k\cos\alpha,\quad \mathbf{k}\perp\mathbf{v}, \qquad (2.8)$$

$$\omega_2^2=gh_0k^2-{}^{16}/_9\nu^2k^4,\quad \mathbf{k}\|\mathbf{v}, \qquad (2.9)$$

$$\omega_3^2=\left(\frac{4}{3}\frac{h_0 v r\Omega_0}{r_0(1-\cos\alpha)^{1/2}}\right)^{2/3}k^2-\frac{16}{9}\nu^2k^4,\quad \mathbf{k}\|\mathbf{v}, \qquad (2.10)$$

where ω_i is the frequency and k the magnitude of the wavevector. We have obtained Eq. (2.8) by taking the curl of (2.3) and Eqs. (2.9) and (2.10) by taking the div and using (2.4). If we neglect the quantity η in the dispersion relations for the longitudinal wave ($\eta\ll h_0$) we obtain Eq. (2.9), if we assume that $\eta\sim h_0$, and we use (2.7), then we obtain Eq. (2.10).

When there is no viscosity ($\nu=0$) Eq. (2.8) changes to the well known spectrum of inertial waves, but in that case $2\Omega(r)$ signifies an average vorticity rather than an angular velocity, so that one can call such waves vorticity waves. When $\nu=0$ holds, Eq. (2.9) changes to the spectrum of long-wavelength gravitational waves occurring due to the elastic oscillations of the surface of the liquid. A wave with the spectrum (2.10) is possible only when there is viscosity and is therefore called an entropy wave.

We get from the condition for resonant interactions between two longitudinal waves ($\omega_2=\omega_3$) an estimate for the value of the velocity of the liquid

$$v\geqslant\frac{3}{4}(gr_0)^{1/2}\frac{gr_0}{\Omega_0\nu}, \qquad (2.11)$$

for which the hydrodynamic analog of the Cherenkov effect occurs and a discontinuity starts in the behavior of several dynamical characteristics of the vortex.

On the basis of these results we may assume that the stream function to a first approximation satisfies a wave equation:

$$\Delta\psi=\frac{1}{v_{ph}^{2}}\frac{\partial^{2}\psi}{\partial t^{2}}=-k^{2}\psi.\quad \Delta\psi+k^{2}\psi=0, \tag{2.12}$$

where v_{ph} is the phase velocity of the wave. We find the required relation

$$\Delta\psi=\Delta\psi(\psi)$$

in the form (2.12). This relation can also be considered as an equation to determine ψ. Its solution in polar coordinates can be expressed in terms of oscillating Bessel functions. However, these solutions do not take into account the role played by $\Omega(r)$ and ν, so that it is necessary to use Eq. (2.5) together with conditions (2.7) and (2.12). In order to take into account the effect of the viscosity on the induced vorticity,

$$\Delta\psi=-\,\text{curl}\,\mathbf{v}$$

we must then use conditon (2.12) only once on the right-hand side of (2.5). Introducing the damping rate γ we have

$$\Delta\psi-\frac{2\Omega(r)}{f(R)}\frac{\psi}{\psi_0}=0\,, \tag{2.13}$$

where we have put

$$\psi(r,\varphi,t)=\psi(r,\varphi)\exp(-\gamma t),\quad f(R)=1-1/R,$$
$$R=R_0/R_*=\gamma/k^2\nu. \tag{2.14}$$

Here R is the ratio of the Reynolds number $R_0=\Omega_0 r_0^2/\nu$ to its critical value defined in terms of the "wave viscosity" γ/k^2, since $\nabla^2\psi$ changes sign for $R=1$. Condition (2.11) thus leads to a discontinuity in the vorticity, i.e., to a weak tangential discontinuity. The condition $R=1$, or $\gamma=\gamma_*=k_*^2\nu$ is additional to (2.11) and it defines the critical value γ_* of the unknown quantity $\gamma=\gamma(k,R_0)$. We can call the occurrence of a discontinuity for $R=1$ also a second-order phase transition since then the symmetry of the motion (the sign of curl $\mathbf{v}$) changes, while the physical quantity (the velocity) may stay constant but its first derivative shows a jump.

