Periodic nonlinear waves in a uniaxial ferromagnet

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An effective form for the periodic solutions of the Landau–Lifshitz equations describing a uniaxial ferromagnet is derived and various degenerate cases and the soliton limit are studied. The solution technique employed augments the well-known inverse scattering method used for integrating soliton equations.

1. INTRODUCTION

Nonlinear spin waves have started to attract attention, starting, apparently, with Akhiezer and Borovik's paper (Ref. 1). In addition to being applied to the description of ordinary magnetic materials,[2] such waves emerge in the physics of quasi-one-dimensional ferromagnets[3–5] and in the physics of quantum paramagnetic gases.[6,7] Soliton solutions of the appropriate Landau–Lifshitz equation for the magnetization have attracted special attention.[2,8,9] Discovery of the fact that the Landau–Lifshitz[10,11] equation can be integrated by the inverse scattering method has made it possible to study many-soliton for a uniaxial ferromagnet.[2,12]

Some applications, however, require knowing not only the soliton solutions but the periodic as well. Unfortunately, the standard method of finite-zone integration[13,14] developed for finding such periodic solutions of integrable equations has proved insufficiently effective for the Landau–Lifshitz equation, as in many other cases. For example, the singly periodic solution found in Ref. 7 by this method for an isotropic magnetic material yielded effective formulas only in the very special two-parameter case, whereas the general singly periodic solution depends on four parameters. For an adequate description of experimental inhomogeneous and non-steady-state situations one must know the solution that depends on all four parameters.

Such difficulties always arise when the respective L operator of the inverse scattering method is not self-adjoint. In Ref. 15 a way to overcome this difficulty was suggested, and in Ref. 16 an example in which the oscillating regions were employed at the wavefronts of ultrashort pulses in lightguides was discussed. An effective form of the periodic solution for an isotropic magnetic material was found in Ref. 17. In the present paper periodic solutions are obtained for the case of a ferromagnet with uniaxial anisotropy (the easiest magnetization axis).

2. BASIC EQUATIONS OF THE INVERSE SCATTERING METHOD

In this section we set up the equations of the inverse scattering method for the case of the Landua–Lifshitz equation in a form found convenient.

A uniaxial ferromagnet is described by the Landau–Lifshitz equation

$$\partial \mathbf{M}/\partial t=[\mathbf{M},\ \partial^2\mathbf{M}/\partial x^2]+\beta(\mathbf{Mn})[\mathbf{M},\ \mathbf{n}], \tag{1}$$

where $\mathbf{M}(x,t)$ is the local magnetization, β the anisotropy constant, and $\mathbf{n}$ the unit vector along the x axis (the easiest magnetization axis) along which a wave is propagating. It is assumed here that in terms of the chosen variables, vector $\mathbf{M}$ is normalized by the condition

$$\mathbf{M}^2=1. \tag{2}$$

As shown in Refs. 10 and 11, Eq. (1) is integrable, that is, can be represented in the form of the capability condition for the following two linear systems of equations:

$$\begin{aligned} &\partial\psi_1/\partial x=F\psi_1+G\psi_2, \quad \partial\psi_1/\partial t=A\psi_1+B\psi_2,\\ &\partial\psi_2/\partial x=H\psi_1-F\psi_2, \quad \partial\psi_2/\partial t=C\psi_1-A\psi_2, \end{aligned} \tag{3}$$

where in the given case

$$\begin{aligned} &F=-(i\lambda/2)M_3, \quad G=-(i/2)(\lambda^2+\beta)^{1/2}M_-,\\ &H=-(i/2)(\lambda^2+\beta)^{1/2}M_+, \end{aligned} \tag{4}$$

$$\begin{aligned} &A=(i/2)(\lambda^2+\beta)M_3+(\lambda/4)[(M_-)_xM_+-M_-(M_+)_x],\\ &B=(i/2)\lambda(\lambda^2+\beta)^{1/2}M_-+(1/2)(\lambda^2+\beta)^{1/2}[(M_3)_xM_--M_3(M_-)_x],\\ &C=(i/2)\lambda(\lambda^2+\beta)^{1/2}M_+-(1/2)(\lambda^2+\beta)^{1/2}[(M_3)_xM_+-M_3(M_+)_x] \end{aligned} \tag{5}$$

with $M_\pm=M_1\pm iM_2$ and $(M_3)_x=\partial M_3/\partial x$, and λ the spectral parameter. Substituting Eqs. (4) and (5) into the condition

$$\partial^2\psi_1/\partial x\partial t=\partial^2\psi_1/\partial t\partial x$$

and equating the coefficients of equal powers of λ, we arrive at Eq. (1).

