Theory and applications of radial orbit instability in collisionless gravitational systems

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We derive integral equations for the low-frequency modes in a gravitating cylinder, disk, and sphere. An analytic theory is proposed for the instability of radial orbits, the most important instability in collisionless gravitational systems. Radially extended orbits arise naturally, and they become the dominant type of orbit as a heterogeneous population of stellar systems is produced by collapse of an initially highly tenuous cloud. Such systems include galaxies, their individual components, clusters of galaxies, and so forth. Based on this theory, the instability in question harks back to the classical Jeans instability; in the present case, it entails the deformation of a system by virtue of the gravitational attraction of orbits that extend toward one another. This embodiment of the Jeans instability has a very interesting feature: the theory predicts that even in the limit of perfectly radial orbits, instability will only develop under one additional condition—that the precession of low angular momentum orbits in the gravitational potential of a particular stellar system take place in the same direction as the stars' orbital motion. The final dispersion of orbital precession rates will remove the instability. Here we obtain simple equations for the relationship between the minimum scatter in angular precession rates necessary for stability and the instability growth factor for exactly radial orbits. We demonstrate the importance of disk and ellipsoidal deformation of spherical systems with radially extended orbits, which relates directly to the role played by radial orbit instability in the formation of barred spirals and elliptical galaxies. Finally, we briefly discuss the possible explanation of ellipticity of thin planetary rings as a manifestation of the appropriate instability in systems with quasicircular orbits.

1. INTRODUCTION

Radial-orbit instability has attained its status as the principal instability of collisionless gravitational systems by virtue of the fact that highly elongated orbits are unavoidable when a collection of gravitating masses collapses out of a tenuous initial state. Essentially all modern scenarios for the formation of galaxies, clusters of galaxies, and other such systems predict a collapse of this type. The ensuing instability in collisionless gravitational systems dominated by radially extended orbits heats the system in the transverse direction (i.e., perpendicular to the radius), and thereby smoothes out any anisotropy in the "particle" (i.e., star or galaxy) radial and transverse velocity distributions, reducing it to some critical value.

In contrast to, say, plasma systems, it is already known[1] that in gravitating systems there are no instabilities that serve solely to equalize the temperature of a medium (isotropize it). In fact, it is usually some manifestation of the Jeans instability that takes on that role. Radial orbit instability is likewise basically a Jeans phenomenon. Recall that the classical Jeans instability of a gaseous medium leads to the gravitational compression of a mass of gas large enough that the pressure force is incapable of withstanding the gravitational force (details may be found, for example, in Ref. 2). Clearly, the colder the medium, the smaller the scales on which perturbations that are unstable "*a la* Jeans" commence, and the more unstable the medium overall. So it is in the present case: If we imagine a system with radial orbits, and those orbits approach some conic section at some instant, then their subsequent coalescence (i.e., something other than radial orbit instability) will be entirely natural, since the system is cold in the transverse direction. Actually, this is not a complete explanation, as we shall see below: apart from the existence of the radial orbits, the presence or absence of instability will also depend on the nature of the orbital precessional motion induced by the perturbations. Instability will develop only for prograde precession (i.e., the direction of precession is the same as that of the stars' orbital motion); there will be none for retrograde precession. In those cases when instability is present, however, it will unquestionably be essential Jeans instability; but here, the elementary objects will most likely be orbits as a whole, rather than individual stars. In terms of its significance for collisionless gravitational systems, radial orbit instability is entirely comparable to the more familiar Jeans instability in a gaseous medium (which is thought to be responsible for star formation, for example).

Thus, in the world of astronomical objects, collisionless systems are at least as well represented as gaseous systems. Similarly, the manifestations of radial orbit instability are quite multifaceted. Radial orbit instability is responsible, for example, both for the shape of elliptical galaxies and that of the spheroidal component of spirals. There is no particular reason to doubt that it is also related to the apparent flattening of compact clusters of galaxies. Less obvious (but no less true) is the assertion that essentially the same instability bears a direct relationship to the central bars of many spiral galaxies (which in general are rapidly rotating). As yet another instance, we note the possible role played by radial orbit instability in the formation of galactic superclusters.[3] Here it is obviously important that this instability does not depend on the sign of the radial velocity, and it can therefore develop unimpeded by the Hubble expansion of the universe.

What we have said thus far points up the importance of a detailed investigation of radial orbit instability.

