Generation of large-scale vortices through the action of spiral turbulence of a convective nature

E. A. Lupyan, A. A. Mazurov, P. V. Rutkevich, and A. V. Tur

Space Research Institute, Russian Academy of Sciences

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We consider the problem of the generation of large-scale vortex structures such as a tropical cyclone under conditions which simulate the turbulence of a tropical depression. We assume that the meso-scale humid convection arising on a locally unstable stratification profile is the main source of the turbulence.

1. INTRODUCTION

Models which take into account the spiral properties of turbulence have opened up the possibility of solving a broad class of hydrodynamical problems about the generation of large-scale structures. Special interest in those arises when one describes processes of the development of cyclonic formations in the earth's atmosphere. Recently it has been shown that one can use them to construct models for the initial stage of the development of tropical cyclones. A whole series of papers[1–4] have been devoted to this problem; it was shown in those that not only the characteristic shape and the parameters for the appearance of large-scale vortex structures in the model agree well with the parameters of tropical cyclones, but also that such a kind of scenario for the development of instabilities describes the characteristic stages of the development of cyclones, starting from a tropical depression.

However, the models constructed up to now neglect a number of features of the motions which develop in a tropical atmosphere. For instance, the basis of these models is the averaging over a characteristic scale of the pulsations of the turbulent velocity and the averaged equations of motion contain large-scale convective terms. The reason is that convection is regarded as a large-scale process on the background of small-scale turbulence. At the same time it is well known that powerful hypercell convection, consisting of a set of separate cells arising as the result of the release of latent heat by vapor condensation, is indicative of a tropical depression.[5–8] The motion in these cells can be considered to be turbulence. One should, however, note that on sufficiently large scales the average density stratification is nearly neutral and therefore the large-scale structures arising in the atmosphere do, generally speaking, not have a convective nature. This fact must be taken into account when constructing a model describing the large-scale motions in the real atmosphere.

In order to model the conditions of a tropical depression one must assume the turbulent motions to develop on a locally unstable profile. Such a turbulence has a convective nature and guarantees that the average profile tends to become neutral according to the theory of convective accommodation.[5,8,9] The large-scale motions are therefore assumed to develop on a neutral profile. Such an approach makes it possible to use successfully the model previously proposed for the appearance of large-scale structures and approximating it substantially to actual atmospheric conditions arising in the region of a tropical depression.

In the present paper we consider the problem of the generation of large-scale structures under conditions modelling a tropical depression on the background of spiral convective turbulence.

2. STATEMENT OF THE PROBLEM

Atmospheric turbulence in the region of a tropical depression is characterized by volume heating due to release of latent heat by vapor condensation.[5,7] If a sufficient amount of heat is released, a local convection process arises which lasts a time τ and has a characteristic size λ. Convective motions in the air soon become turbulent because of its low viscosity coefficient. It is therefore natural to assume that the turbulent motions start from the convective scale λ and are transferred to smaller scales. It is well known that the characteristic size of a convective cloud cell (the largest reach 2 km) is much smaller than the thickness of the troposphere (approximately 10 to 15 km).[5]

The convective accommodation model makes it possible under conditions of a tropical depression to determine such turbulence parameters in the atmosphere as the turbulence viscosity and the velocity of the turbulent pulsations. The amount of heat released determines the strength of the turbulence and the turbulent viscosity ν_T increases so much that the Rayleigh number of the small-scale convection remains on the level of its critical value Ra_c:[5,10,11]

$$Ra_c \sim (\gamma-\gamma_a) g\beta\lambda^4/\nu_T^2,$$

where γ_a is the moist-adiabatic temperature gradient and γ the average value of the gradient of the actual temperature profile existing after a time τ in a region of size λ. Hence, the velocity of the turbulent pulsations can be estimated using the formula

$$u_T=\nu_T/\lambda=[g\beta(\gamma-\gamma_a)\lambda^2 Ra_c^{-1}]^{1/2}.$$

To describe the large-scale motions in a layer of the atmosphere with a thickness h appreciably larger than the characteristic energy-scale λ of the convective turbulence we must specify a model describing the convection process. The simplest problem in convection theory is that of the stability of a horizontal plane-parallel layer of fluid with a vertical temperature gradient A.[10,11] Since details of the process of the appearance and the course of moist convection is not important for the development of large-scale processes, it is sufficient to restrict ourselves in the calculation of the Reynolds stresses to the simplest model, assuming that the

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quantity A corresponds to the difference between the gradients of the actual and the moist-adiabatic temperature profiles:

$$A=\gamma-\gamma_a.$$

The convection process in the model hence proceeds on a given temperature profile and lasts a time τ which may be assumed to be comparable to the correlation time of the turbulent motions caused by the convection.