3. FUNDAMENTAL SOLUTIONS FOR THE STREAM FUNCTION

We shall look for two kinds of solution of (2.13) which occur for $R<1$ and for $R>1$ and which satisfy the conditions

$$\psi(r_0,\varphi)=\psi_{01}=\Omega_0 r_0^2,\quad \psi(r,\varphi+2\pi)=\psi(r,\varphi),$$
$$\psi(r\to\infty)=0,\quad r^2\Omega(r)=r_0^2\Omega_0. \tag{3.1}$$

We use the method of separation of variables, putting

$$\psi(r,\varphi)=L(r)\Phi(\varphi).$$

We choose the constant λ in the separation of variables so that

$$\lambda-2/f(R)=m^2>0. \tag{3.2}$$

Here we have used the fact that the parameter g does not dominate for an induced vortex, and also the fact that we assumed in (2.13) that $\psi_0=\psi_{01}$. If $R<1$ [$f(R)<0$] holds we must take λ with a negative sign in order to avoid satisfying (3.2) ($R=0$) trivially. It then follows from (2.13) that for the $R<1$ case we have

$$r^2L''(r)+rL'(r)+\lambda L(r)=0,\quad \Phi''(\varphi)-m^2\Phi(\varphi)=0, \tag{3.3}$$

and for the $R>1$ case

$$r^2L''(r)+rL'(r)-\lambda L(r)=0,\quad \Phi''(\varphi)+m^2\Phi(\varphi)=0. \tag{3.4}$$

Using the solutions of those equations corresponding to (3.1) we can write down expressions for the stream function including the time-dependence:

$$\psi_1(r,t)=\psi_{01}\cos\left(\lambda^{1/2}\ln\frac{r}{r_0}\right)\exp(-\gamma t),\quad R<1,\quad m=0, \tag{3.5}$$

$$\psi_2(r,\varphi,t)=\psi_{01}\left(\frac{r_0}{r}\right)^{\lambda^{1/2}}\exp(im\varphi-\gamma t),\ R>1,$$
$$m=0,\pm1,\pm2,\dots \tag{3.6}$$

These solutions reflect actual physical phenomena and are of fundamental value for further discussions. If we assume that for the basic perturbation mode we have $k=1/r$ the condition $\gamma_*=k_*^2/\nu$ ($R=1$) determines the boundaries of the spatial separation of the regions where the solutions (3.5) and (3.6) can be applied. Inside the circle $r=r_*=(\nu/\gamma_*)^{1/2}$ we have periodicity in the polar angle with the formation of a discrete number (m) of structures. The continuous and discrete spatial motions in the vortex are mutually exclusive. The solution (3.6) describes a vortex cluster with m "branches."

We find the values of r_* from the contition that (3.5) and (3.6) must be matched at that point. To do that we must renormalize the function ψ_2. The renormalization coefficient and the value of r_* are found from the conditions that the stream function and the absolute magnitude of the velocity are continuous:

$$\psi_1(r_*)=\psi_2(r_*),\quad \left(\left|\frac{\partial\psi_1}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial\psi_1}{\partial\varphi}\right|^2\right)_{r=r_*}$$
$$=\left(\left|\frac{\partial\psi_2}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial\psi_2}{\partial\varphi}\right|^2\right)_{r=r_*}, \tag{3.7}$$

$$\psi_2(r,\varphi,t,r_*)=\psi_{02}\left(\frac{r_0}{r}\right)^{\lambda^{1/2}}\exp(im\varphi-\gamma t),$$

$$\psi_{02}=\psi_{01}\cos\left(\lambda^{1/2}\ln\frac{r_*}{r_0}\right)\left(\frac{r_*}{r_0}\right)^{\lambda^{1/2}}, \tag{3.8}$$

$$r_*=r_0\exp\left[\frac{1}{\lambda^{1/2}}\arctan\left(1+\frac{m^2}{\lambda}\right)^{1/2}+n\pi\right],\quad n=0,1,2,\dots \tag{3.9}$$

Since the argument of (.39) is multivalued, the wavelike and structural characteristics of the motion in the vortex on multiple scales are superposed on the shorter-scale perturbations. This means that the vortex motion, as a structural element for turbulence, also has hierarchical properties which are characteristic of the turbulent motion, such as the formation of subharmonics (Ref. 8, p. 180).