The system (3) has two basic solutions, (ψ_1,ψ_2) and (φ_1,φ_2), which can be used to build a vector with the spherical components

$$f=-(i/2)(\psi_1\varphi_2+\psi_2\varphi_1), \quad g=\psi_1\varphi_1, \quad h=-\psi_2\varphi_2, \tag{6}$$

satisfying the following linear systems:

$$\begin{aligned} &\partial f/\partial x=-iHg+iGh, \quad \partial f/\partial t=-iCg+iBh,\\ &\partial g/\partial x=2iGf+2Fg, \quad \partial g/\partial t=2iBf+2Ag,\\ &\partial h/\partial x=-2iHf-2Fh, \quad \partial h/\partial t=-2iCf-2Ah. \end{aligned} \tag{7}$$

In the process of evolution the length of the vector with components (6) is preserved, that is the quantity

$$f^2-gh=P(\lambda) \tag{8}$$

is independent of x and t. Periodic solutions are specified by the condition that $P(\lambda)$ be a polynomial of λ. Aided with (7), we can easily verify that in our case the single-phase solutions correspond to the fourth-degree polynomial

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$$P(\lambda)=\prod_{i=1}^{4}(\lambda-\lambda_i)=\lambda^4-s_1\lambda^3+s_2\lambda^2-s_3\lambda+s_4. \quad (9)$$

For many physical applications knowledge of just such solutions is sufficient.

Function of the following type correspond to systems (7):

$$f=M_3\lambda^2-f_1\lambda+f_2,\quad g=-iM_-(\lambda^2+\beta)^{1/2}(\lambda-\mu), \quad (10)$$
$$h=-iM_+(\lambda^2+\beta)^{1/2}(\lambda-\mu^*).$$

Identity (8) then yields the following equations:

$$\begin{aligned} &2f_1M+(1-M^2)(\mu+\mu^*)=s_1,\\ &2f_1f_2+(1-M^2)\beta(\mu+\mu^*)=s_3,\\ &f_1^2+2f_2M+(1-M^2)(\beta+\mu\mu^*)=s_2,\\ &f_2^2+(1-M^2)\beta\mu\mu^*=s_4, \end{aligned} \quad (11)$$

with $M\equiv M_3$. The equations for the variable μ introduced in (10) can easily be obtained from (7):

$$\partial\mu/\partial x=-if(\mu)=-i(P(\mu))^{1/2},\quad \partial\mu/\partial t=-(s_1/2)\partial\mu/\partial x. \quad (12)$$

Thus, μ depends only on the phase

$$W=x-(s_1/2)t,\quad \partial\mu/dW=-if(\mu)=-i(P(\mu))^{1/2}. \quad (13)$$

The components of vector **M** can be found from equations that follow from (7):

$$\begin{aligned} &\partial M_3/\partial x=-(i/2)(1-M_3^2)(\mu-\mu^*),\\ &\partial M_-/\partial x=-i(f_1-\mu M_3)M_-,\\ &\partial M_-/\partial t=-i(f_2-\beta M_3)M_--(s_1/2)\partial M_-/\partial x. \end{aligned} \quad (14)$$

We now turn to solving these equations.

3. THE PERIODIC SOLUTION

As in Refs. 15–17, we assume from the start that μ moves only along such trajectories on which identity (8) is always satisfied. A convenient variable parametrizing such trajectories is the quantity $M\equiv M_3$. Using system (11), we can find the link between μ and M.

