# Quasilongitudinal nonlinear dispersing MHD waves

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The study of nonlinear dispersing waves in a magnetized plasma is a vital topic in theoretical physics. Its importance is due for the most part to the variety of its physical application, noteworthy among which are interplanetary shock waves and nonlinear Alfven waves in solar wind.

The use of exact analytic methods to solve problems of dispersive magnetohydrodynamics was restricted until recently to several classical equations, such as the Korteweg–de Vries (KdV) equation and its modification (mKdV) or the nonlinear Schrödinger equation (NSE). These equations can be derived using the standard procedures of expanding in powers of the nonlinearity (KdV and mKdV) or the method of multiscale expansion of the NSE from exact systems of equations describing the dynamics of a magnetized plasma.

Whereas, however, wave propagation at relatively large angles to the magnetic field is well enough described by single-wave equations (KdV for magnetosonic waves and mKdV for Alfven waves), the quasilongitudinal state turns out to be degenerate (Ref. 1, Sec. 69): in the limit of ideal magnetohydrodynamics, the two linear waves move with near-Alfven velocities and differ only in polarization. Joint propagation of such waves under conditions of weak nonlinearity and dispersion (which lift the degeneracy) are described by a nonlinear Schrödinger equation with a derivative (DNSE).

A DNSE describing degenerate Alfven waves was first derived[2] by using kinetic theory. A simpler derivation of DNSE from the equations of two-fluid dispersive magnetohydrodynamics, with electron inertia neglected, was proposed later.[3] It should also be noted that the DNSE describes not only dispersing quasilongitudinal MHD waves, but can also be derived as a universal equation for the envelope if the quadratic response of the medium is not uniform (see the Appendix).

The DNSE is an equation that can be exactly integrated by the inverse-scattering-problem method (ISPM). The corresponding inverse scattering transformations were obtained in Refs. 4 and 5 for DNSE with decreasing as well as do nondecreasing initial conditions at infinity.

In addition to the ISPM, Whitham's averaging method[6] is being used of late to investigate nonlinear equations with dispersion. Integration of Whitham's modulation systems yields in a number of cases more effective results than the ISPM. The most interesting case of this kind is that a nondissipative shock wave (NSW) in the form of an expanding region filled with nonlinear small-scale undamped oscillations. This problem was first considered by Gurevich and Pitaevskii (GP) in Ref. 7, where an exact solution was obtained for self-similar decay of an initial discontinuity in KdV hydrodynamics. An analogous solution for NSE was obtained in Ref. 8.

An important property of Whitham-modulation systems for KdV and NSE is the presence of a diagonal (Riemann) form.[9,10] An effective theory for the integration of such systems exists at present[11–16] and is made essentially feasible by a generalization of the classical hodograph method to include the multidimensional case (in the space of invariants).

A modulation DNSE system in Riemann form was obtained in Ref. 17 and was found to be identical with the NSE modulation system (this was not noted in the cited reference). This makes many NSE results for DNSE; a difference occurs only in the stage of recalculation to physical variables and interpretation of the resultant equations.

There are, however, important differences due primarily to the presence of a wide instability region in the DNSE system. The modulation system in this region is elliptic.[18] The present paper is devoted to an investigation of the stable (hyperbolic) region in which the DNSE has properties indicative also of other dispersion-hydrodynamic systems. The general analysis presented in Sec. 1 shows that in contrast to the NSE, which describes two identical waves having positive dispersion and propagation in opposite directions, DNSE hydrodynamics contains two opposite-dispersion waves propagating in the same direction, with the wave with the positive dispersion propagates ahead of the negative. Comparison with formal limiting cases, in which DNSE goes over into KdV and mKdV, makes it possible to identify the waves as magnetosonic and Alfven, respectively.

The nondispersive limit in DNSE hydrodynamics is equivalent to ideal Euler hydrodynamics with $\gamma = 2$. Both Riemann invariants are positive, ensuring that the system has modulation stability and real physical variables (transverse-magnetic-field energy density and polarization angle).

In Sec. 2 we use Whitham's method to average the DNSE over the period of the stationary wave, and express the modulation system in Riemann form. In Sec. 3 we obtain for the GP problem solutions that describe self-similar Alfven and magnetosonic NSW. We show that a fourfold jump of the magnetic-field energy produces on the solition front of a magnetosonic NSW a singularity wherein phase conjugation of the NSW sets in. Section 4 is devoted to a classification of various decays of the initial discontinuity in dispersive DNSE dynamics.

## 1. DNSE. DERIVATION AND ELEMENTARY PROPERTIES

**a**) The proposed brief formal derivation of DNSE for quasilongitudinal dispersive MHD waves is close to the one


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described in Ref. 3. Consider the one-dimensional dynamics equations of a quasineutral two-fluid plasma in a constant magnetic field $B_x$:

$$\partial_t\rho+\partial_x(\rho v_x)=0, \qquad \partial_t v_x+v_x\partial_x v_x+\frac{1}{\rho}\partial_x\left(P+\frac{B_\perp^2}{8\pi}\right)=0, \quad (1)$$

$$\partial_t \mathbf{v}_\perp+v_x\partial_x\mathbf{v}_\perp=\frac{B_x}{4\pi\rho}\partial_x\mathbf{B}_\perp,$$

$$\partial_t\mathbf{B}_\perp+v_x\partial_x\mathbf{B}_\perp=B_x\partial_x\mathbf{v}_\perp-\mathbf{B}_\perp\partial_x v_x-\partial_x\left(\frac{cB_x m_i}{4\pi\rho e}[\mathbf{e}_x\partial_x\mathbf{B}_\perp]\right).$$

Here $\mathbf{B}_\perp=(B_y,B_z)$ and $\mathbf{v}_\perp=(v_y,v_x)$ are the transverse magnetic field and the hydrodynamic velocity, $\rho$ is the hydrodynamic density, $P/\rho^\gamma$ is the adiabatic equation of state and is assumed to be the same for electrons and ions, $m_i$ is the ion mass, and $\mathbf{e}_x$ is a unit vector in the direction of the $x$ axis.