Since the convective motions in a tropical depression region are influenced by the Coriolis force we shall assume that the turbulence arising under those conditions is spiral. The concept of spiral turbulence is these days well known.[1,12–14] In the present model we shall assume the spiral turbulence to be given through an external random force F_i. We shall assume the parameters of the turbulent velocity field caused by the action of the force F_i to be compatible with the parameters of the convection process. To describe convection with spiral turbulence in the tropical depression model we shall use the following set of equations:[1]

$$\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\nabla_k v_i+\frac{\nabla_i P}{\rho}-ge_i\beta\theta=F_i. \quad (1)$$

$$\frac{\partial\theta}{\partial t}-\chi\Delta\theta+v_k\nabla_k\theta-e_kv_kA=0. \quad (2)$$

$$\nabla_k v_k=0,$$

$$\left[\nabla_k=\frac{\partial}{\partial x_k}\right],\quad \Delta=\nabla_k\nabla_k,\quad \mathbf{e}=(0,0,1). \quad (3)$$

Here g is the free-fall acceleration, ν the kinematic viscosity, χ the thermal conductivity, and β the thermal expansion coefficient. We shall assume the Prandtl number to be equal to unity:

$$\nu/\chi=1.$$

The temperature field $T(x,t)$ is given in the form of a sum of the basic state,

$$T_0(z)=-Az,$$

existing after a time τ and the turbulent pulsations $\theta(x,t)$:

$$T(x,t)=T_0(z)+\theta(x,t).$$

3. EQUATION FOR THE AVERAGE FLOW

The set of Eqs. (1) to (3) enables us to obtain, accurate to second-order nonlinear terms, the equation describing the velocity field under the action of the random force F_i:

$$DL_{ij}v_j=F_i-P_{im}\nabla_k(v_kv_m)-\beta AgP_{im}e_me_j\nabla_k\frac{1}{D}\left(v_k\frac{1}{D}v_j\right). \quad (4)$$

We have introduced here the operators

$$L_{ij}=\delta_{ij}-\beta AgD^{-2}P_{im}e_me_j,$$
$$D=\frac{\partial}{\partial t}-\nu\Delta,\quad P_{im}=\delta_{im}-\frac{\nabla_i\nabla_m}{\Delta}, \quad (5)$$

where P_{im} is the projection operator eliminating the potential part of the velocity field.

We evaluate for the set (1) to (3) the Reynolds stresses caused by the spirality of the turbulence and irreducible to turbulent dissipative terms. To do this we use statistical averaging over small-scale pulsations, assuming the background spiral turbulence to be weak (see, e.g., Ref. 1). We write the velocity field as a sum

$$v_i=\langle v_i\rangle+v_i^T+\tilde v_i.$$

Here $\langle v_i\rangle$ is the average large-scale flow, v_i^T are the turbulent velocity pulsations caused by the external random force F_i and $\tilde v_i$ is a small inhomogeneous correction—the result of the interaction between the average field and the turbulent pulsations, $\langle v\rangle\ll v^T$, $\tilde v\ll v^T$. Here $\tilde v$ is a functional of v^T and $\langle v\rangle$.

We get for the unknown fields $\langle v_i\rangle$ and $\tilde v_i$ from Eq. (4), by standard methods to lowest approximation, the equations

$$D\langle v_j\rangle=-P_{im}\nabla_k(\langle v_k^T\tilde v_m\rangle+\langle\tilde v_kv_m^T\rangle), \quad (6)$$

$$DL_{ij}\tilde v_j=-P_{im}\nabla_k(\langle v_k\rangle\tilde v_m+\tilde v_k\langle v_m\rangle)$$
$$-\beta AgP_{in}e_ne_m\nabla_k\frac{1}{D}\left(\tilde v_k\frac{1}{D}\langle v_m\rangle+v_k\frac{1}{D}\tilde v_m\right). \quad (7)$$

We assume, as we discussed already earlier, the temperature profile to be neutral in Eq. (6) for the large-scale velocity $\langle v_i\rangle$.