From the first and second pair of Eqs. (11) we get

$$2f_1(M-f_2/\beta)=s_1-s_3/\beta,$$
$$f_1^2/\beta-(M-f_2/\beta)^2=(s_2-s_4/\beta)/\beta-1,$$

which yield

$$f_1^2=[(P_2(\beta))^{1/2}-\beta^2+s_2\beta-s_4]/(2\beta), \quad (15)$$

where

$$P_2(\beta)=\prod_{i=1}^{4}(\beta+\lambda_i^2) \quad (16)$$

and

$$f_2=(s_3-s_1\beta)/(2f_1)+\beta M. \quad (17)$$

If we allow for (17), the last equation in (14) assumes the form

$$\partial M_-/\partial t=-i[(s_3-s_1\beta)/2f_1]M_--(s_1/2)\partial M_-/\partial x,$$

which together with the second equation in (14) yield

$$M_-=\exp[it(s_1\beta-s_3)/2f_1]\hat{M}_-, \quad (18)$$

where $\hat{M}_-$ satisfies the following equation

$$d\hat{M}_-/dW=-i(f_1-\mu M)\hat{M}_-. \quad (19)$$

We now seek μ and μ^*. Equations (11), (15), and (17) imply that

$$\mu+\mu^*=(s_1-2f_1M)(1-M^2)^{-1},$$
$$\mu\mu^*=\{s_2-f_1^2-[(s_3-s_1\beta)/f_1]M-(1+M^2)\beta\}(1-M^2)^{-1},$$

from which we find that

$$\mu=[s_1-2f_1M+2(-\beta R(M))^{1/2}]/2(1-M^2), \quad (20)$$

where

$$R(M)=M^4+[(s_3-s_1\beta)/f_1\beta]M^3-(s_2/\beta)M^2$$
$$+[s_1f_1/\beta-(s_3-s_1\beta)/f_1\beta]M+(4s_2-4f_1^2-s_1^2-4\beta)/4\beta, \quad (21)$$

the constant f_1 is specified by Eq. (15), and μ^* is obtained by changing the sign in front of the sequence root in (20). We call polynomial $R(M)$ the resolvent of $P(\lambda)$, since as $\beta\to 0$ it transforms into its well-known third-degree resolvent,[17] and its zeros ν_i,

$$R(\nu_i)=0,\quad i=1,2,3,4 \quad (22)$$

are linked to the zeros λ_i $(i=1,2,3,4)$ by symmetric expressions derived in the Appendix. We have

$$\begin{aligned} \nu_1=&(4f_1\beta)^{-1}[(\lambda_1-\lambda_3)(\lambda_2'-\lambda_4')+(\lambda_2-\lambda_4)(\lambda_1'-\lambda_3')]\\ &\times\{(\lambda_1-\lambda_3)[2(\lambda_1+\lambda_3)(\lambda_2'-\lambda_4')\beta+(\lambda_2\lambda_4'-\lambda_4\lambda_2')((\lambda_1+\lambda_3)^2\\ &-(\lambda_1'-\lambda_3')^2)]\\ &+(\lambda_2-\lambda_4)[2(\lambda_2+\lambda_4)(\lambda_1'-\lambda_3')\beta+(\lambda_1\lambda_3'-\lambda_3\lambda_1')((\lambda_2+\lambda_4)^2\\ &-(\lambda_2'-\lambda_4')^2)]\}, \end{aligned} \quad (23)$$

where

$$\lambda_i'=(\lambda_i^2+\beta)^{1/2}, \quad (24)$$

ν_2 and ν_3 are obtained from ν_1 by interchanging indices $3\leftrightarrow 4$ and $3\leftrightarrow 2$, respectively, and ν_4 can be found by the formula

$$\nu_4=(s_1\beta-s_3)/f_1\beta-(\nu_1+\nu_2+\nu_3). \quad (25)$$

By introducing the resolvent (21) we were able to describe explicitly the trajectory of μ in the complex plane via Eq. (20) and to rid ourselves of the need to allow for additional identities. Let us find the equation for the third component of the spin, $M=M_3$, which parametrizes curve (20). From (8) it is clear that when μ is given by (20) and $\lambda=\mu$, so that $g=0$, we have $P(\mu)=f^2(\mu)$, and differentiating (20) with respect to M yields