The system (1) describes nonlinear dispersing waves in a magnetized plasma, with the electron inertia neglected. Allowance for the ion inertia in the generalized Ohm's law leads to a dispersion term in the equation for the transverse magnetic field. By changing to the Lagrangian independent coordinates $(x_0, t)$, where $x_0$ is the coordinate of the particle at a fixed instant of time $t_0$, we can eliminate all the velocities from the system (1) (see, e.g., Ref. 19) and write instead

$$\frac{\partial^2 N}{\partial t^2}+\frac{\partial^2}{\partial x_0^2}\left(P_0N^{-\gamma}+\frac{b_\perp^2}{2}\right)=0, \quad (2)$$

$$\frac{\partial^2(N\mathbf{b}_\perp)}{\partial t^2}-\frac{\partial^2\mathbf{b}_\perp}{\partial x_0^2}-\frac{\partial^3}{\partial x_0^2\,\partial t}[\mathbf{e}_x\mathbf{b}_\perp]=0. \quad (3)$$

Here $N=\rho_0/\rho$ and $\rho_0$ is the density in the unperturbed state; all the quantities are renormalized to their characteristic values: the independent spatial and temporal variables are renormalized to $\Omega_i^{-1}$ and $c_a\Omega_i^{-1}$ respectively, where $\Omega_i=eB_x/mc$ is the ion-cyclotron frequency and $c_a=B_x/4\pi\rho_0)^{1/2}$ is the Alfven velocity; the magnetic field is renormalized to a constant value $B_x$:

$$\mathbf{b}_\perp=(B_yB_x,\ B_z/B_x)$$

the hydrodynamic pressure is renormalized to $B_x^2/4\pi$.

Linearizing Eqs. (2) and (3) with respect to $N=1$ and $b_\perp=b_{\perp 0}$ we obtain a dispersion equation that describes six waves: fast and slow magnetosonic waves and an Alfven wave, all propagating in the positive and negative directions. It is known[1] that in the ideal MHD limit one magnetosonic wave moves in the case of longitudinal propagation with velocity $c_a$ a fast magnetosonic wave (if $c_a>c_s$, where $c_s=[(\partial P/\partial\rho)_{\rho 0}]^{1/2}$ is the speed of sound, this is a fast magnetosonic wave, and a slow one if $c<c_s$), and the waves differ only in polarization. To describe the dynamics of such a two-wave system, with allowance for weak nonlinearity and dispersion, it is necessary to change to a reference frame moving with Alfven (unity) velocity $\eta=x_0-t$ and introduce a "slow" time $\tau=\varepsilon t$, where $\varepsilon$ is a small nonlinearity parameter. Representing the dependent variables as asymptotic series in $\varepsilon$:

$$N(\eta,\tau)=1+\varepsilon N_1+\ldots,\quad b_\perp^2=b_{\perp 0}^2+\varepsilon b_1+\ldots,$$

we obtain from (2) the relation

$$N_1=(b_\perp^2-b_{\perp 0}^2)/2\varepsilon(1-c_s^2),\quad c_s=(\gamma P_0)^{1/2}.$$

Eliminating $N$ from (3) and putting $\varepsilon=1$, we arrive at a nonlinear Schrödinger equation with a derivative

$$\partial_\tau\mathfrak{b}_\perp+\alpha\partial_\eta[\mathfrak{b}_\perp(|\mathfrak{b}_\perp|^2-b_{\perp 0}^2)]+\frac{i\mu}{2}\partial_{\eta\eta}^2\mathfrak{b}_\perp=0, \quad (4)$$

where

$$\mathfrak{b}_\perp=b_y\pm ib_z,\qquad \alpha=\frac{1}{4(1-c_s^2)},\qquad \mu=\pm1.$$

The $\pm$ signs correspond here to left- and right-hand polarization. Clearly, in quasilongitudinal propagation the wave polarization is close to circular. From the foregoing conclusion, however, it is clear that DNSE describe also elliptically polarized waves propagating at an angle to the magnetic field. It is important only that the quantity

$$\varepsilon\sim|b_\perp|^2-b_{\perp 0}^2,$$

where $b_{\perp 0}$ is the value of the field at infinity, be small enough.

**b) Dispersive-hydrodynamic representation and linear waves**

Consider, to be specific, a DNSE with $b_{\perp 0}=0$ (change to a moving coordinate system), $\mu=1$ (left-hand polarization), and $\alpha>0$ (fast magnetosonic wave). The case $\alpha<0$ reduces to the studied one by reversing the coordinate, $\eta\to-\eta$, while the case $\mu=-1$ reduces to inversion of both the coordinate and the time

$$\eta=-\eta,\quad \tau\to-\tau.$$

We introduce the normalization

$$b=\alpha^{1/2}(b_y+ib_z).$$

Equation (4) takes then the form (we reintroduce for convenience $x$ and $t$ in place of $\tau$ and $\eta$

$$\partial_t b+\partial_x(b|b|^2)+\frac{i}{2}\partial_{xx}^2 b=0. \quad (5)$$

The change of variables

$$b=w^{1/2}\exp(i\theta),\quad u=\partial_x\theta$$

reduces Eq. (5) to a dispersion-hydrodynamic system [cf. Ref. 8):

$$\partial_t w+\partial_x(w(^3/_2w-u))=0,$$
$$\partial_t u+\partial_x\left(wu-\frac{1}{2}u^2+\frac{1}{2w^{1/2}}\partial_{xx}^2w^{1/2}\right)=0. \quad (6)$$

The variable $w$ has here the meaning of the energy density of the transverse magnetic field, while the quantity $u$ is indicative of the change of the polarization angle $\theta$ along the wave ($u>0$ in the investigated case of left-hand polarization). An important property that distinguishes (6) from NSE-hydrodynamics[8] is the non-invariance to the Galileo transformation:

$$x\to x-ct,\quad u\to u+c.$$

The linear waves of the system (6) are characterized by the dispersion relation

$$\omega=(2w_0-u_0)k\pm k(w_0(w_0-u_0)+k^2/4)^{1/2}. \quad (7)$$

where $\omega$ is the frequency and $k$ the wave number. Relation (7) can be equated to the NSE[8] dispersion relation by the substitutions

$$\rho_0=w_0(w_0-u_0),\ v_0=2w_0-u_0.$$

There is, however, an important difference between the linear waves in the NSE and DNSE cases. The $\pm$ signs in the NSE dispersion relation simply correspond to different wave propagation directions relative to a uniform flow with velocity $v_0$, (which can always be assumed to be zero by virtue of the Galilean invariance). These waves have identical properties (in particular, a dispersion of the same—positive—sign). In fact, the NSE

$$2i\partial_t\psi+\partial_{xx}\psi-2|\psi|^2\psi=0$$

is invariant to the transformation $x\to -x$. The DNSE, on the other hand, has no such invariance, so that waves corresponding to opposite signs in the dispersion relation (7) are not equivalent and have dispersions of opposite sign.