To determine the averages on the right-hand side of Eq. (6) we use the Furutsu–Novikov formula,[15] using the functional dependence of the $\tilde v_i$ field on the turbulent v_i^T field:

$$\langle v_k^T(t,\mathbf{x})\tilde v_m(t,\mathbf{x})\rangle=\lim_{\substack{t_1\to t\\ \mathbf{x}_1\to\mathbf{x}}}\int ds\int d\mathbf{y}\,Q_{kr}^T(t-s,\mathbf{x}-\mathbf{y})\times\left\langle\frac{\delta\tilde v_m(t_1,\mathbf{x}_1)}{\delta v_r^T(s,\mathbf{y})}\right\rangle, \quad (8)$$

where

$$Q_{kr}^T(t-s,\mathbf{x}-\mathbf{y})=\langle v_k^T(t,\mathbf{x})v_r^T(s,\mathbf{y})\rangle$$

is the correlation tensor of the spiral turbulent field $v_k^T(t,x)$.

In the derivation of the Furutsu–Novikov formula it is assumed that the turbulence is Gaussian. The assumption can be considered to be satisfied in the present model since we study the range of energy-scale turbulence for which the Gaussian assumption is best applicable.[16]

The variational derivative in Eq. (8) is evaluated using Eq. (7) and neglecting derivatives of the large-scale velocity field (they give a small correction to the turbulent viscosity):

$$\left\langle\frac{\delta\tilde v_m(t,\mathbf{x})}{\delta v_r^T(s,\mathbf{y})}\right\rangle=-\langle v_a\rangle L_{mi}^{-1}D^{-1}P_{il}\nabla_a(\delta_{lr}+\beta AgD^{-2}e_le_r)\delta(t-s)\delta(\mathbf{x}-\mathbf{y}), \quad (9)$$

where the inverse operator L_{mi}^{-1} has the form

$$L_{mi}^{-1}=\delta_{mi}+\frac{\beta AgP_{mn}e_ne_i}{D^2-\beta AgP_{nm}e_me_n}.$$

It is convenient, when carrying out calculations using Eq. (8), to change to a Fourier transformation in $\mathbf{x}$ and $\mathbf{x}_1$ and to integrate over dy, using $\delta(\mathbf{x}_1-\mathbf{y})$ in Eq. (9). This makes is possible to write the quadratic combination

$$\langle v_k^T(t,\mathbf{x})\tilde v_m(t,\mathbf{x})\rangle$$

as follows:

$$\langle \tilde{v}_k^T(t,\mathbf{x})\tilde{v}_m(t,\mathbf{x})\rangle = -\langle v_a\rangle \lim_{t_1\to t}\int ds\int (2\pi)^{-3}$$

$$\times\ d\mathbf{k}\hat{Q}_{kr}^T(t-s,k)$$

$$\times L_{mi}^{-1}\left(\frac{\partial}{\partial t_1}+\nu k^2\right)^{-1}P_{il}ik_a\Big(\delta_{lr}$$

$$+\beta A g\left(\frac{\partial}{\partial t_1}+\nu k^2\right)^{-2}e_l e_r\Big)\delta(t_1-s). \quad (10)$$

We give the spiral part of the Fourier transform of the turbulence correlator by the formula

$$\hat{Q}_{kr}^T(t-s,k)=G_0\lambda^4 u_T^2\exp\left(-\frac{|t-s|}{\tau}\right)f(k)\varepsilon_{krg}ik_g. \quad (11)$$

To take into account the characteristic properties of the convective turbulence it is necessary to make the parameters of the correlation tensor $\hat{Q}_{kr}^T$ agree with the parameters of the linear L_{ij} operator of (5) which describes the process of the development of convection. It is natural to assume that the characteristic scale of turbulent motions caused by convection does not exceed the size of the convective cells and to give the spectral distribution $f(k)$ of the external force in the form of a function of the absolute magnitude of the wave number k, which vanishes when

$$k\lambda\leqslant\varkappa,$$

where the quantity $\varkappa$ guarantees the compatibility of the energy-scale turbulence with the convection. We choose the way the function $f(k)$ decreases with increasing wave number k to be a power law with exponent n:

$$f(k)=\theta(\lambda k-\varkappa)(\lambda k)^{-n}. \quad (12)$$

We must choose the exponent n such that the magnitude of the topological invariant

$$H=\langle \mathbf{v}^T \text{ curl } \mathbf{v}^T\rangle$$

turns out to bounded:[16,17]

$$H=-(2\pi)^{-3}\int \varepsilon_{ksr}ik_s\hat{Q}_{kr}^T\,d\mathbf{k}<\infty.$$

Since the main contribution to the integral in Eq. (10) comes from the energy-scale turbulent convection region, the explicit form of the decrease of the turbulence spectrum with increasing wave number k in this case does not affect greatly the value of the integral. It is natural to take the turbulence correlation time in the present model to be equal to the average value of the time τ during which the local temperature gradient exists.