$$dM/d\mu=(-\beta R(M))^{1/2}/f(\mu). \quad (26)$$

Multiplying (26) by (13), we find that

$$dM/dW=(\beta R(M))^{1/2}. \quad (27)$$

Since we are considering the case of $\beta>0$ for the easiest magnetization axis, the values of M vary within the interval $\nu_3\leqslant M\leqslant\nu_2$, where $R(M)\geqslant 0$. Polynomial $R(M)$ is of the fourth degree, and the solution of Eq. (27) can be expressed in a standard manner in terms of elliptic functions:

$$M_3 = M = \frac{(\nu_2-\nu_4)\nu_3-(\nu_2-\nu_3)\nu_4\,\mathrm{sn}^2\{[\beta(\nu_1-\nu_3)(\nu_2-\nu_4)]^{1/2}(W+W_0)/2,k\}}{\nu_2-\nu_4-(\nu_2-\nu_3)\,\mathrm{sn}^2\{[\beta(\nu_1-\nu_3)(\nu_2-\nu_4)]^{1/2}(W+W_0)/2,k\}}, \quad (28)$$

where W_0 is the initial phase value, and

$$k^2 = \frac{(\nu_2-\nu_3)(\nu_1-\nu_4)}{(\nu_1-\nu_3)(\nu_2-\nu_4)}. \quad (29)$$

In what follows to simplify notation we put $W_0 = 0$.

Substitution of (20) and (27) into (19) yields

$$M_- = (1-M^2)^{1/2}\exp\left\{(i/2)\int_0^W \frac{s_1M-2f_1}{1-M^2}\,dW\right\}, \quad (30)$$

where the function $M(W)$ is specified by (28). It is convenient to express the integral in (30) in terms of Weierstrass's $\wp$ function. If we use the formula

$$\mathrm{sn}^2\{[\beta(\nu_1-\nu_3)(\nu_2-\nu_4)]^{1/2}W/2,k\} = \frac{e_1-e_3}{\wp(W)-e_3},$$

where

$$e_1 = [s_2-3\beta(\nu_1\nu_4+\nu_2\nu_3)]/12,$$
$$e_2 = [s_2-3\beta(\nu_1\nu_3+\nu_2\nu_4)]/12, \quad (31)$$
$$e_3 = [s_2-3\beta(\nu_1\nu_2+\nu_3\nu_4)]/12,$$

then for the integrand in (30) we obtain

$$\frac{s_1M-2f_1}{1-M^2} = \frac{s_1-2f_1}{2(1-\nu_3)}\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\varkappa)} - \frac{s_1+2f_1}{2(1+\nu_3)}\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\tilde\varkappa)}$$

where the parameters ρ, $\varkappa$, and $\tilde\varkappa$ are defined via the following formulas:

$$\wp(\rho) = e_3+\beta(\nu_1-\nu_3)(\nu_2-\nu_3)/4,$$
$$\wp(\varkappa) = e_3+\beta(\nu_1-\nu_3)(\nu_2-\nu_3)(1-\nu_4)/4(1-\nu_3), \quad (32)$$
$$\wp(\tilde\varkappa) = e_3+\beta(\nu_1-\nu_3)(\nu_2-\nu_3)(1+\nu_4)/4(1+\nu_3).$$

Integration is carried out using the formula

$$\int_0^W \frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\varkappa)}\,dW = W + \frac{\wp(\rho)-\wp(\varkappa)}{\wp'(\varkappa)}\left[\ln\frac{\sigma(W+\varkappa)}{\sigma(W-\varkappa)} - 2\zeta(\varkappa)W\right],$$

where ζ and σ are Weierstrass's ζ and σ functions. Simple calculations lead to the following results:

$$M_-(x,t) = (1-\nu_3^2)^{1/2}\exp\{it(s_1\beta-s_3)/2f_1 + 2iW(s_1\nu_3-2f_1)/(1-\nu_3^2) - (\zeta(\varkappa)+\zeta(\tilde\varkappa))W\}\frac{\sigma^2(\rho)\sigma(W+\varkappa)\sigma(W+\tilde\varkappa)}{\sigma(\varkappa)\sigma(\tilde\varkappa)\sigma(W+\rho)\sigma(W-\rho)}, \quad (33)$$

with $W = x - s_{1/2}t$. Formulas (28) and (33) give the general expression for the single-phase periodic solution of the Landau–Lifshitz equation in the case of a uniaxial ferromagnet.