4. ENERGY INTEGRAL AND SPATIAL TURBULENCE SPECTRUM

One can prove that Eq. (2.13), written in Cartesian coordinates, has an invariant functional

$$E(\psi(x,y))=\int\int_S\left(\left|\frac{\partial\psi}{\partial x}\right|^2+\left|\frac{\partial\psi}{\partial y}\right|^2+\frac{2\Omega(r)}{\psi_0 f(R)}|\psi|^2\right)dx\,dy. \quad (4.1)$$

Indeed, if we put the functional derivative of the integrand with respect to ψ equal to zero, which is the condition that it is conserved, we get the initial equation (2.13). We call the functional $E(\psi)$ the energy of the vortex since the first two terms are essentially proportional to the kinetic energy and the last term is proportional to the interaction energy of the particles of the liquid. Replacing $\partial/\partial x$, $\partial/\partial y$ by $\partial/\partial r$, $\partial/\partial\varphi$ we get an expression for the energy in coordinates:

$$E(r)=\int_0^r\int_0^{2\pi}\left(\left|\frac{\partial\psi}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial\psi}{\partial\varphi}\right|^2+\frac{2\Omega(r)}{\psi_0 f(R)}|\psi|^2\right)r\,dr\,d\varphi. \quad (4.2)$$

If we change to function of the wavenumber ($r\to 1/k$), the energy of the vortex packet determines the spectral energy density of the turbulent pulsations according to the normalization

$$\int_{k_{min}}^{k_0}E(k)dk=\frac{1}{2S}\int\int_S u_i' u_i'\,dx\,dy, \quad (4.3)$$

where the u_i' are the pulsations of the component of the translational velocity of the vortex, $k_{\min}$ is the minimum value of the wavenumber corresponding to the area S of the surface of the vortex, and we have $k_0 = 1/r_0$.

We first consider the small wavenumber region where we have

$$0\leqslant k\leqslant k_* = 1/r_*.$$

Using (3.8) we have from (4.2)

$$E(k,k_*)=E_0[\lambda+m^2+2/f(R)](k_*^{2\lambda^{1/2}}-k^{2\lambda^{1/2}}),$$
$$E_0=\psi_{0z}^2\pi r_0^{2\lambda^{1/2}}/\lambda^{1/2}. \quad (4.4)$$

If from (3.2)

$$m^2+2/f(R)=\lambda,$$

holds in (4.4), the result will not depend explicity on the number of structures in the cluster. However, the solution (3.6) allows the number of structures to change spontaneously. One can therefore consider a virtual state of a vortex with a number $m\pm j$ of structures, where $j=0,\pm 1,\pm 2,...$ is the number of virtual secondary structures. In that case (4.4) takes the form

$$E(k,k_*,m,j)=E_0(2\lambda\pm 2mj+j^2)(k_*^{2\lambda^{1/2}}-k^{2\lambda^{1/2}}). \quad (4.5)$$

One can treat this expression as the binding energy necessary to form j secondary vortex structures from m primary structures in the wavenumber range from k to k_*. We denote by $E(k_*)$ the maximum energy of a single vortex in a localized state ($j=0$) and by $E(k,m,j)$ the energy of a vortex cluster containing $m\pm j$ structures. From the definition of the binding energy we then have

$$E(k,m)=E(k_*)+E(k,k_*,m,j)$$
$$=E_0(2\lambda k^{2\lambda^{1/2}}\pm(j^2\pm 2mj)(k_*^{2\lambda^{1/2}}-k^{2\lambda^{1/2}}). \quad (4.6)$$

It is necessary to take into account that $E(k\to 0,m)\geqslant 0$, which reflects that there are no pulsations with infinitely large spatial scales. This condition determines the sign and maximum value of $j=j(k)$ ($j<0,|j|\leqslant 2m$) and indicates that it is possible spontaneously to form coherent structures with a complex configuration in the $0\leqslant k\leqslant k_*$ range. The properties of these structures, formed when vortices are coupled were considered in Ref. 9.