One must bear in mind that the convection process and hence the convective turbulence is not strictly isotropic. This manifests itself in our model in the presence of the scalar $P_{mn}e_m e_n$ in the integrand. The integration over the wave numbers in Eq. (10) must thus be carried out taking the shape of the convective cell into account. This can be done most simply by giving its aspect ratio η which is such that

$$k_z=\eta k_\perp.$$

The integration over the wave numbers then takes the form

$$\int d\mathbf{k}=2\int_0^\infty dk_z\int_0^\infty k_\perp dk_\perp\int_0^{2\pi}d\varphi\,\delta(\lambda k_z-\eta\lambda k_\perp).$$

We perform the integration over the angle $d\varphi$ and over the vertical wave number dk_z and let $\mathbf{x}_1\to\mathbf{x}$ in Eq. (10). The right-hand side of the large-scale Eq. (6) takes the form

$$-P_{im}\nabla_k(\langle v_k^T\tilde{v}_m\rangle+\langle\tilde{v}_k v_m^T\rangle)$$
$$=-\frac{1}{8\pi^2}\frac{G_0u_T^2}{\lambda}P_{im}(e_m\varepsilon_{kra}+e_k\varepsilon_{mra})\nabla_k e_r\langle v_a\rangle I. \quad (13)$$

Here

$$I=\lim_{t_1\to t}\int ds\int \lambda^5 k_\perp dk_\perp\theta(\lambda k-\varkappa)(\lambda k)^{-n}\exp\left(-\frac{|t-s|}{\tau}\right)$$
$$\times\frac{D_1^2+g\beta A(1+\eta^2)^{-1}}{D_1^2-g\beta A(1+\eta^2)^{-1}}k_\perp\cdot\frac{g\beta A}{D_1^3}\delta(t_1-s),$$

where

$$D_1=\frac{\partial}{\partial t_1}+\nu k^2,\qquad k^2=(1+\eta^2)k_\perp^2.$$

One can easily estimate the quantity I by putting

$$\varkappa^4=Ra_c$$

and assuming the correlation time τ to be sufficiently long, which corresponds to the conditions for the generation of convective turbulence:

$$I=\frac{\lambda^2}{\nu}\varkappa^{2-n}(1+\eta^2)^{-1-n/2}\ln\left[\frac{2\varkappa^2\tau\nu(1+\eta^2)}{\lambda^2}\right]. \quad (14)$$

The equation for the large-scale motions (6) must be made dimensionless, expressing lengths in terms of the thickness of the atmospheric layer $h\gg\lambda$ and the times in terms of h^2/ν_T. As a result we obtain the Reynolds equation describing large-scale motions on the background of spiral convective turbulence:

$$\left(\frac{\partial}{\partial t}-\Delta\right)\langle v_i\rangle=SP_{im}(e_m\varepsilon_{kra}+e_k\varepsilon_{mra})\nabla_k e_r\langle v_a\rangle, \quad (15)$$

where we have introduced the quantity S which we shall call the convective spirality coefficient. It can be expressed in terms of the macroparameters of the problem as follows:

$$S=-\frac{G_0}{8\pi^2}Re\frac{h}{\lambda}\varkappa^{2-n}(1+\eta^2)^{-1-n/2}\ln\left[\frac{2\varkappa^2\tau\nu(1+\eta^2)}{\lambda^2}\right]. \quad (16)$$

The Reynolds equation we have obtained turns out to be significantly simpler than the corresponding large-scale equation studied in Refs. 1 to 4, in which the random external force was not made compatible with the convection and the temperature gradient was specified in the whole layer. These equations contained convection on the large scale and also a second spirality coefficient. The absence of these parameters in the large-scale Eq. (15) is connected with taking the feedback of the small-scale convection on the temperature gradient into account, which leads to the establishment of a neutral stratification profile on the large scale.