#### c) Nonlinear nondispersive limit

The dispersion term with the highest-order derivative can be neglected in the second equation of (6) in the case of sufficiently smooth large-scale motions. The corresponding ideal hydrodynamics is equivalent to Euler hydrodynamics with density

$$\rho=(w-u)w,$$

velocity

$$v=2w-u$$

and equation of state

$$P(\rho)=\rho^2/2.$$

It will be more convenient to continue using the invariant Riemann representation of the hydrodynamic equations:

$$\partial_t r_\pm+V_\pm(r)\partial_x r_\pm=0, \tag{8}$$

where the relations between the invariants $r_\pm$ and the characteristic velocities $V_\pm(r)$ are

$$r_\pm=2w-u\pm2[(w-u)w]^{1/2}=[w^{1/2}\pm(w-u)^{1/2}]^2, \tag{9a}$$

$$V_\pm=2w-u\pm[(w-u)w]^{1/2}=3/4\,r_\pm+1/4\,r_\mp. \tag{9b}$$

Evidently, the structure of Eqs. (8) depends on the relation between $w$ and $u$. If $w<u$ the invariants and the velocities are complex, and the system (8) is elliptic and describes the case of modulation instability.[18] Here we confine ourselves to the study of a steady (hyperbolic) case. The hyperbolicity condition is

$$w\geqslant u. \tag{10}$$

It follows then from (9) that in the investigated region we have $r_+\geqslant r_-\geqslant 0$, $V_+\geqslant V_-\geqslant 0$; i.e., both waves propagate in the positive direction. We present now expressions for the physical variables $w$ and $u$ in terms of the Riemann invariant which we shall need below:

$$w=(r_+^{1/2}+r_-^{1/2})^2/4,\quad u=(r_+r_-)^{1/2}. \tag{11}$$

#### d) Nonlinear small-amplitude weakly dispersive waves

Formal asymptotic expansions in the small parameter $\delta$ near the hydrodynamic simple wave with $r=$ const, viz.,

$$w^{1/2}=w_0^{1/2}+\delta a_1(\tau,\xi)+\delta^2a_2(\tau,\xi)+\ldots,$$

$$u=\delta^{1/2}(u_1(\tau,\xi)+\delta u_2(\tau,\xi)+\ldots),$$

$$\xi=\delta^{1/2}(x-c_+t),\ c_+=V_+(w=w_0,\ u=0)=3w_0,\ \tau=\delta^{3/2}t$$

lead, when substituted in (6), to a KdV equation with positive dispersion:

$$\partial_\tau a_1+6w_0^{1/2}a_1\partial_\xi a_1-\frac{1}{8w_0}\partial_{\xi\xi\xi}{}^3a_1=0, \tag{12}$$

which describes fast magnetosonic waves.

The expansions

$$w^{1/2}=w_0^{1/2}+\delta a_1(\tau,\xi)+\delta^2a_2(\tau,\xi)+\ldots,$$

$$u=\delta u_1(\tau,\xi)+\delta^2u_2(\tau,\xi)+\ldots,$$

$$\xi=\delta(x-c_-t),\ c_-=V_-(w=w_0,\ u=0)=w_0,\ \tau=\delta^3t$$

near a simple wave with $r_+=$ const yield the mKdV equation

$$\partial_\tau a_1+3a_1^2\partial_\xi a_1+\frac{1}{8w_0}\partial_{\xi\xi\xi}{}^3a_1=0,$$

which describe approximately nonlinear dispersive Alfven waves.[18]

The KdV and mKdV approximations are physically suitable in the case of elliptic polarization ($\delta^\alpha\sim b_z/b_y$, $\alpha\geqslant 1$), i.e., for the propagation of dispersive MHD waves at an angle to the magnetic field. The MNSE provides in this case a more correct description of waves of sufficient amplitude (the restrictions will be formulated below).[18,3] For quasilongitudinal propagation (circular polarization of MHD waves), however, an adequate description is possible only with the aid of DNSE. The approximations (11) and (12) are nonetheless useful also in this case, since they have been thoroughly studied, making possible a qualitative analysis and an indication of the possible effects.

## 2. STATIONARY WAVES AND THE WHITHAM EQUATION

The DNSE has solutions in the form of stationary waves $f(x-Ut)$, where $U$ is the phase velocity and is constant. Stationary DNSE are described by the ordinary differential equation

$$\left(\frac{dw}{d\xi}\right)^2=Q(w),\quad \xi=x-Ut, \tag{13}$$

$$Q(w)=-w^4+4Uw^3-Aw^2+Bw-4C^2, \tag{14}$$

where $A$, $B$, and $C$ are integration constants. Equation (13) describes the motion of a "particle" with velocity $dw/d\xi$ in a potential $-Q(w)/2$ (Fig. 1). The variable $u$ is connected with $w$ in a stationary wave by the expression

$$u=3/2\,w+U-C/w. \tag{15}$$

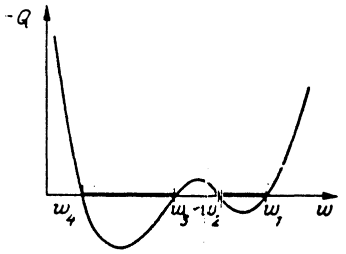


FIG. 1. The potential $Q(W)$ in the stationary MNSE wave.