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The possibility of radial orbit instability was first pointed out in Ref. 4, and it was subsequently found in direct numerical simulations[5,6] and by numerically solving the linearized kinetic equation.[7] Curiously enough, in the many papers published in 1985 modeling the collapse of collisionless gravitational systems (except for Ref. 5), no mention was made of the deformation that accompanies the collapse of an ellipsoidal system, even though such a deformation must have been encountered, for example, by the authors of Refs. 8 and 9, inasmuch as they studied collapse in extremely cold systems. They failed to draw attention to the phenomenon, since they obviously did not entertain even the possibility of instability (the sole reasonable explanation!) which alters the shape of an initially spherically symmetric, nonrotating system. Indeed, a spherical system is traditionally supposed to be the stablest possible. This mass hypnosis (besides Refs. 8 and 9, mentioned above, we cite Refs. 10 and 11) continued right up through appearance in 1984 of the English edition[12] of Ref. 13, where these problems had already been addressed in detail. In Ref. 7 and in other papers by the present author (details and references can be found in Refs. 12 and 13), several criteria were identified for stabilizing the instability in question—in other words, various values of the necessary minimum transverse stellar kinetic energy. In all of the papers cited, however, numerical methods were employed exclusively. We feel the need, therefore, for a more analytical approach, perhaps to assist in filling in the overall picture and providing the key to identifying more general criteria for instability and its stabilization.

The first (and as yet the only) attempt to prove analytically the existence of instability in systems with radial orbits was undertaken in Ref. 14. However, despite the author's claims,[14] it is untrue that such systems—with strictly radial stellar motion—are automatically unstable, and that they possess no stable modes at all, a point that we shall return to subsequently. For example, with predominantly retrograde precession of orbits, the instability is purely oscillatory, i.e., it is manifestly a stable mode. The correct result can only be obtained by approaching the limiting case of purely radial motion from a system with highly elongated but nevertheless slightly nonradial orbits. By attempting to deal directly with the limiting system from the outset, one inevitably encounters integrals that diverge as $r\to 0$.

In Sec. 2 below, we develop a general stability theory for collisionless gravitational systems, a task made feasible by two simplifying assumptions. The first is that the mass of the active group of stars (i.e., those affected by the perturbations) is small compared with the mass of the passive "halo," which determines the equilibrium potential Φ_0 but remains unaffected by the perturbations. The second is that the spread in angular rates of precession (about some mean) is small compared either with typical rates of stellar radial oscillation or with the small values of the precession rates themselves, as in systems with almost radial orbits. In such systems we shall study slow, low-frequency modes. For example, the orbital frequencies of quasiradial orbits are of the same order of magnitude as the precession frequencies.

It is important to emphasize that both conditions are fulfilled in many realistic gravitational systems. The proposed approach, which should make radial orbit instability more comprehensible, therefore has immediate practical applications.

As a result, we have obtained a description of gravitational systems analogous to the drift approximation in plasma physics (see, e.g., Ref. 15), but for orbits of a much more general form. The general integral equation for low-frequency modes presented in Sec. 2 will be applied to the theory of radial orbit instability in cylindrical and disk systems (§3.1), and a modified version will be applied to spherical systems (§3.2).

Although the present paper is chiefly concerned with radial orbit instability, we comment in Sec. 4 on low-frequency modes in ring and disk systems with quasicircular orbits. Low-frequency modes are probably also important in normal spiral galaxies, where stellar orbits are closer to being circular than radial, on average (except possibly near the center). We have already remarked that radial orbit instability and its associated low-frequency modes bear a direct relation to the formation of two large classes of galaxies, the barred spirals (see §3.1) and the ellipticals (§3.2). Normal spirals comprise the third (and last) large class of galaxies (leaving out the so-called irregulars, which are not nearly so well organized structurally), and they are consequently also a suitable target for the theory. Our theory therefore encompasses all of the basic types of galaxies, and it may also account for the elliptical deformation of the thin ring systems around Saturn and Uranus (Sec. 4).

2. DERIVATION OF THE GENERAL INTEGRAL EQUATION FOR LOW-FREQUENCY MODES IN A CYLINDER, DISK, AND RING

It makes sense to treat planar systems (disks and rings) or planar (flute) perturbations in a gravitating cylinder separately from three-dimensional systems (spherical in the present case). We begin with the former—as might be expected, the equations for perturbations in spherical systems can be obtained virtually automatically from the corresponding equations for cylindrical systems by taking advantage of simple group-theoretic considerations (see Sec. 3).