The convective spirality coefficient S can be written in terms of the topological invariant H, in terms of which we can express the coefficient G_0:

$$G_0=-\pi^2\frac{H\lambda}{u_T^2}\varkappa^{n-4}(1+\eta^2)^{-1+n/2}(n-4). \quad (17)$$

The formula for the coefficient S then takes the form

$$S=\frac{H\lambda}{u_T^2}\frac{h}{\lambda}Re\frac{1}{8}(n-4)\varkappa^{-2}(1+\eta^2)^{-2}\ln\left[\frac{2\varkappa^2\tau\nu(1+\eta^2)}{\lambda^2}\right]. \quad (18)$$

The spirality coefficient S is according to Eq. (18) proportional to the quantity $H\lambda/u_T^2$ which characterizes the energy fraction of the spirality of the turbulence. The parameter h/λ shows that the coefficient S increases with increasing thickness of the tropical depression region which is entrained by the turbulence. When the strength of the turbulence increases S also increases and although the model was constructed for small Reynolds numbers one can draw qualitative conclusions about the amplification of the large-scale instability for large Reynolds numbers by assuming that the tensor structure of the large-scale equations does not change in that case.

4. STUDY OF THE LARGE-SCALE EQUATIONS OF MOTION

The equations obtained for the average field contain in contrast to their laminar analogues terms in which the tensor ε_{ira} occurs. These terms lead to a positive feedback between the toroidal and poloidal components of the velocity field and as a consequence to large-scale instability.

In order to study these effects we write the velocity field in the form

$$\langle \mathbf{v}\rangle=\langle \mathbf{v}_t\rangle+\langle \mathbf{v}_p\rangle,$$

$$\langle \mathbf{v}_t\rangle=\mathrm{curl}(\mathbf{e}\psi),\quad \langle \mathbf{v}_p\rangle=\mathrm{curl}\,\mathrm{curl}\,(\mathbf{e}\varphi). \tag{19}$$

Here $\langle \mathbf{v}_t\rangle$ and $\langle \mathbf{v}_p\rangle$ are, respectively, the toroidal and poloidal components of the $\langle \mathbf{v}\rangle$ field and ψ and φ are, respectively, a pseudoscalar and a scalar function.

Substituting formulas (19) into Eq. (15) we get for the large-scale fields ψ and φ the set

$$\begin{gathered}(\partial/\partial t-\Delta)\psi=-S(\mathbf{e}_z\nabla)^2\varphi,\\ (\partial/\partial t-\Delta)\Delta\varphi=-S[(\Delta_\perp-(\mathbf{e}_z\nabla)^2]\psi,\end{gathered} \tag{20}$$

where $\Delta_\perp$ is the Laplace operator for the horizontal coordinates.

It is clear from the set (20) that the connection between the toroidal and poloidal fields is realized in terms of the spirality parameter S.

We place the origin of the vertical axis halfway between the upper and the lower boundaries of the layer so that the upper boundary corresponds to the value $z=\frac{1}{2}$ and the lower one to $z=-\frac{1}{2}$. By virtue of the horizontal homogeneity of the problem, we check for stability a solution of the form:

$$\psi(\mathbf{r})=\psi(z)\exp[i(\mathbf{k}_\perp\mathbf{r}_\perp)+\gamma t],$$

$$\varphi(\mathbf{r})=\varphi(z)\exp[i(\mathbf{k}_\perp\mathbf{r}_\perp)+\gamma t]. \tag{21}$$

Substituting the explicit forms (21) of the functions ψ and φ into the set of Eqs. (20) we get

$$\left[\gamma+k_\perp^2-\left(\frac{\partial}{\partial z}\right)^2\right]\psi=-S\left(\frac{\partial}{\partial z}\right)^2\varphi,$$

$$\left[\gamma+k_\perp^2-\left(\frac{\partial}{\partial z}\right)^2\right]\varphi=S\left[k_\perp^2+\left(\frac{\partial}{\partial z}\right)^2\right]\psi. \tag{22}$$