It is convenient to introduce in lieu of the constants $A$, $B$, $C$, and $U$ the roots of the polynomial $Q(w)$:

$$Q(w)=-\prod_{i=1}^{4}(w-w_i), \quad w_1\geqslant w_2\geqslant w_3\geqslant w_4>0. \tag{16}$$

Comparing with (14), we have

$$4U=\sum_{i=1}^{4} w_i, \quad A=\sum_{i<j} w_iw_j, \quad B=\sum_{i<j<k} w_iw_jw_k,$$
$$4C^2=w_1w_2w_3w_4. \tag{17}$$

In the hyperbolic region all the roots are real (Fig. 1) and Eq. (13) has two solutions[20]

$$w(\xi)=w_2-\frac{w_2-w_3}{1-\nu_1\,\mathrm{sn}^2(\alpha\xi,m)} \tag{18}$$

$$w(\xi)=w_3+\frac{w_2-w_3}{1-\nu_2\,\mathrm{sn}^2(\alpha\xi,m)}, \tag{19}$$

where

$$\alpha^2={}^1/_4(w_1-w_2)(w_2-w_4),$$
$$m^2=\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)}, \tag{20}$$
$$\nu_1^2=\frac{w_3-w_4}{w_2-w_4}, \quad \nu_2^2=\frac{w_1-w_2}{w_1-w_3}$$

and $\mathrm{sn}(\alpha\xi,m)$ is the elliptic sine.

Equation (18) describes a magnetosonic wave (positive dispersion) oscillating between the roots $w_4$ and $w_3$, while (19) describes an Alfven wave ($w_2\leqslant w\leqslant w_1$, negative dispersion). The wave number $k$ is given in both cases by

$$k^{-1}=\frac{1}{\pi}\int_{w_2}^{w_1}\frac{dw}{Q^{1/2}(w)}=\frac{1}{\pi}\int_{w_4}^{w_3}\frac{dw}{Q^{1/2}(w)}$$
$$=\frac{2K(m)}{\pi[(w_1-w_3)(w_2-w_4)]^{1/2}}, \tag{21}$$

where $K(m)$ is a complete elliptic integral of the first kind.

Having defined the averaging over the period of the stationary wave as

$$\bar{f}=\frac{k}{2\pi}\oint\frac{f\,dw}{Q^{1/2}(w)}, \tag{22}$$

where the integration is over the periodicity interval of the corresponding function [i.e., between $w_4$ and $w_3$ in a magnetosonic (m) wave and between $w_2$ and $w_1$ in an Alfven (a) wave], we obtain the mean values

$$\bar{w}_m=w_2-(w_2-w_3)\frac{\Pi_1(\nu_1^2,m)}{K(m)}, \tag{23}$$

$$\bar{u}_m=\frac{3}{2}w_2-U-\frac{C}{w_2}-\frac{w_2-w_3}{K(m)}$$
$$\times\left(\frac{3}{2}\Pi_1(\nu_1^2,m)+\frac{C\Pi_1((w_2/w_3)\nu_1^2,m)}{w_2w_3}\right) \tag{24}$$

in a magnetosonic wave and

$$\bar{w}_a=w_3+(w_2-w_3)\frac{\Pi_1(\nu_2^2,m)}{K(m)}, \tag{25}$$

$$\bar{u}_a=\frac{3}{2}w_3-U-\frac{C}{w_3}+\frac{w_2-w_3}{K(m)}$$
$$\times\left(\frac{3}{2}\Pi_1(\nu_2^2,m)+\frac{C\Pi_1((w_3/w_2)\nu_2^2,m)}{w_2w_3}\right) \tag{26}$$

in an Alfven one.

Here $\Pi_l(\nu,m)$ is a complete elliptic integral of the third kind and the functions $U(w_i)$ and $C(w_i)$ are specified by (17), where $c$ must be chosen to be positive to ensure positive values of $u$ [see (15)] for small values of $w$.

Under quasistationary conditions, the slow evolution of the parameters $w_i$ is described by Whitham's modulation equations, which can be obtained by averaging (22) of the conservation laws of the system (6).[6,21] Just as for the NSE, in the present case they can be represented in the diagonal (Riemann) form[17]

$$\partial_t r_i+V_i(r)\partial_x r_i=0, \quad i=1,\ \ldots,\ 4. \tag{27}$$

It is remarkable that the characteristic DNSE velocities coincide with those obtained in Ref. 10 for NSE and are given

$$V_i(r)=U(r)+W_i(r),$$

by

$$r_1\geqslant r_2\geqslant r_3\geqslant r_4\geqslant 0, \quad U=\frac{1}{4}\sum_{i=1}^{4} r_i,$$

$$W_1=\Delta_{12}\left[2\left(1-\frac{\Delta_{24}}{\Delta_{14}}\mu(m)\right)\right]^{-1},$$
$$W_2=-\Delta_{12}\left[2\left(1-\frac{\Delta_{13}}{\Delta_{23}}\mu(m)\right)\right]^{-1},$$
$$W_3=\Delta_{34}\left[2\left(1-\frac{\Delta_{24}}{\Delta_{23}}\mu(m)\right)\right]^{-1}, \tag{28}$$
$$W_4=-\Delta_{34}\left[2\left(1-\frac{\Delta_{13}}{\Delta_{14}}\mu(m)\right)\right]^{-1},$$

$$\Delta_{ij}=r_i-r_j, \quad m^2=\Delta_{12}\Delta_{34}/\Delta_{13}\Delta_{24}, \quad \mu(m)=E(m)/K(m),$$

$E(m)$ is a complete elliptic integral of the second kind. It is important that the modulation equations have the same form in both averagings (i.e., for the magnetosonic and Alfven waves). The connection between the Riemann invariants and the parameters $w_i$ of the stationary wave, however, is different, namely

$$r_1={}^1/_4(w_1^{1/2}+w_2^{1/2}+w_3^{1/2}-w_4^{1/2})^2, \quad r_2={}^1/_4(w_1^{1/2}+w_2^{1/2}-w_3^{1/2}+w_4^{1/2})^2, \tag{29}$$
$$r_3={}^1/_4(w_1^{1/2}-w_2^{1/2}+w_3^{1/2}+w_4^{1/2})^2, \quad r_4={}^1/_4(-w_1^{1/2}+w_2^{1/2}+w_3^{1/2}+w_4^{1/2})^2$$

for the magnetosonic wave and

$$r_1={}^1/_4(w_1^{1/2}+w_2^{1/2}+w_3^{1/2}+w_4^{1/2})^2,\quad r_2={}^1/_4(w_1^{1/2}+w_2^{1/2}-w_3^{1/2}-w_4^{1/2})^2,$$
$$r_3={}^1/_4(w_1^{1/2}-w_2^{1/2}+w_3^{1/2}-w_4^{1/2})^2,\quad r_4={}^1/_4(w_1^{1/2}-w_2^{1/2}-w_3^{1/2}+w_4^{1/2})^2 \tag{30}$$

for the Alfven wave.