4. DEGENERATE CASES

At $f_1 = 0$ formulas (28) and (33) require a complicated passage to the limit, so that it is better to consider such degenerate cases separately. Note that f_1^2 is always nonnegative since

$$P_2(\beta)-(\beta^2-s_2\beta+s_4)^2 = \beta(s_1\beta-s_3)^2.$$

Combining this with (15), we find that f_1 vanishes in two cases:

$$a)\ s_1=s_3=0,\quad б)\ \beta=s_3/s_1. \quad (34)$$

We start with case (a), which corresponds to two pairs of zeros λ_i on the imaginary axis:

$$\lambda_1=i\gamma_1,\quad \lambda_2=i\gamma_2,\quad \lambda_3=-i\gamma_1,\quad \lambda_4=-i\gamma_2. \quad (35)$$

For values of β such that

$$\beta^2-s_2\beta+s_4\geqslant 0,$$

we find the system (11) yields

$$f_1=0,\quad f_2=M\beta+(\beta^2-s_2\beta+s_4)^{1/2}, \quad (36)$$

$$\mu = i[(1-M^2)(s_2-\beta-2(\beta^2-s_2\beta+s_4)^{1/2}M-\beta M^2)]^{1/2}/(1-M^2),$$

that is, μ moves along the imaginary axis. Suppose that $\gamma_1\geqslant\gamma_2$. Then we must distinguish between three cases,

$$1)\ \beta\leqslant\gamma_2^2,\quad 2)\ \beta\geqslant\gamma_1^2,\quad 3)\ \gamma_2^2\leqslant\beta\leqslant\gamma_1^2, \quad (37)$$

with $f_1 = 0$ only for cases (1) and (2). The zeros of the resolvent in (36) form the following patterns:

$$\nu_4=-\{\gamma_1\gamma_2+[(\gamma_1^2-\beta)(\gamma_2^2-\beta)]^{1/2}\}/\beta\leqslant\nu_3=-1<\nu_2=1\leqslant\nu_1 = \{\gamma_1\gamma_2-[(\gamma_1^2-\beta)(\gamma_2^2-\beta)]^{1/2}\}/\beta \quad (38)$$

for case (1) ($\beta\leqslant\gamma_2^2$), and

$$\nu_4=-1\leqslant\nu_3=-\{\gamma_1\gamma_2-[(\beta-\gamma_1^2)(\beta-\gamma_2^2)]^{1/2}\}/\beta<\nu_2 = \{\gamma_1\gamma_2+[(\beta-\gamma_1^2)(\beta-\gamma_2^2)]^{1/2}\}/\beta\leqslant\nu_1=1 \quad (39)$$

for case (2) ($\beta\geqslant\gamma_1^2$). The solutions for M_3 are obtained by substituting these values of ν_i into (28).

For case (3) ($\gamma_2^2\leqslant\beta\leqslant\gamma_1^2$) we have

$$(P_2(\beta))^{1/2} = -(\beta^2-s_2\beta+s_4),$$

that is,

$$f_1=(s_2\beta-s_4-\beta^2)/\beta\geqslant 0,\quad f_2=\beta M,$$
$$R(M)=M^4-(s_2/\beta)M^2+s_4/\beta^2, \quad (40)$$

and the resolvent's zeros are

$$\nu_4=-\gamma_1/\beta^{1/2},\quad \nu_3=-\gamma_2/\beta^{1/2},\quad \nu_2=\gamma_2/\beta^{1/2},\quad \nu_1=\gamma_1/\beta^{1/2}, \quad (41)$$

which leads to the appropriate solution after substitution into (28).

We now turn to case (b) ($\beta = s_3/s_1$). This corresponds to the following pattern of the zeros of $P(\lambda)$:

$$\lambda_1=V+i\gamma,\quad \lambda_2=i\beta^{1/2},\quad \lambda_3=V-i\gamma,\quad \lambda_4=-i\beta^{1/2}. \quad (42)$$

so tht $\lambda_2' = \lambda_4' = 0$. Simple calculations lead to the resolvent

$$R(M) = M^4 - [(V^2+\gamma^2+\beta)/\beta]M^2 + \gamma^2/\beta \quad (43)$$

with zeros

$$\begin{aligned} \nu_1 = -\nu_4 = \{[V^2+\gamma^2+\beta+((V^2+\gamma^2+\beta)^2-4\gamma^2\beta)^{1/2}]/2\beta\}^{1/2}, \\ \nu_2 = -\nu_3 = \{[V^2+\gamma^2+\beta-((V^2+\gamma^2+\beta)^2-4\gamma^2\beta)^{1/2}]/2\beta\}^{1/2} \end{aligned} \quad (44)$$

and the appropriate solution (28).