We check the set (22) for stability of solutions with large horizontal dimensions, i.e., we shall work in the limit as $k_\perp^2\to 0$. We assume the upper and lower boundaries to be rigid, i.e., the vertical component of the velocity to vanish on them:

$$v_z|_{z=-1/2}=v_z|_{z=1/2}=0.$$

For the horizontal component of the velocity at the upper boundary we take the condition of free flow along the boundary:

$$\frac{\partial}{\partial z}v_\perp|_{z=1/2}=0.$$

It is natural to take into account at the lower boundary slipping with a friction coefficient $(1-\xi)/\xi(0\leqslant\xi\leqslant1)$:

$$\xi\frac{\partial}{\partial z}v_\perp|_{z=-1/2}-(1-\xi)v_\perp|_{z=-1/2}=0.$$

In terms of ψ and φ these conditions take the form

$$\varphi(z)|_{z=-1/2}=\varphi(z)|_{z=1/2}=0. \tag{23}$$

$$\xi\varphi''(z)|_{z=-1/2}-(1-\xi)\varphi'(z)|_{z=-1/2}=\varphi''(z)|_{z=1/2}=0. \tag{24}$$

$$\xi\psi'(z)|_{z=-1/2}-(1-\xi)\psi(z)|_{z=-1/2}=\psi'(z)|_{z=1/2}=0. \tag{25}$$

The set (22) of equations with constant coefficients with the boundary conditions (23) to (25) is an eigenvalue problem. The eigenvalue is in this case the value of the growth rate γ.

We consider the characteristic equation of the system (22):

$$(\gamma+k_\perp^2+q^2)^2(-k_\perp^2-q^2)+S^2q^4-S^2k_\perp^2q^2=0, \tag{26}$$

where the q are the characteristic roots.

For small wave numbers $k_\perp$ Eq. (26) can be solved by asymptotic methods. Expanding q and γ in series in the small parameter $k_\perp$,

$$q=A_0+A_1k_\perp+A_2k_\perp^2+A_3k_\perp^3+\dots,$$

$$\gamma=\gamma_0+\gamma_1k_\perp+\gamma_2k_\perp^2+\gamma_3k_\perp^3+\dots,$$

we get the first terms in the expansion of the roots q:

$$q_1=ik_\perp+O(k_\perp^2),$$

$$q_2=-ik_\perp+O(k_\perp^2),$$

$$q_3=\frac{-S+(S^2-4\gamma_0)^{1/2}}{2}+O(k_\perp),$$

$$q_4=\frac{-S-(S^2-4\gamma_0)^{1/2}}{2}+O(k_\perp),$$

$$q_5=\frac{S+(S^2-4\gamma_0)^{1/2}}{2}+O(k_\perp),$$

$$q_6=\frac{S-(S^2-4\gamma_0)^{1/2}}{2}+O(k_\perp).$$

Following the theory of linear differential equations, we look for the eigenfunctions of the set (22) in the following form:

$$\begin{gathered}\psi(z)=\sum_{n=1}^{6}\psi_n\exp[iq_nz],\\ \varphi(z)=\sum_{n=1}^{6}\varphi_n\exp[iq_nz],\end{gathered} \tag{27}$$

where ψ_n and φ_n are coefficients. The relations between ψ_n and φ_n are found from the set (22):

$$\psi_n=\frac{Sq_n^2}{\gamma+q_n^2+k_\perp^2}\varphi_n. \tag{28}$$

To find the values of the coefficients ψ_n and φ_n we substitute the explicit form (27) of the eigenfunctions into the boundary conditions (23) to (25). We get a linear homogeneous algebraic set of equations of sixth order to find the coefficients ψ_n and φ_n. The necessary condition for the existence of a solution is the vanishing of the determinant consisting of the matrix elements of the linear system. This determinant has in an expansion in $k_\perp$ the following form:

$$I=I_0+I_1k_\perp+I_2k_\perp^2+\ldots$$

By successively setting the I_n equal to zero we get equations for the coefficients in the expansion of the growth rate γ. This problem has a discrete eigenvalue spectrum. We shall be interested in the lowest mode, the generation of which starts with the smallest value of the parameter S (we assume that $S>0$). Assuming that in this determinant the growth rate γ is small and the friction is weak ($\xi\sim1$), we get the following formula for the expansion of the growth rate γ to lowest orders in $k_\perp$:

$$\gamma=\frac{\pi}{4}(1-\xi)(S-\pi)-\frac{1}{4}(1-\xi)^2-k_\perp^2. \tag{29}$$

Hence it is clear that in the considered region of small wave numbers $k_\perp$ the modes with $k_\perp=0$ have the best conditions for generation. The critical value of the parameter S for which the growth rate γ becomes positive is easily found from (29):

$$S_{cr}=\pi+\frac{1-\xi_1}{\pi}.$$

To study the horizontal behavior of the solutions of the problem we consider a spirality parameter which depends on the horizontal coordinates. We shall study a system for the case when in some region of space S is just above its critical value by an amount δS which depends weakly on the horizontal coordinates:

$$S(\mathbf{r}_\perp)=S_{cr}+\delta S(\mathbf{r}_\perp). \tag{30}$$

In the simplest case we can write $\delta S(\mathbf{r})$ in the form

$$\delta S(r_\perp)=\delta S_0\left(1-\frac{r_\perp^2}{r_0^2}\right),$$

$$\delta S_0\ll S_{cr}\qquad r_0\gg h. \tag{31}$$

Using Eq. (29) for small wave numbers $k_\perp$ we can in this case establish an equation for the horizontal dependence of the potentials φ and ψ:

$$\gamma Q(\mathbf{r}_\perp)=\frac{\pi}{4}(1-\xi)(S(r_\perp)-\pi)Q(\mathbf{r}_\perp)$$
$$-\frac{1}{4}(1-\xi_1)^2Q(\mathbf{r}_\perp)+\Delta_\perp Q(\mathbf{r}_\perp). \tag{32}$$

We have obtained an equation of the time-independent Schrödinger equation type with an effective potential well which is determined by the shape of the function (30).

Using the explicit dependence (31) of the parameter S on the radius vector $r_\perp$ we construct an axisymmetric solution of the form

$$Q(\mathbf{r}_\perp)=\exp[im\Phi]Q(r)$$

using Laguerre polynomials $L_n^{(m)}$:

$$Q(r)=\exp[-A^{1/2}r^2/2]r^mL_n^{(m)}(A^{1/2}r^2), \tag{33}$$

where

$$A=\frac{\pi}{4}\frac{(1-\xi)}{r_0^2}\delta S_0.$$

The eigenvalues of Eq. (32) determine the growth rates of the various modes:

$$\gamma_{n,m}=\frac{\pi}{4}(\delta S_0)(1-\xi)-\frac{(\pi(1-\xi)\delta S_0)^{1/2}}{r_0}(2n+m+1). \tag{34}$$

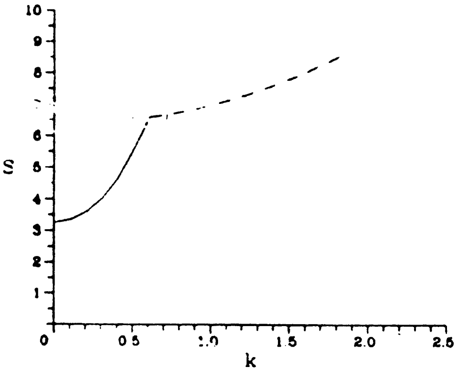

FIG. 1. The neutral curve. The critical value of the spirality S as a function of the horizontal wave number $k_\perp$.

We easily find from (33) and (34) the characteristic size L and time T for the development of the instability for the main mode, assuming a large radius r_0:

$$L\approx\frac{(r_0)^{1/2}}{(\delta S_0)^{1/4}}\frac{h}{(1-\xi)^{1/4}},$$

$$T\approx\frac{h^2}{\nu_T}\frac{4}{\pi(\delta S_0)(1-\xi)}$$

In the estimates we have found we substitute values which are typical of a turbulent atmosphere under the conditions of a tropical depression. We take the size of the supercritical region to be of order 2×10^3 km ($r_0\approx100$), the height of the tropopause in the tropics to be $h\approx16$ km, $\nu_T\approx3\times10^3$ m^2/s, the most probable value for the supercriticality to be $\delta S\approx\frac{1}{10}-\frac{1}{20}$, and the friction parameter to be $\xi\approx0.9$ (weak friction). In that case we get

$$L\approx800-1000\text{ km},$$
$$T\approx10-20\text{ days}.$$

These values of the parameters are in good agreement with observational data, although one should remember that our system does not take into account the decrease in density with height which is characteristic for the atmosphere.

We checked Eq. (22) numerically for the stability of normal perturbations in a horizontal layer of unit thickness. We show in Fig. 1 the neutral curve characterizing the critical value of the turbulent spirality parameter S for different horizontal wave numbers $k_\perp$. The minimum of this curve is reached in the point $k_\perp=0$. This means that when the spirality increases the instability appears on a larger horizontal scale. For small values of $k_\perp$ the instability is monotonic in character. The neutral curve has a kink caused by the merging of two monotonic modes into a complex conjugate pair.