We present also the inverse expression for the parameters of a stationary wave in terms of Riemann invariants:

$$w_1={}^1/_4(r_1^{1/2}+r_2^{1/2}+r_3^{1/2}-r_4^{1/2})^2,\quad w_2={}^1/_4(r_1^{1/2}+r_2^{1/2}-r_3^{1/2}+r_4^{1/2})^2,$$
$$w_3={}^1/_4(r_1^{1/2}-r_2^{1/2}+r_3^{1/2}+r_4^{1/2})^2,\quad w_4={}^1/_4(-r_1^{1/2}+r_2^{1/2}+r_3^{1/2}+r_4^{1/2})^2. \tag{31}$$

It is easy to verify that the two families of Whitham's characteristics are joined together at the singular points of the paired coalescence of the invariants (corresponding to coalescence of the corresponding roots $w_i$). For the remaining two invariants, the equations are identical with Eqs. (8) of ideal hydrodynamics (see Ref. 8):

$$m=0,\quad V_1=V_2={}^1/_4(2r_1+r_3+r_4)+\frac{\Delta_{13}\Delta_{14}}{\Delta_{13}+\Delta_{14}},$$
$$r_3=r_+,\ r_4=r_-,\ V_3=V_-,\ V_4=V_-. \tag{32}$$

For $r_2=r_3$ we have

$$m=1,\ V_2=V_3={}^1/_4(r_1+2r_2+r_4),$$
$$r_1=r_+,\ r_4=r_-,\ V_1=V_+,\ V_4=V_-. \tag{33}$$

and for $r_3=r_4$

$$m=0,\quad V_3=V_4={}^1/_4(r_1+r_2+2r_4)-\frac{\Delta_{14}\Delta_{24}}{\Delta_{14}+\Delta_{24}},$$
$$r_1=r_+,\ r_2=r_-,\ V_1=V_+,\ V_2=V_-. \tag{34}$$

The oscillations vanish in this case in the following manner: their amplitude vanishes at $m=0$ and they are transformed at $m=1$ into individual solitons the distance between which tends to infinity $[k(m\to1)\to0]$, viz., into solitons which are negative in a magnetosonic wave and positive in an Alfven wave.

## 3. NONDISSIPATIVE SHOCK WAVES

It is known[7] that phase conjugation of a finite-amplitude large perturbation leads in dispersion hydrodynamics to formation of a nondissipative shock wave (NSW)—a region that expands with time and is filled with small-scale undamped oscillations. These oscillations are quasistationary and are described by the modulation system (28). Outside the NSW the flow is described by the Euler equations. We formulate now the general conditions for joining the Riemann invariants $r_i$ ($i=1,...,4$) of the Whitham equations to the Euler invariants $r_\pm$ at the boundaries $x^\pm(t)$ of the NSW (which are likewise to be determined—the GP conditions, see Refs. 7 and 8).

For an Alfven NSW (Fig. 2a) we have:
on the leading (soliton) edge $x=x_a^+(t)$

$$r_3=r_2,\ r_1(x_a^+,t)=r_+(x_a^+,t),\ r_4(x_a^+,t)=r_-(x_a^+,t), \tag{35}$$

on the trailing edge $x=x_a^-(t)$

$$r_3=r_4,\ r_1(x_a^-,t)=r_+(x_a^-,t),\ r_2(x_a^-,t)=r_-(x_a^-,t). \tag{36}$$

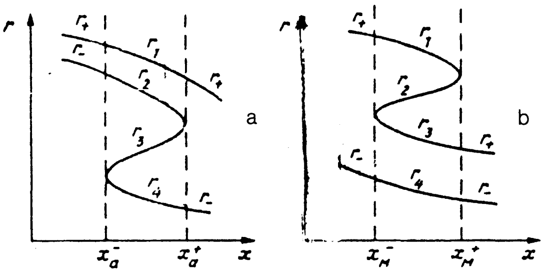

FIG. 2. Plots of Riemann invariants and of the Alfven (a) and magnetosonic (b) NSW.

For a magnetosonic NSW (Fig. 2b) we have:
on the leading (linear) edge $x=x_M^+(t)$

$$r_2=r_1,\ r_3(x_M^+,t)=r_+(x_M^+,t),\ r_4(x_M^+,t)=r_-(x_M^+,t). \tag{37}$$

on the trailing (soliton) edge $x=x_M^-(t)$

$$r_2=r_3,\ r_1(x_M^-,t)=r_+(x_M^-,t),\ r_4(x_M^-,t)=r_-(x_M^-,t). \tag{38}$$

The methods developed to date for analytic solution of the GP problem[14–16] make it possible to obtain a solution that describes an NSW in DNSE hydrodynamics with arbitrary initial data (both monotonic and localized), but we confine ourselves in the present paper to a derivation of the simplest family of self-similar solutions. These solutions describe a simple centered NSW (Ref. 8) that joins homogeneous-flow regions in which the invariants $r_\pm$ are constant (but one of them takes on different values on the left- and right-hand boundaries of the NSW), corresponding to initial data in the form of a "step" that satisfies the relation of a hydrodynamic simple wave. The physical quantities—the energy density $w$ and the change of the polarization angle $u$—undergo a constant jump on going through the NSW. This accords with the classical formulation of the shock-wave problem in the standard hydrodynamics (Ref. 22, Sec. 101).