Depending on the actual magnitude of the difference

$$j^2-2mj\leqslant 0,$$

$E(k,m)$ takes on different values. Extremal values of $E(k,m)$ are reached for $m=j/2$ and $m=j$. In the first case $E(k,m)$ is a minimum [the binding energy occurring in (4.6) with a negative sign takes on its maximum value, zero].

We use (4.6) in Fig. 2 for the $k<k_*$ region to show the curves corresponding to $\lambda=18$, 6, and 3. We took these values to describe the coupling of vortices ($m=4$, 2, and 1) taking into account that for large Reynolds numbers we have $f(R)\to 1$ in Eq. (3.2). The restriction on the maximum value of m is connected with the fact that a system with a large number of vortices becomes stochastic (in an ideal liquid for[10] $m\geqslant 4$). The lowest curve ($\lambda=18, j=0$) was chosen to show the pumping energy which causes states with $m=2$, 1 to appear spontaneously. The positions of the minima ($k/k_* = \frac{1}{2},\frac{3}{4}$) correspond to the multiple change of the localization region of a system of vortices when they are coupled. The value of r_* for $m=1$, $\lambda=3$ is determined from (3.9). The dashed line shows the possible behavior of $E(k,m)$. The turbulence spectrum in the long-wavelength region, discussed in the literature from various points of view (Ref. 1, Vol. 2, p. 151) may thus turn out to be irregular.

We find the energy of the turbulent pulsations in the $k>k_*$ region by using the solution (3.5):

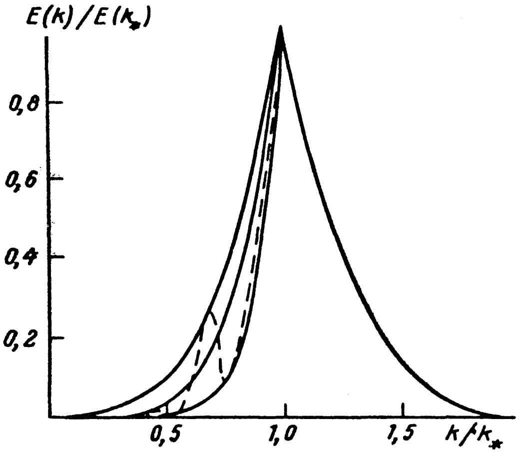

FIG. 2. Theoretical behavior of the homogeneous turbulence spectrum.

$$E(k, k_0)=2\pi\psi_{01}^2 \int_{1/k_0}^{1/k} \left[\lambda \sin^2\left(\lambda^{1/2} \ln \frac{r}{r_0} \right) + \frac{2}{f(R)} \cos^2\left(\lambda^{1/2} \ln \frac{r}{r_0} \right) \right] \frac{dr}{r}. \quad (4.7)$$

If we take into account the condition $m = 0$ corresponding to (3.5) and assume

$$2/f(R)=\lambda=\text{const},$$

which is possible for $\gamma \sim k^2$, we have

$$E(k, k_0)=2\pi\psi_{01}^2\lambda \ln \frac{k_0}{k} \quad (4.8)$$

for the vortices on the small scales determined by the viscosity.

The scale-invariant (inertial) region of the turbulence spectrum is determined by the quasistationary properties of the vortices. A quasistationary vortex is a virtual system where direct (splitting up) and inverse (merging) cascades of changes in the number of structures take place.[11] Then, although $m = 0$ holds, one can describe the cascade processes in some k range thanks to the $\gamma = \gamma(k, R_0)$ dependence which may cause λ to differ from $2/f(R)$.