The most convenient description is in terms of action-angle variables $\mathbf{I} = (I_1,I_2)$ and $\mathbf{w} = (w_1,w_2)$, which suitably takes account of the double periodicity of stellar motion in the equilibrium potential. We start out with the linearized kinetic equation in its usual form (see, e.g., Ref. 12):

$$\frac{\partial f_1}{\partial t} + \Omega_1\frac{\partial f_1}{\partial w_1} + \Omega_2\frac{\partial f_1}{\partial w_2} = \frac{\partial f_0}{\partial I_1}\frac{\partial \Phi_1}{\partial w_1} + \frac{\partial f_0}{\partial I_2}\frac{\partial \Phi_1}{\partial w_2}, \qquad (1)$$

where $f_0(\mathbf{I})$ and $f_1(\mathbf{I},\mathbf{w},t)$ are the unperturbed and perturbed distribution functions, Φ_1 is the perturbation of the gravitational potential, Ω_1 and Ω_2 are the frequencies of the radial and azimuthal oscillations of stars in the equilibrium potential $\Phi_0(r)$, $\Omega_i = \partial E(\mathbf{I})/\partial I_i$ (E is the energy in terms of $\mathbf{I}, i = 1,2$). The change of variables $\overline{w}_2 = w_2 - w_1/2$, $\overline{w}_1 = w_1$ in (1) yields

$$\frac{\partial f}{\partial t} + im\Omega_{pr}f + \Omega_1\frac{\partial f}{\partial w_1} = \frac{\partial f_0}{\partial I_1}\frac{\partial \Phi}{\partial w_1} + im\Phi\left(\frac{\partial f_0}{\partial I_2} - \frac{1}{2}\frac{\partial f_0}{\partial I_1}\right), \qquad (2)$$

where we have assumed that the perturbations are proportional to $\exp(im\overline{w}_2)$:

$$\Phi_1 = \Phi\exp(im\overline{w}_2), \quad f_1 = f\exp(im\overline{w}_2),$$

m is the azimuthal index (an even integer), and $\Omega_{pr}(E,L) = \Omega_2 - \Omega_1/2$ is the rate at which an orbit with energy E and angular momentum L precesses. If we also transform from the action (I_1,I_2) to (E,L), we obtain a form of the linearized kinetic equation that will be more convenient for subsequent use:

$$\frac{\partial F}{\partial t}+im\Omega_{pr}F+\Omega_1\frac{\partial F}{\partial w_1}$$
$$=\Omega_1\frac{\partial F_0}{\partial E}\frac{\partial \Phi}{\partial w_1}+im\Phi\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right). \tag{3}$$

where $F_0(E,L)=f_0(I_1,I_2)$. Note that in this form, Eq. (3) also holds for a quasi-Coulomb potential Φ_0, i.e., one that is due principally to a large central mass. All that is necessary then is to redefine $\overline{w}_2$ and $\Omega_{\rm pr}$:

$$\overline{w}_2=w_2-w_1,\ \Omega_{pr}=\Omega_2-\Omega_1.$$

Then, neglecting self-gravitation in the system and the quadrupole moment of the central object, we have $\Omega_{\rm pr}=0$, as one would assume for closed Keplerian orbits. Here the azimuthal index m can be either odd or even.

We assume that the rms deviation of the precession rates about the mean $\overline{\Omega}_{\rm pr}$, given by $\Delta\Omega_{\rm pr}=\left[\overline{(\Omega_{\rm pr}-\overline{\Omega}_{\rm pr})^2}\right]^{1/2}$, and the typical gravitational frequency ω_G are both small, $\Delta\Omega_{\rm pr},\ \omega_G\ll\Omega_1$. Note, for example, that for cylindrical (and spherical) systems, ω_G is of the order of the Jeans frequency: $\omega_G\sim\omega_j=(4\pi G\overline{\rho}_0)^{1/2}$ where ρ_0 is the density and G the gravitational constant. The above inequality also means immediately that we are dealing with a system of stars that have essentially equal precession rates within a massive halo which, while remaining unpertubed itself, furnishes the dominant contribution to the equilibrium potential Φ_0. Under these circumstances, then, there may exist a low-frequency mode ($\propto\exp(-i\overline{\omega}t)$, with $\overline{\omega}\equiv\omega-m\overline{\Omega}_{\rm pr}\sim\omega_G,\Delta\Omega_{\rm pr}$) in a coordinate system rotating at angular velocity $\overline{\Omega}_{\rm pr}$ such that the slow precessional dispersal of orbits is canceled by their mutual gravitational attraction. It would be natural to suppose that if self-gravitation were to win out over the spread in orbital precession rates, an instability might develop that could eventually deform the system (under the influence of the largest-scale growing modes).