5. THE SOLITON LIMIT

In the soliton limit the two pairs of complex conjugate zeros λ_i merge into one pair,

$$\lambda = \lambda_1 = \lambda_2 = \lambda_3^* = \lambda_4^* = V/2 + i\gamma, \quad \lambda' = (\lambda^2+\beta)^{1/2}. \quad (45)$$

In this limit the zeros (23) and (25) of the resolvent transform into

$$\begin{aligned} \nu_1 = \nu_2 = 1, \quad \nu_3 = (\lambda\lambda'+\lambda^*\lambda'^*)/(\lambda'\lambda^*+\lambda\lambda'^*), \\ \nu_4 = -(\lambda\lambda^*+\lambda'\lambda'^*)/\beta. \end{aligned} \quad (46)$$

For such values of ν_i formula (28) yields

$$M_3 = \frac{\nu_4(1-\nu_3)+(\nu_3-\nu_4)\,\mathrm{ch}^2(\gamma W)}{1-\nu_3+(\nu_3-\nu_4)\,\mathrm{ch}^2(\gamma W)}, \quad W = x - Vt. \quad (47)$$

By introducing the angle θ between vectors $\mathbf{M}$ and $\mathbf{n}$, so that $M_3 = \cos\theta$, the soliton solution can be written as[2,10]

$$\mathrm{tg}^2\frac{\theta}{2} = \frac{\gamma^2/\beta}{\Omega\,\mathrm{ch}^2(\gamma W) - (\Omega-\Omega_1)/2}, \quad (48)$$

where

$$\Omega = [(V^2/4-\gamma^2+\beta)^2+V^2\gamma^2]^{1/2}/\beta, \quad \Omega_1 = (V^2/4-\gamma^2+\beta)/\beta. \quad (49)$$

For the applications of the theory developed here it is important that the soliton parameters be expressed in terms of the values of the spectral parameter λ_i, since the Whitham equations, which describe the weakly inhomogeneous periodic waves, assume the simplest form when expressed in terms of just the variables λ_i.

6. CONCLUSION

The periodic solution obtained here has a fairly effective form that makes it possibe to trace the variations of the solution caused by the evolution of the parameters λ_i in inhomogeneous and non-steady-state problems. Employing the methods used in Refs. 16 and 17, we can easily verify that the corresponding Whitham equations have the same form as in the isotropic case,[17] where $\beta = 0$. Thus, the theory forms a basis for applications.

It can be assumed that this approach, in which the equations of the inverse scattering method are re-parametrized via the algebraic resolvent of the initial polynomial $P(\lambda)$, which specifies the periodic solution, can be generalized to other integrable equations, including those that that are not within the scope of the Zakharov–Shabat scattering problem.

APPENDIX

Let us find the zeros of the resolvent (21). If we combine (10), (17), and (20) with identity (8), we get

$$\begin{aligned} P(\lambda) = (M\lambda^2 - f_1\lambda + (s_3-s_1\beta)/2f_1 + \beta M)^2 + (1-M^2)(\lambda^2+\beta) \\ \times[\lambda - (s_1-2f_1M+2(-\beta R(M))^{1/2})/2(1-M^2)] \\ \times[\lambda - (s_1-2f_1M-2(-\beta R(M))^{1/2})/2(1-M^2)]. \end{aligned} \quad (A1)$$

If we assume here that M is equal to one of the zeros ν of the resolvent, the right-hand side of (A1) takes on the form of the difference of two squares, so that the four zeros of the polynomial $P(\lambda)$ prove to be the roots of two equations,

$$\begin{aligned} \nu\lambda_i^2 - f_1\lambda_i + (s_3-s_1\beta)/2f_1 + \beta\nu \\ = \pm i(1-\nu^2)^{1/2}(\lambda_i^2+\beta)^{1/2}[\lambda_i - (s_1-2f_1\nu)/2(1-\nu^2)]. \end{aligned} \quad (A2)$$