We find the approximate form of $E(k,k_0)$ in the inertial region. We can take the function in the square brackets in (4.7) from under the integral sign because of the slow, logarithmic nature of the change of its argument. Moreover, we require that this function be a homogeneous function of degree β, i.e., proportional to k^β. We find the proportionality coefficient from the condition for matching with expression (4.6) for $k = k_*$:

$$E(k, k_0) = E(k_*)\, k^\beta \ln \frac{k_0}{k}, \quad E(k_*) = E_0 2\lambda k_*^{2\lambda^{1/2}+\beta} / \ln \frac{k_0}{k_*}. \quad (4.9)$$

From the way it is introduced, we have $\beta < 0$. For $\beta = -5/3$ Eq. (4.9) goes over into the Kolmogorov–Obukhov spectrum with the logarithmic correction factor established in Ref. 12. Applying Euler's theorem for homogeneous functions to the integrand in (4.7) gives us a connection between $\gamma(k)$ and the degree of homogeneity β:

$$\frac{\gamma(k)}{\gamma(k)-k^2\nu} = \int_{k_*}^{k} \frac{\gamma(k)}{\gamma(k)-k^2\nu} \frac{\beta}{k}\, dk. \quad (4.10)$$

We have obtained an integral relation, typical for the characteristics of strong turbulence.[13] We restrict ourselves to the first approximation for $\gamma(k)$. Substituting into the right-hand side of (4.10) the value of the absolute magnitude

$$\left| \frac{\beta f(R)}{k} \right| = \frac{2|\beta|}{\lambda k},$$

we get

$$\gamma(k)=k^2\nu \frac{|\beta|\lambda \ln k/k_*}{|\beta| \ln k/k_* - 2}, \quad k_* \leqslant k \leqslant k_0. \quad (4.11)$$

Hence the inertial region range satifies the condition

$$\ln \frac{k_0}{k_*} \geqslant \frac{2}{\lambda|\beta|} \quad (4.12)$$

guaranteeing that we have $\gamma(k) \geqslant 0$.

From (4.9) follows the possibility of determining the parameter λ for given Ω_0 and r_0 in terms of the empirical constant occurring in the Kolmogorov–Obukhov spectrum. In the $k < k_*$ range that parameter was determined in terms of the number of structures in the vortex cluster. In Fig. 2 we show the function (4.9) in the $k > k_*$ range for $\beta = -\frac{5}{3}$.

5. EXPERIMENT

The conclusions of the theory were checked through a specially designed experiment. Quasi-two-dimensional vortices were produced on the surface of the liquid in a rectangular tank of dimensions $38 \times 60 \times 6.5$ cm by the steady rotation of 10–20 mm diameter disks. The thickness of the liquid above the disk was 1–3 mm in the state of rest. Taking the centrifugal effect into account, it was chosen so that liquid remained at the edge of the disk and identical conditions were created to induce vortex motion at the surface of the liquid at rest. In accordance with the aim of the experiment the depth h_0 of the liquid in the tank had to satisfy the condition of the "shallow water" theory (neglect of changes in the average characteristics of the flow in the transverse direction). We chose $h_0 = 2$ cm since for smaller values of h_0 ($h_0 \lesssim 1$ cm) the effects of the boundary layer formed on the bottom of the tank affected the picture of the flow.

The rotational frequency of the disks was varied from 20 to 40 s^{-1} and was measured by electron counters supplied by light detectors. The change in the level h of the surface of the liquid was fixed with an accuracy of ± 0.01 mm by checking the contact between the point of an ordinary needle and the surface of the liquid. The contact was observed by optical and electrical means. The velocity of the liquid was measured with an accuracy of ± 0.02 m/s using a 1.2 cm diameter vane made from a light foil. To reduce the friction of the rotation of the vane we used the bearings of a watch mechanism. The rotation frequency of the vane was determined by fixing the interruption of a light beam passing through holes in the vane equally spaced along a circle. The velocity detector was calibrated by comparing the measured velocity with the rate at which liquid passed through special channels of different cross-sections.

The kinematic viscosity of the working liquid—glycerine—could be varied over two orders of magnitude by adding the necessary amount of water. The flow was visualized by introducting to the liquid an amount of the same liquid of a somewhat different viscosity (density).