It is clear, however, that even in a system with essentially radial orbits, this holds true only if the torque that alters the orbital angular momentum of the stars forces their orbital precession rates to change in the same direction (upper part of the "tuning-fork" diagram, Fig. 1c′). If the orbits exhibit "asinine" behavior (a term we owe to Lynden-Bell and Kalnajs[16], i.e., they tend to move in the direction opposite that toward which they are being pushed, then rather than instability we will have pure oscillations (lower tine of the "tuning fork," Fig. 1c″). The type of precession that takes place (prograde or retrograde) will depend on the form of the potential $\Phi_0(r)$. It is not possible to claim that retrograde precession requires some sort of unusual potential, even for low angular momentum stars, although in most actual systems, such orbits in fact undergo prograde precession (see §3.1). For the opposite limiting case—quasicircular orbits—the situation is considerably more complicated than for quasiradial orbits, and the nature of the precession that takes place is not so directly related to the possibility or impossibility of instability.

We employ perturbation theory to derive the desired solution. Let $F=F^{(1)}+F^{(2)}$, where $F^{(1)}$ corresponds to the reversible mode obtained from (3) by neglecting terms proportional to $\Delta\Omega_{\rm pr}$ and $\Phi\propto G$: $\omega=0$ (of $\omega=m\overline{\Omega}_{\rm pr}$ for $\overline{\Omega}_{\rm pr}\neq 0$), $\partial F^{(1)}/\partial w_1=0$. In other words, $F^{(1)}=F^{(1)}(E,L)$ is an as-yet arbitrary function of the integrals of the motion, which we will subsequently make more specific by requiring that the solution of the next approximation be periodic. The equation for $F^{(2)}$ takes the form

$$-i\omega F^{(1)}+im\Omega_{pr}F^{(1)}+\Omega_1\frac{\partial F^{(2)}}{\partial w_1}$$
$$=\Omega_1\frac{\partial F_0}{\partial E}\frac{\partial\Phi}{\partial w_1}+im\Phi\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right). \tag{4}$$

Averaging (4) over w_1 from 0 to 2π, and bearing in mind the periodicity of $F^{(2)}$ and Φ, we have

$$-(\overline{\omega}-m\delta\Omega_{pr})F^{(1)}\approx m\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right)\frac{1}{2\pi}\int_0^{2\pi}\Phi\,dw_1, \tag{5}$$

where $\delta\Omega_{\rm pr}=\Omega_{\rm pr}-\overline{\Omega}_{\rm pr}$.

Invoking the Poisson equation, some minor manipulations yield

$$\Phi=-G\int d\mathbf{I}'F^{(1)}(\mathbf{I}')\int d\mathbf{w}'\Gamma(r,r',\varphi'-\varphi)\exp[im(\overline{w}_2'-\overline{w}_2)], \tag{6}$$

where $d\mathbf{I}'=dI_1'dI_2'$, $d\mathbf{w}'=dw_1'dw_2'$, and Γ is the Green's function, which is