Suppose that the zeros λ_1 and λ_2 correspond to the " + " on the right-hand side of (A2) and the zeros λ_3 and λ_4 to the " − ." We introduce the following notation:

$$(i) = 2f_1\lambda_i'\nu - (2f_1^2\lambda_i + s_1\beta - s_3)/\lambda_i'. \quad (A3)$$

Dividing the four relations in (A2) by each other, we get six formulas of the type

$$\frac{(i)}{(j)} = \pm\frac{\lambda_i'}{\lambda_j'}\,\frac{2\lambda_i(1-\nu^2)-s_1+2f_1\nu}{2\lambda_j(1-\nu^2)-s_1+2f_1\nu}, \quad (A4)$$

where the " + " corresponds to (1)/(2) and (3)/(4), and the " − " to the other combinations. From (A4) we can obtain for $1-\nu^2$ six expressions of the type

$$1-\nu^2 = \frac{s_1-2f_1\nu}{2}\,\frac{(i)\pm(j)}{\lambda_j(i)\pm\lambda_i(j)}, \quad (A5)$$

where we have used the same sign convention. Equating (A5) to each other pairwise, we get four equations that are linear in (i),

$$\begin{aligned} (\lambda_1-\lambda_2)(3)+(\lambda_1-\lambda_3)(2)+(\lambda_3-\lambda_2)(1)=0, \\ (\lambda_1-\lambda_2)(4)+(\lambda_1-\lambda_4)(2)+(\lambda_4-\lambda_2)(1)=0, \\ (\lambda_1-\lambda_3)(4)+(\lambda_4-\lambda_1)(3)+(\lambda_4-\lambda_3)(1)=0, \\ (\lambda_2-\lambda_3)(4)+(\lambda_4-\lambda_2)(3)+(\lambda_4-\lambda_3)(2)=0, \end{aligned} \quad (A6)$$

and three equations quadratic in (i),

$$\begin{aligned} (1)(2)(\lambda_3-\lambda_4)+(2)(3)(\lambda_1-\lambda_4)+(3)(4)(\lambda_1-\lambda_2) \\ +(4)(1)(\lambda_3-\lambda_2)=0, \\ (1)(2)(\lambda_3-\lambda_4)+(2)(4)(\lambda_3-\lambda_1) \\ +(4)(3)(\lambda_2-\lambda_1)+(3)(1)(\lambda_2-\lambda_4)=0, \\ (1)(3)(\lambda_2-\lambda_4)+(3)(2)(\lambda_4-\lambda_1)+(2)(4)(\lambda_1-\lambda_3) \\ +(4)(1)(\lambda_3-\lambda_2)=0. \end{aligned} \quad (A7)$$

Only three equations in (A6) are linearly independent and Eqs. (A7) are corollaries of (A6). Since (A3) is linear in ν, Eqs. (A6) are the linear equations for calculating ν. For instance, from (A6) it follows that

$$[(1)-(2)](\lambda_3-\lambda_4)+[(3)-(4)](\lambda_1-\lambda_2)=0, \quad (A8)$$

so that ν_3 can be found by solving the following symmetric equation:

$$\begin{aligned} 2f_1[(\lambda_1-\lambda_2)(\lambda_3'-\lambda_4')+(\lambda_3-\lambda_4)(\lambda_1'-\lambda_2')]\nu \\ = 2f_1^2[(\lambda_1-\lambda_2)(\lambda_3/\lambda_3'-\lambda_4/\lambda_4')+(\lambda_3-\lambda_4)(\lambda_1/\lambda_1'-\lambda_2/\lambda_2')] \\ -(s_3-s_1\beta)[(\lambda_1-\lambda_2)(1/\lambda_3'-1/\lambda_4')+(\lambda_3-\lambda_4)(1/\lambda_1'-1/\lambda_2')]. \end{aligned} \quad (A9)$$

After simple transformations that use formula (15) for f_1^2 we find the root ν_3 of the resolvent in a form similar to (23).

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