The experiments showed the complex nature of the motion in a hydrodynamic surface vortex. For several values of the Reynolds number

$$R_0=\Omega_0 r_0^2/\nu \quad (R_0 \geqslant 10^2)$$

we observed oscillations of the level h of the surface of the liquid inside a closed line. For $R_0 \gtrsim 2\times 10^2$ wave perturbations appear beyond this line in the shape of separate vortex structures; i.e., a discontinuity of the vorticity or a weak tangential discontinuity occurs. The evolution of the new structures is nonlinear in nature.[14]

We shall discuss the data from the measurements of the pressure and the velocity necessary for the determination of the energy of the vortex, i.e., the turbulent spectrum. We show in Figs. 3 and 4, respectively, typical changes along the radial coordinate of the relative values of the pressure and of

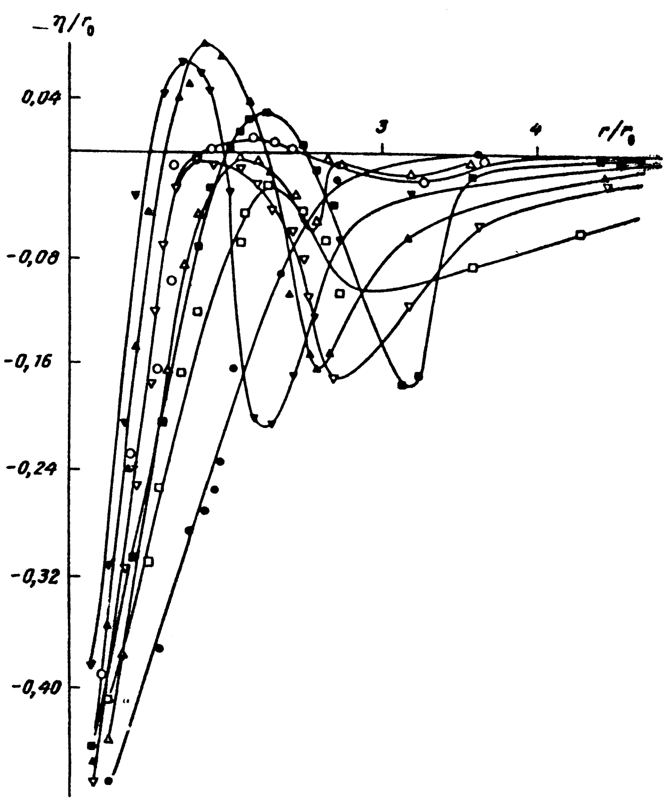

FIG. 3. Measurement of the pressure in a vortex as function of the radial coordinate. $R_0 = 20.9$ (●), 28.7 (○), 41.2 (■), 54.6 (□), 77.2 (△), 108 (▽), 184 (▲), 236 (▼).

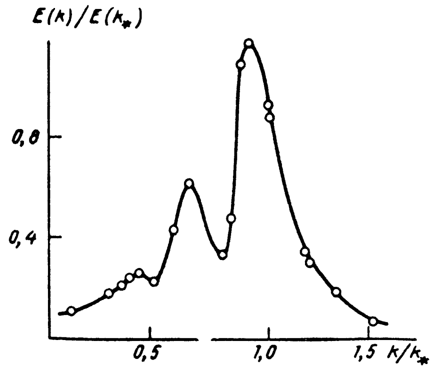

FIG. 5. Experimental dependence of the energy of a vortez on its spatial scale. $R_* = 3.17$, $E(k_*) = 0.957\ \Omega_0^2 r_0^4$.