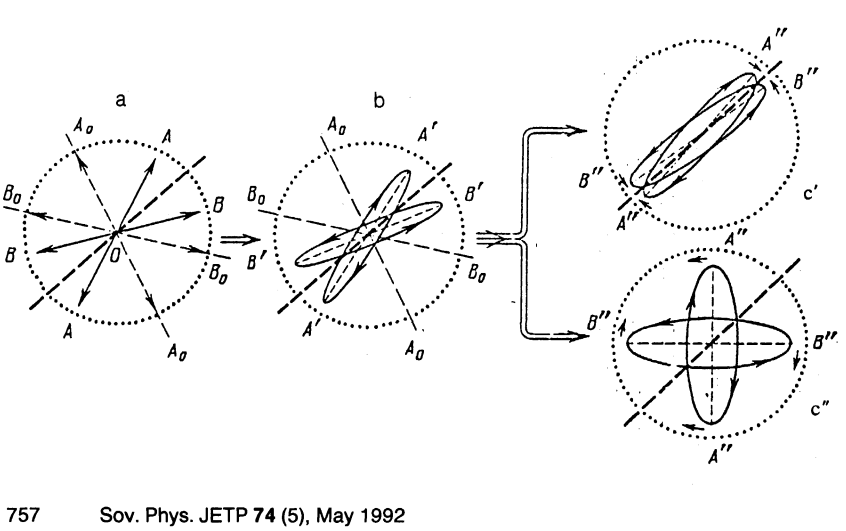

FIG. 1. "Tuning fork" diagram elucidating the physical mechanism for radial orbit instability. a) Two typical radial orbits A_0 and B_0 in an equilibrium axially symmetric state. A and B are their initial positions in the perturbed state on either side of the minimum of the perturbed potential (heavy dashed lines); b) the stars in orbits A (now A') and B (now B') have acquired a small amount of angular momentum ($L<0$ on A' and $L>0$ on B'); c′) the major axes of orbits A' and B' have drawn closer together (to positions A'' and B''), i.e., radial orbit instability has set in (in the case of prograde orbital precession, the direction of precession is shown by the double arrows); c″) return of the orbital major axes to their original equilibrium position, i.e., oscillation in the case of retrograde precession.

$$\Gamma=2\ln(1/r_{12}),\quad r_{12}=[r^2+r'^2-2rr'\cos(\varphi'-\varphi)]^{1/2} \tag{7'}$$

for a cylinder and

$$\Gamma=1/r_{12} \tag{7''}$$

for a disk or ring. Equation (6) is an integral equation for $\Phi(\mathbf{I},w_1)$ which takes into account (5) for $F^{(1)}$ in terms of Φ. Particle coordinates r,φ,r',φ' in (6) and (7) must be expressed in terms of **I**, **I**′, **w**, **w**′, where

$$r=r(I,\ w_1),\ r'=r'(\mathbf{I}',\ w_1'),\ \bar{w}_2'-\bar{w}_2=(w_2'-w_2)-(w_1'-w_1)/2,$$
$$\varphi'-\varphi\equiv\delta\varphi=w_2'-w_2+\phi(\mathbf{I},\ \mathbf{I}',\ w_1,\ w_1'),$$

and we refrain from writing out the expression for ϕ. Since the perturbed potential can always be written out as

$$\Phi_1(r,\ \varphi)=\bar{\Phi}_1(r)\exp(im\varphi)=\Phi\exp(im\bar{w}_2),$$

we have

$$\Phi=\bar{\Phi}_1(r)\exp[im\delta(\mathbf{I},\ w_1)]$$

($\delta=\varphi-\bar{w}_2$ is a known function of **I** and w_1). In actual fact, then, (6) is an integral equation for the unknown function $\bar{\Phi}_1(r)$ in only one variable. That equation, however, is not in a terribly symmetric form, a situation that can be improved somewhat. The right-hand side of (6) depends on w_1 through Γ and $\exp(im\bar{w}_2)$ alone, so by averaging (6) over w_1, we obtain for the function

$$\chi(\mathbf{I})=\bar{\Phi}=\int_0^{2\pi}\Phi\, dw_1/2\pi$$

the expression

$$\chi(\mathbf{I})=\frac{Gm}{2\pi}\int d\mathbf{I}'\Pi(\mathbf{I},\mathbf{I}')\,\frac{\partial f_0(\mathbf{I}')/\partial I_2'-{}^1/{}_2\partial f_0(\mathbf{I}')/\partial I_1'}{\omega-m\delta\Omega_{pr}(\mathbf{I}')}\chi(\mathbf{I}'), \tag{8}$$

where

$$\Pi(\mathbf{I},\mathbf{I}')=\int dw_1\,dw_1'\,d\delta w_2\Gamma(r,r',\delta\varphi)$$
$$\times\exp(im\delta w_2)\exp[-im(w_1'-w_1)/2], \tag{8'}$$

$$\delta w_2\equiv w_2'-w_2.$$

Physically, $\Pi(\mathbf{I},\mathbf{I}')$ is proportional to the torque δM acting upon some selected (test) orbit with action **I** and resulting from all orbits with fixed action **I**′; these all have the same shape, but their major axes are oriented in all possible directions:

$$\delta M\propto -imG\exp(im\bar{w}_2)\Pi(\mathbf{I},\ \mathbf{I}')F^{(1)}(\mathbf{I}')d\mathbf{I}'. \tag{8''}$$

For a quasi-Coulomb field $\Phi_0(r)$, instead of $(1/2)\ \partial f_0/\partial I_1'$ in (8) we have simply $\partial f_0/\partial I_1'$, and instead of $\exp[-im(w_1'-w_1)/2]$ (with m required to be even) in (8′) we have $\exp[-im(w_1'-w_1)]$ (with arbitrary m).