### Alfven NSW

Let

$$r_+^{(l)}=r_+^{(r)}=r_{01}, \tag{39}$$

$$r_-^{(l)}=r_{02},\ r_-^{(r)}=r_{04}, \tag{40}$$

where the superscripts $(l)$ and $(r)$ designate the regions on the left and on the right of the NSW, with $r_0$, $r_{02}$, $r_{04}$ constants, $r_{01}>r_{02}>r_{04}$.

The solution of Eqs. (28) with boundary conditions (35), (36), (39), and (40) is obvious:

$$r_1=r_{01},\ r_2=r_{02},\ r_4=r_{04},$$
$$V_3=(r_{01},r_{02},r_3,r_{04})=\tau,$$
$$x=\tau t+P(\tau). \tag{41}$$

where $P(\tau)$ is an arbitrary function. We confine ourselves hereafter to the self-similar case

$$P(\tau)\equiv0,\ \tau=x/t.$$

Equations (41), (25), (26), and (31) determine the variation of the mean values of $\bar{w}_a$ and $\bar{u}_a$ in an Alfven NSW, and also its oscillator structure (19). The self-similar boundaries of the NSW are defined with the aid of (41), (33), and (35) as follows:

Trailing edge ($r_3 = r_4$):

$$\tau_a^- = V_3(r_{01}, r_{02}, r_{04}, r_{04}). \tag{42}$$

Leading front ($r_3 = r_2$):

$$\tau_a^+ = V_3(r_{01}, r_{02}, r_{02}, r_{04}). \tag{43}$$

Transition through a simple NSW, as shown by Gurevich and Meshcherkin,[23] is characterized by definite relations between the hydrodynamic quantities on both sides of the shock wave. The GM relations are similar to the Rankine–Hugoniot conditions in standard dissipative hydrodynamics. Let us obtain similar relations in our case for the physical variables $w$ and $u$. Let their (constant) values be $w_l$ and $u_l$ on the left and $w_r$ and $u_r$ on the right. The condition (39) under which the invariant $r_+$ is constant on going through the simple NSW and Eqs. (11) lead to the relation

$$w_l^{1/2} - w_r^{1/2} = (w_r - u_r)^{1/2} - (w_l - u_l)^{1/2}, \tag{44}$$

which means that the four quantities describing the flow on the left and on the right of the simple NSW are not independent. We express with the aid of (39)–(44), (33), and (9a) the width of the self-similar NSW in terms of three of them: the discontinuity $w_l^{1/2} - w_r^{1/2}$ and the values of $w_r$ and $u_r$ on the leading front:

$$\Delta\tau_a = \tau_a^+ - \tau_a^- = (w_l^{1/2} - w_r^{1/2})(w_l^{1/2} - (w_r - u_r)^{1/2}) \times \left(1 + \frac{4(w_r - u_r)^{1/2} w_r^{1/2}}{w_l^{1/2}(w_l^{1/2} - w_r^{1/2} - (w_r - u_r)^{1/2}) + 2w_r^{1/2}(w_r - u_r)^{1/2}}\right). \tag{45}$$

Evidently, the NSW width increases with increase of the field discontinuity. The qualitative structure of the Alfven NSW is perfectly analogous to that investigated in Ref. 7 for the KdV equation (note that the mKdV, which describes approximately nonlinear Alfven waves, has, after averaging in the hyperbolic region, the same Riemann form as the KdV equation.[24]

### Magnetosonic NSW

Consider now a simple NSW cooresponding to a constant invariant $r_-$:

$$r_-^{(l)} = r_-^{(r)} = r_{04}, \tag{46}$$

$$r_+^{(l)} = r_{01}, \quad r_+^{(r)} = r_{03}, \tag{47}$$

where $r_{01} > r_{03} > r_{04}$, and the constants $r_{01}$, $r_{03}$, and $r_{04}$ are, generally speaking, not equal to those defined in (39) and (40).

A self-similar solution of Eqs. (28) with boundary conditions (37), (38), (46), and (47) is

$$r_1 = r_{01}, \quad r_3 = r_{03}, \quad r_4 = r_{04}, \tag{48a}$$

$$V_2(r_{01}, r_2, r_{03}, r_{04}) = \tau, \quad \tau = x/t. \tag{48b}$$

Equations (48), (23), (24), and (31) determine the behavior of $\overline{W}_m$ and $\bar{u}_m$ in a magnetosonic NSW and its oscillatory structure (16). The coordinates of the NSW fronts [see (48), (32), and (33)] are defined as follows:

Trailing edge ($r_2 = r_3$):

$$\tau_M^- = V_2(r_{01}, r_{03}, r_{03}, r_{04}). \tag{49}$$

Leading front:

$$\tau_M^+ = V_2(r_{01}, r_{03}, r_{03}, r_{04}). \tag{50}$$

The GM relation for the variables $w$ and $u$ on going through a simple magnetosonic NSW is obtained from (46) and (9a) and takes the form

$$w_l^{1/2} - w_r^{1/2} = (w_l - u_l)^{1/2} - (w_r - u_r)^{1/2}. \tag{51}$$

The width of the magnetosonic NSW is

$$\Delta\tau_M = \tau_M^+ - \tau_M^- = (w_l^{1/2} - w_r^{1/2})(w_r^{1/2} + (w_l - u)^{1/2}) \times \left(1 + \frac{4w_l^{1/2}(w_l - u_l)^{1/2}}{w_r^{1/2}(w_l^{1/2} - w_r^{1/2} - (w_l - u_l)^{1/2} - 2w_l^{1/2}(w_l - u_l)^{1/2})}\right). \tag{52}$$