An important role for the construction of a solution of the set of Eqs. (22) is played by the sign of the spirality parameter S, since the direction in which the vortex rotates depends on it (in accordance with Eq. (28)). It is clear from Eq. (18) that the sign of the parameter S depends on the sign of

$$H=\langle v\,\text{curl}\,v\rangle.$$

The appearance of an average value of the product $\langle\mathbf{v}\,\text{curl}\,\mathbf{v}\rangle$ in the atmosphere is connected with the expansion of ascending and the compression of descending air flows in the pres-

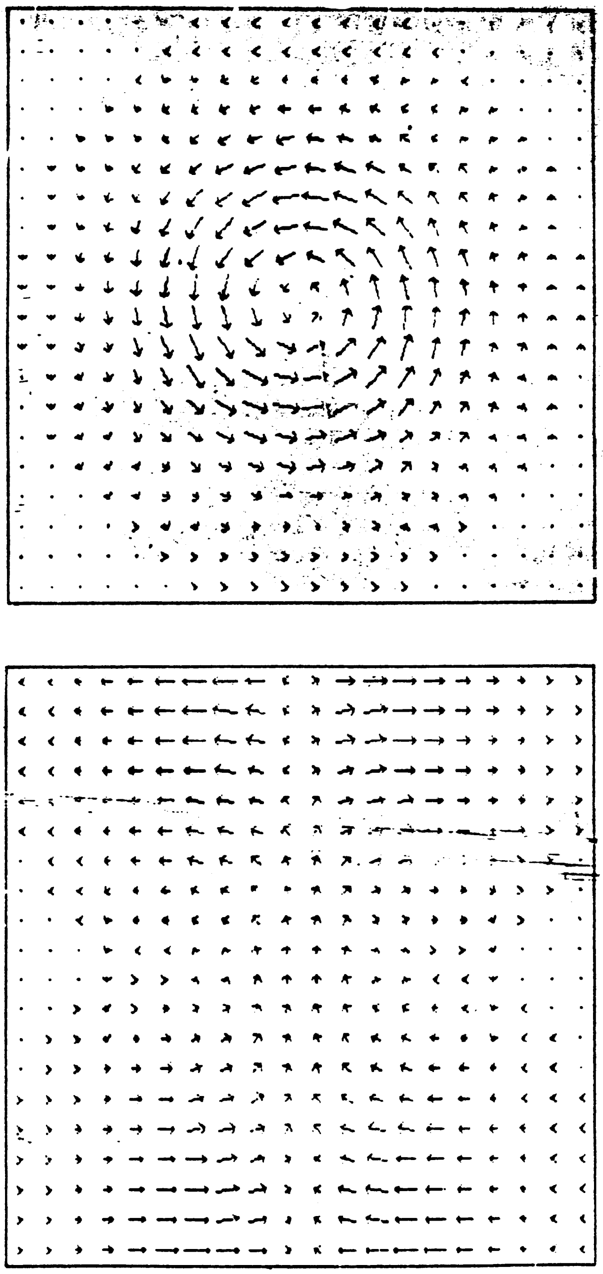

FIG. 2. Velocity field in horizontal (upper picture) and vertical (lower picture) sections. The velocity is normalized by the maximum value of the projected component. In the horizontal section $v_{max} = 1$ and in the vertical section $v_{max} = 0.07$.

ence of the Coriolis force.[17] For the earth's northern hemisphere we have $H>0$. In that case the rotation of the resulting vortex agrees with the rotation of the air in tropical cyclones.

The configuration of the large-scale velocity field is given in Fig. 2 for the axisymmetric lowest mode ($m = 0$, $n = 0$). The upper figure is the top view of a horizontal section of the structure near the lower boundary of the layer, while in the lower figure we show a vertical cut of the vortex. For the given parameter values the toroidal (directed along a circle) component of the velocity field turns out to be larger by an order of magnitude than the poloidal (directed to the center and vertically) component. The eigenfunction of the mode considered is thus a strongly rotating vortex with a horizontal velocity component which is practically uniform along the height. The presence of poloidal motions is important in the structure of this vortex. For instance, it is very clear in Fig. 2 that air flows upwards in the center of the vortex, and that there is an ascent and an outflow at the upper boundary. At the periphery of the vortex, on the other hand, we observe a descent of the air. It was noted in Refs. 1 and 4 that the poloidal and toroidal motions in such a vortex turn out to be linked to one another.

The study reported here shows that the present model can qualitatively describe large-scale instabilities under conditions of a neutral density stratification. In the framework of convective accommodation this makes it possible to use this model to describe the generation of vortices in the region of a tropical depression.

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