One might hope to reduce (8) to a set of one-dimensional integral equations in two limiting cases: 1) the distribution function $f_0(\mathbf{I})$ is close to a delta function in I_2 near some value $I_2=I_2^{(0)}=L_0$; we will be commenting on this circumstance, writing $f_0=\Delta_2(I_1,I_2-L_0)\approx\delta(I_2-L_0)\varphi_0(I_1)$; 2) the system in question has quasicircular orbits, whereupon $f_0=\Delta_1(I_1,I_2)$, with $\Delta_1\approx\delta(I_1)\bar{\varphi}_0(I_2)$. Below we will concern ourselves mainly with the first case. The second is technically somewhat more complicated, and we will simply restrict ourselves to a few general remarks (see Sec. 4).

3. THEORY OF RADIAL ORBIT INSTABILITY

3.1. Cylindrical and disk systems

We take $f_0=\Delta_2(I_1,I_2-I_2^{(0)})$ and neglect the term $\partial f_0/\partial I_1'$, which is small in the present case. Using the fact that Π and χ hardly vary in I_2' over the characteristic scale length of the function $(\partial f_0/\partial I_2')/[\bar{\omega}-m\delta\Omega_{\rm pr}(I_1',I_2')]$, we can reduce Eq. (8) to an integral equation for the function of one variable $\psi(I_1)\equiv\chi(I_1,I_2'=I_2^{(0)})$:

$$\psi(I_1)=\frac{Gm}{2\pi}\int dI_1'P(I_1,I_1')S_0(I_1')\psi(I_1'), \tag{9}$$

where

$$P(I_1,\ I_1')=\Pi(I_1,\ I_1',\ I_2=I_2^{(0)},\ I_2'=I_2^{(0)}), \tag{9'}$$

$$S_0(I_1')=\int dI_2'\,\frac{\partial f_0(I_1',I_2')/\partial I_2'}{\bar{\omega}-m\delta\Omega_{pr}(I_1',I_2')}. \tag{9''}$$

If we are dealing with orbits that are almost radial (and assume that $\bar{\Omega}_{\rm pr}=0$, so that $\bar{\omega}=\omega$ and $\delta\Omega_{\rm pr}=\Omega_{\rm pr}$), we can put $I_2^{(0)}=0$ when we calculate $P(I_1,I_1')$. Furthermore, $\delta\varphi\equiv\varphi'-\varphi\approx\bar{w}_2'-\bar{w}_2$ for such orbits, so the function Π can then be substantially simplified:

$$\Pi(\mathbf{I},\mathbf{I}')=2\pi\int dw_1\,dw_1'J_m[r(\mathbf{I},w_1),\,r'(\mathbf{I}',w_1')], \tag{10}$$

where

$$J_m(r,r')=\frac{1}{2\pi}\int_0^{2\pi}d\alpha\Gamma(r,r',\alpha)\cos m\alpha. \tag{10'}$$

When the orbits are exactly radial ("cold" system), i.e., $f_0=\delta(I_2)\varphi_0(I_1)$,

$$S_0(I_1')=-\frac{m}{\omega^2}A(I_1')\varphi_0(I_1'),$$

where

$$A(I_1')\equiv\left.\frac{\partial\Omega_{pr}(I_1',I_2')}{\partial I_2'}\right|_{I_2'=0}. \tag{11}$$

Accordingly, the integral equation (9), which then describes radial orbit instability in a cold system, is

$$\psi(I_1)=-\frac{Gm^2}{2\pi\omega^2}\int dI_1'P(I_1,I_1')A(I_1')\varphi_0(I_1')\psi(I_1'). \tag{12}$$

By making use of (10′), it can easily be shown that $J_m(r,r')$ is a positive function; consequently, so is $\Pi(\mathbf{I},\mathbf{I}')$ in (10), and most importantly, so is $P(I,I_1')$ from (9′) when $I_2^{(0)}=0$, which enters into the derived equation (12). This can be made explicit for a disk system, for example. Expanding $(r^2+r'^2-2rr'\cos\alpha)^{-1/2}$ in a harmonic series, we obtain

$$J_m(r,r')=\sum_{n=0}^{\infty}F_n(r,r')\frac{1}{\pi}\int_0^{\pi}d\alpha P_n(\cos\alpha)\cos m\alpha.$$

where

$$F_n(r,r')=r_<^n/r_>^{n+1},\quad r_<\equiv\min(r,r'),\quad r_>\equiv\max(r,r'). \tag{13}$$

and the P_n are Legendre polynomials. Now the positive definiteness of J_m follows from the fact that $P_n(\cos\alpha)$ can in turn be expanded in cosines of multiples of the angle, with positive coefficients:[17]

$$P_n(\cos\alpha)$$
$$=\frac{(2n-1)!!}{2^{n-1}n!}\left[\cos n\alpha+\frac{1}{1}\frac{n}{2n-1}\cos(n-2)\alpha+\ldots\right]$$
$$=\sum_k{}' A_k^{(n)}\cos k\alpha,$$

with all $A_k^{(n)}>0$ (a prime indicates that the parity of k and n must be the same), and from

$$\frac{1}{\pi}\int_0^\pi d\alpha\cos m\alpha\cos n\alpha=\frac{\delta_{mn}}{2}.$$