The qualitative structure of the magnetosonic NSW is similar to that investigated in Ref. 8 for NSE (KdV with positive dispersion), which describes qualitatively nonlinear magnetosonic waves [see Eq. (12)], can be obtained by a limiting transition from NSU [see (8)]. Just as in the case of an Alfven NSW, the width of a magnetosonic NSW increases with increase of the field-amplitude jump. The equations describing NSW (both magnetosonic and Alfven) can be substantially simplified by using the invariance of the GP problem (28), (35)–(38) with respect to the linear transformation

$$r_i \to r_i - s, \quad x \to x + ct$$

(recall that the initial equation has no such invariance), we can therefore assume without loss of generality that

$$r_{04} = 0.$$

It follows from (51) then directly for a magnetosonic NSW that

$$u_l = u_r = 0.$$

If we put now $w_r = 1$ (we measure $w$ in units of $w_r$), we readily see that an NSW is characterized in the corresponding reference frame by a single quantity—the field-amplitude jump

$$a_0 = w_l^{1/2} / w_r^{1/2}.$$

Assume that the width of the magnetosonic NSW is expressed in terms of $a_0$ as follows:

$$\Delta\tau_M = (a_0^2 - 1)\frac{6a_0^2 - 1}{2a_0^2 - 1}. \tag{53}$$

### Phase conjugation of magnetosonic NSW

Phase conjugation of NSW due to vanishing of the density in the peak of the first soliton, was observed[8] in media with positive dispersion. A similar phenomenon is observed also in MNSW magnetosonic NSW dispersion hydrodynamics. It follows from (18), (31), and (48a) that

$$w_{min} = w_4|_{m=1} = {}^1/_4(-r_{01}^{1/2} + 2r_{02}^{1/2} + r_{04}^{1/2})^2.$$

Putting $r_{04} = 0$, we obtain with the aid of (51) and (9a)

$$w_{min} = (2 - a_0)^2. \tag{54}$$

Evidently the field in the peak of the first soliton vanishes at $a_0 = 2$. It follows then from (15) that the quantity $u$ that defines the rotation of the polarization vector becomes infi-

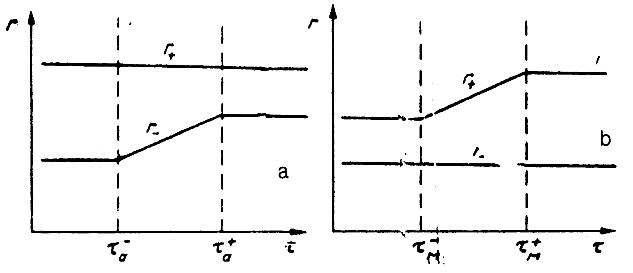


FIG. 3. Plots of dynamic invariants in Alfven (a) and magnetosonic (b) rarefaction waves.

nite. Phase conjugation of FSW takes place thus at $a_0 > 2$ and the conditions for the existence of the single-phase regime investigated in the present paper are therefore violated.

### Rarefaction waves

Another type of wave encountered in dispersion hydrodynamics is the rarefaction wave. We confine ourselves to a brief examination of the self-similar case. The flow in the region of a rarefaction wave is smooth and is described by Eqs. (8) and (9) of ideal hydrodynamics. The invariants $r_+$ and $r_-$ are constant in simple rarefaction waves and magnetosonic waves, respectively (Fig. 3). The condition for passage through a rarefaction wave are the same as (44) and (51) for a NSW. Finally, the self-similar solution is

$$r_+=\text{const},\ V_-=\tau,$$

for a rarefaction wave and

$$r_-=\text{const}.\ V_+=\tau.$$

for a magnetosonic wave.

## 4. DECAY OF INITIAL DISCONTINUITIES

Assume that the following discontinuities are produced at the initial instant $t = 0$ in the energy density $w$ of the magnetic field and in the polarization variable $u$: in the left half-space:

$$w=w^{(2)},\ u=u^{(2)},$$

and in the right

$$w=w^{(1)},\ u=u^{(1)},$$

where $w^{(1)(2)}$ and $u^{(1)(2)}$ are constants and $w^{(1),(2)} \geqslant u^{(1),(2)}$. This is, of course, accompanied by discontinuities of the hydrodynamic invariants; $r_\pm = r_\pm^{(2)}$ in the left half-space and $r_\pm = r_\pm^{(1)}$ in the right-hand side. According to Refs. 23 and 8, a pair of either FSW or rarefaction waves moves out then in both directions from the initial discontinuity. A plateau region in which the flow is constant is produced between these waves:

$$w=\text{const}=w^{(0)},\ u=\text{const}=u^{(0)},\ r_\pm=r_\pm^{(0)}.$$

As shown in Sec. 3, it can be assumed without loss of generality that

$$r_-^{(1)}=0\ (u^{(1)}=0)$$

(i.e., one can change to a coordinate frame moving to the left with velocity $r_-^{(1)}$) and $r_+^{(1)} = 4$ [$w$ is measured in units of $w^{(1)}$, see (11)]. Thus,

$$w(x,0)=\begin{cases} w^{(2)} & \text{for } x\leqslant 0 \\ 1 & \text{for } x>0 \end{cases},\qquad u(x,0)=\begin{cases} u^{(2)} & \text{for } x\leqslant 0 \\ 0 & \text{for } x>0 \end{cases}. \tag{55}$$

Using the conditions (44) and (51) for a transition through Alfven or magnetosonic simple waves (the magnetosonic moves in advance)—an NSW or a rarefaction wave, we obtain the values of $w$ and $u$ on the plateau:

$$u^{(0)}=0,\ w^{(0)}={}^1/_4[(w^{(2)})^{1/2}+(w^{(2)}-u^{(2)})^{1/2}]^2. \tag{56}$$

Next, starting from the condition $w^{(0)} = 4$ for phase conjugation of NSW, we determine from (56) the range of parameters at which the solution (55) of the initial problem exists without appearance of a phase-conjugation singularity:

$$(w^{(2)})^{1/2}+(w^{(2)}-u^{(2)})^{1/2}<4,$$

or, equivalently, $r_+^{(2)} < 16$.