the velocity, averaged over the time and the polar angle, for different values of R_0. One can, according to (2.7) and (4.2), determine from these data (by evaluating the area bounded by the curves in the appropriate coordinates) the dependence of the energy of the vortex (of the turbulent pulsations) on R_0. Using the relations

$$R_0 = \gamma/k^2\nu, \quad R_* = \gamma_*/k_*^2\nu, \quad R_0/R_* = R, \quad k = k_*(\gamma/\gamma_* R)^{1/2}, \tag{5.1}$$

we can obtain from the experimental data the function

$$E = E(k/k_*).$$

For the cluster and quasistationary states of the vortex ($R \gtrsim 1$) we can neglect variatons in the function $\gamma(R)$ and assume that we have $\gamma = \gamma_*$ since in that case the role of the viscosity is small. For $R < 1$ ($k > k_*$) it is necessary to take into account the difference between the functions $\gamma(R)$ and $\gamma(R_*)$. It follows from the theory of hydrodynamic stability (Ref. 8, p. 86) that the scales of the perturbations in the viscous sublayer are proportional to $R^{1/3}$. Hence in the dissipative range we can put

$$\gamma_* = \gamma R^{-1/3}, \quad k = k_* R^{-1/3}. \tag{5.2}$$

We show in Fig. 5 the experimental dependence of the spectral energy density $E(k)/E(k_*)$ on the wavenumber $k/k_* = R^{-1/2}$ in the $k \lesssim k_*$ range and $k/k_* = R^{-1/3}$ for $k > k_*$.

The first, weak minimum of $E(k)$ corresponds to a state of the vortex which is a cluster consisting of three or four vortices. Most often one observes a four-vortex regime. The second minimum corresponds to a stable configuration of two vortices. The value of $E(k_*)$ determines the maximum energy of a localized single ($m = 0$) vortex. By comparing Figs. 2 and 5 we can judge the validity of the basic conclusions of the theory.

To describe Euler turbulence produced, for instance, in a liquid stream, one must determine k_0 in accordance with the Komogorov length scale:

$$k_0^{-1} = (\nu^3/\varepsilon)^{1/4} \sim \nu/u,$$

where ε is the energy dissipation rate and find from (3.9) the ratio k/k_* as function of n. Depending on how the energy of the pulsations is measured (whether or not the energy of the regular motion is included) one must use (2.2) to take into account the change of the Euler velocity.

The irregular nature of the turbulence spectrum in the long-wavelength region is also observed in the experiment described in Ref. 15 which is based on the usual Euler method. The extent k_0/k_* of the inertial region determined in various experiments may differ by several orders of magni-

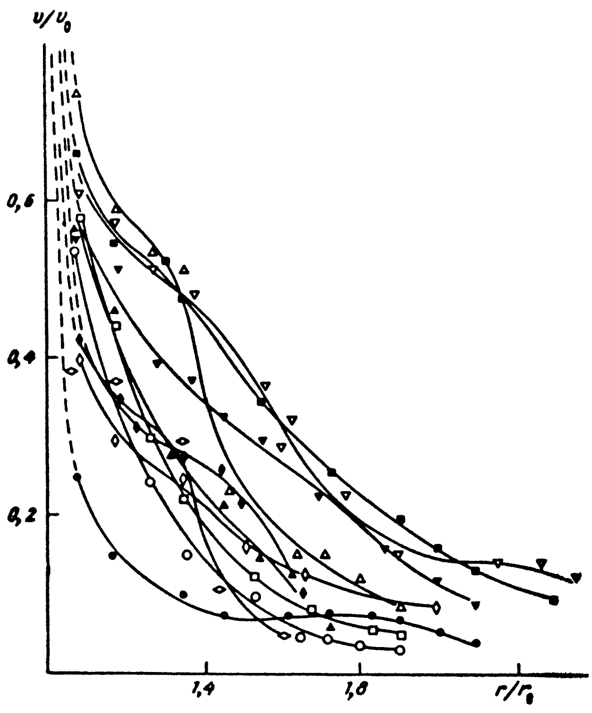

FIG. 4. Measurement of the velocity in a vortex as function of the radial coordinate. $R_0 = 7.31$ (●), 10.7 (○), 14.8 (□), 24.6 (■), 31.7 (△), 39.6 (▽), 41.5 (▲), 61.7 (▼), 116 (◆), 162 (◇), 354 (⊖).

tude. This corresponds to different values of n in (3.9).

Summarizing, we note that models of quasi-two-dimensional structures can apparently be used also for the description of inhomogeneous turbulence. To do this it is necessary, in particular, to know the features of the interaction between vortex packets taking into account their internal degrees of freedom.[16]

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