We finally obtain a convenient representation for $J_m(r,r')$ in simple series form:

$$J_m(r,r')=\frac{1}{2}\sum_{n\geqslant m}{}' F_n(r,r')A_m^{(n)}>0.$$

Given the positivity of $P(I_1,I_1')$, the sign of the integrand in (12), and therefore the sign of ω^2 (i.e., the stability or instability of a system with purely radial orbits), will depend on the sign of $A(I_1')$ as defined by (11). If $A>0$ for all orbits in the system under consideration (in other words, for all values of I_1, or what amounts to the same thing in the present case, for any value of the energy E of radial stellar oscillations), then $\omega^2<0$. As a result, radial orbits are unstable with $A>0$. On the other hand, when $A<0$, it is the purely oscillatory mode that is unstable. The most compact formula for computing $A(E)$ is

$$A(E)=\frac{1}{(2E)^{1/2}}\lim_{r_0\to0}\left\{\int_{r_0}^{r_{\max}}\frac{dx}{x^2[1-\Phi_0(x)/E]^{1/2}}-\frac{1}{r_0}\right\}\Big/\int_0^{r_{\max}}\frac{dx}{[2E-2\Phi_0(x)]^{1/2}},$$

where $\Phi_0(r_{\max})=E$. Consider, for example, the potential $\Phi_0(r)$ near the center of a system. If we assume that no large point mass is present there, it can be represented in the form

$$\Phi_0(r)\approx\Omega^2r^2/2+br^4 \qquad (14)$$

(Ω and b are constant). For the low-frequency modes in question, then, if they are confined to some region where (14) holds, instability ($\omega^2<0$) will develop if $b<0$; conversely, if $b>0$, then $\omega^2>0$, corresponding to oscillations. In actual fact, what is required for instability (by way of illustration) is merely that some mean $\overline{A}$ be positive, a condition that can be met, in principle, even when A is positive for certain values of E and negative for others. Such circumstances are not at all unrealistic.

Suppose, for example, that we have a system whose potential $\Phi_0(r)$ can be represented near $r=0$ in the form (14) with $b>0$. Then, as we have already pointed out, modes of the given type that are localized near the center will be oscillatory. The possibility of such localization, however, is not obvious *a priori*: it might ultimately depend, for instance, on the specific orbits followed by the bulk of the mass in the active system. If those orbits stray far from the region in which (14) holds, a more accurate representation of the potential will be required in place of (14). The precession of low angular momentum orbits may then reverse sign (as compared with orbits that consistently remain close to the center). Radial orbit instability should then ensue. The potentials of real systems are such that sufficiently elongated orbits experience prograde precession. Which type of orbit predominates, however—those confined near the center or those that stray far from the center—depends on the detailed distribution of orbits over energy, and in principle either type of behavior is possible.

We remark here that Lynden-Bell[18] actually proposed the inequality $\partial\Omega_{\rm pr}(I_1,I_2)/\partial I_2>0$ as a condition for formation of a bar—i.e., of an elliptical deformation of the disk of a spiral galaxy. What is in fact responsible for the formation of a bar of the specified type is radial orbit instability in the stellar disk, and the proposed inequality (evaluated at $L=0$) is, as we pointed out above, a necessary condition for that instability. Rather than radial orbits, Lynden-Bell had in mind orbits that are close to being circular. The point is that at the time of Lynden-Bell's paper (1979), such orbits were deemed to be the only admissible type in spiral galaxies (that situation has since changed). For general orbits, however, no such simple condition exists.[1)] Formally, this follows, at the very least, from the necessity of retaining the term proportional to $\partial f_0/\partial I_1'$ in the integrand of (8), since it is not small compared with the term $\partial f_0/\partial I_2'$. Furthermore, for quasicircular orbits, with $f_0\approx\Delta_1(I_1,I_2)$, it is the former term that dominates.