We now consider briefly possible cases of discontinuity

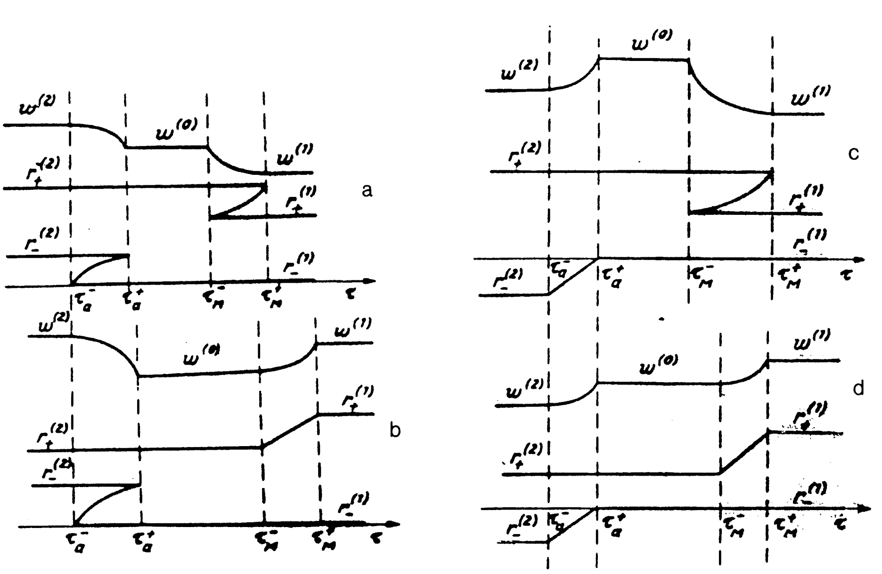


FIG. 4. Decay of initial discontinuities a — $r_-^{(2)} > r_-^{(1)}$, $r_+^{(2)} > r_+^{(1)}$; b — $r_-^{(2)} < r_-^{(1)}$, $r_+^{(2)} > r_+^{(1)}$; c — $r_-^{(2)} > r_-^{(1)}$, $r_+^{(2)} > r_+^{(1)}$; d — $r_-^{(2)} > r_-^{(1)}$, $r_+^{(2)} > r_+^{(1)}$.

decay, as function of the relation between the quantities $r_{\pm}^{(1),(2)}$.

$$1.\ r_-^{(2)}>0,\ r_+^{(2)}>4\ \ (w^{(2)}>w^{(0)}>1)$$

The decay of such a discontinuity produces two NSW. Plots of the invariants and of the average magnetic-field energy density in this situation are shown in Fig. 4a. The width of the plateau is

$$\Delta\tau_p^w=\tau_M^- - \tau_a^+ = {}^1/_2 (r_+^{(1)} - r_-^{(2)}) = 2 - r_-^{(2)}/2.$$

The plateau vanishes if $r_-^{(2)} = 4$. This is the particular solution obtained in Ref. 17. Clearly, however, it does not describe the general case of discontinuity decay. For $r_-^{(2)} > 4$ the plateau width is negative—two NSW are nonlinearly superimposed. A two-stream-solution region, not describable by the one-phase Whitham theory, is then produced. Note that the possiblity of superposition of two NSW exists also in NSE hydrodynamics.

$$2.\ r_-^{(2)}>0,\ r_+^{(2)}<4\ \ (w^{(0)}<w^{(2)},\ w^{(0)}<1)$$

An Alfven NSW and a magnetosonic rarefaction wave are produced (Fig. 4b).

$$3.\ r_-^{(2)}<0,\ r_+^{(2)}>4\ \ (w^{(0)}>1,\ w^{(0)}>w^{(2)})$$

A magnetosonic NSW and an Alfven rarefaction wave are produced (Fig. 4c).

$$4.\ r_-^{(2)}<0,\ \ r_+^{(2)}<4\ \ \ (w^{(2)}<w^{(0)}<1)$$

Two rarefaction waves are produced (Fig. 4a).

Note that in cases 3 and 4 the invariant $r_-^{(2)}$ is negative. This does not contradict the hyperbolicity condition, since we have changed over to a moving coordinate frame with simulataneous shift $r_{i,\pm} \to r_{i,\pm} - r_-^{(1)}$.

## APPENDIX

### MNSU as the universal equation for the envelope (informal derivation)

Consider a nonlinear wave packet with a dispersion relation

$$\omega=\omega(k,\ |\psi|^2). \tag{A1}$$

where $\psi$ is the complex amplitude. We expand the function $\omega$ near $|\psi|^2 = 0$ and $k = k_0$:

$$\omega-\omega_0=(k-k_0)\left(\frac{\partial\omega}{\partial k}\right)_0+|\psi|^2\left(\frac{\partial\omega}{\partial|\psi|^2}\right)_0$$
$$+(k-k_0)|\psi|^2\left(\frac{\partial^2\omega}{\partial|\psi|^2\partial k}\right)_0+(k-k_0)^2\left(\frac{\partial^2\omega}{\partial k^2}\right)_0+\dots \tag{A2}$$

Here $\omega_0 = \omega(k_0,0)$, and the subscript "0" denotes that the corresponding function is taken at the point $(k = k_0, |\psi|^2 = 0)$.

Using the formal correspondence

$$\omega-\omega_0\to-i\partial_\tau,\quad k-k_0\to i\partial_x.$$

we write down an equation corresponding to the expansion (A2):

$$i\partial_\tau\psi+ic\partial_x\psi+d\partial_{xx}{}^2\psi+g|\psi|^2\psi+ih\partial_x(|\psi|^2\psi)=0. \tag{A3}$$

where $c$, $d$, $g$, and $h$ are parameters that depend on $k_0$.

Changing to the moving coordinate frame $x \to x - c\tau$ and introducing a new variable defined as[25]

$$\psi=\left(\frac{2d}{h}\right)^{1/2}\exp\left(\frac{it}{2\gamma^2}+\frac{ix}{\gamma}\right)\varphi\left(-\frac{t}{2},-\left(x+\frac{t}{\gamma}\right)\right). \tag{A4}$$

where

$$t=-2d\tau,\qquad \gamma=\frac{h}{g},$$

we obtain the MNSU

$$\partial_t\varphi+\partial_x(|\varphi|^2\varphi)+\frac{i}{2}\partial_{xx}{}^2\varphi=0.$$

The substitution

$$\varphi=I^{1/2}\exp(i\theta),$$

which leads to a system of the dispersive-hydrodynamic type (6), contains the following functions: $I$—intensity (envelope), $\varkappa = \partial_x\theta$—effective wave number of the carrier.


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