Physically, this formal result corresponds to the fact that systems with quasicircular orbits evolve primarily through changes in the orbital energy of stars, rather than in their angular momentum. In the same vein as Lynden-Bell,[18] we can imagine an orbit that has outdistanced a bar that has formed in the disk, having acquired a somewhat higher precession rate $\Omega_{\rm pr}$ (compared with the angular rotation rate of the bar, $\Omega_b=\overline{\Omega}_{\rm pr}$). The sign of the torque exerted on such an orbit by the bar is then obvious: the bar will tend to turn the orbit backwards, bringing it closer to the bar. If this were the only mechanism in operation, the precession of the orbit would be retarded, and the orbit would tend back toward the bar if $\partial\Omega_{\rm pr}/\partial I_2>0$. But the bar also induces a change in the orbital energy of a star, and for quasicircular orbits that effect is the decisive one. The orbit in question will then precess at a rate corresponding to the new energy E (or action I_1). In other words, the rate of orbital precession is altered not so much through the direct influence of the bar-induced torque as through the aforementioned indirect agency. In the present case, where we have a group of active stars embedded in a massive halo, the orbits perturbed by the bar drift slowly among the admissible orbits belonging to the given set, which itself is determined by the pre-existing equilibrium potential of the system. Accordingly, for quasiradial orbits, this drift results primarily from changes in $I_2(L)$, while for quasicircular orbits, it results from changes in $I_1(E)$.

The inequality $(\partial\Omega_{\rm pr}/\partial I_2)_{I_2=0}>0$ is merely a necessary (and in no way sufficient) condition for radial orbit instability, and in particular, for the formation of a bar. The insufficiency of this criterion is immediately obvious from the fact that the retarding torque due to the bar can turn out to be ineffectual in the face of large orbital precession rates. To derive valid conditions for bar formation, it is necessary to stabilize a set of radial orbits whose precession rates have some finite spread [having demonstrated once again the predominance of the bar mode ($m=2$)]. The bar formation criterion is in fact none other than the condition for bar-mode instability of the type under consideration. We therefore now proceed to derive the stabilization conditions for systems with quasiradial orbits.

For definiteness, we assume a Maxwell distribution in $I_2 = L$:

$$f_0 = \frac{1}{\pi^{1/2} L_T} \exp(-I_2^2/I_T^2)\,\varphi_0(E), \tag{15}$$

where $L_T = I_T$ is the thermal rms deviation. If we then assume in (9″) that $\bar{\omega} = \omega = 0$, $I_2^{(0)} = 0$, $\delta\Omega_{pr} = \Omega_{pr} \approx A(E)L$, we will then have for the system stability limit

$$S_0(I_1') = 2\varphi_0(I_1')/mL_T^2 A(E'),$$

so that the integral equation (9) becomes

$$\psi(I_1) = \frac{G}{\pi L_T^2} \int dI_1' P(I_1, I_1') \frac{\varphi_0(I_1')}{A(I_1')} \psi(I_1'). \tag{16}$$

This is almost the same as Eq. (12). A comparison of these two equations suggests a simple relationship between the instability growth rate γ for a system with strictly radial orbits, $\gamma^2 = -\omega^2$, and the minimum dispersion in orbital angular momentum required to curtail that instability:

$$(L_T)_{min} = 2^{1/2}\gamma/m\bar{A}, \tag{17}$$

where $\bar{A}$ is some mean over the stellar orbits of the different energies E.

Equation (17) acquires a precisely defined meaning when all stars have almost the same energy, $E \approx E_0$, since we can then take $\bar{A} = A(E_0)$. If in (15) we go to a precession rate distribution $\Omega_{pr} = AL$, we then obtain a more tractable relation in place of (17):

$$(\Omega_{pr})_T = 2^{1/2}\gamma(m)/m, \tag{18}$$

where $(\Omega_{pr})_T$ denotes the thermal spread in precession rates, and the growth rate γ is given in the form $\gamma(m)$ to emphasize that in general it depends on the azimuthal index m. Note that Eqs. (17) and (18) hold both for a disk and a cylinder (for all m). Since $\gamma(m)$ is only a weak function of m,[2)] it follows from (18) that the most difficult modes to stabilize (and in that sense, the most unstable) are those with the smallest possible m. For almost radial orbits we have $m_{min} = 2$, which corresponds precisely to formation of an elliptical bar out of an initially circular disk. All the modes with odd m, particularly the $m = 1$ mode, are suppressed in this case, two oppositely directed (but equal) moments of forces would act on the two halves of an elongated orbit. The forces break, but do not rotate such "spoke" orbits.

For quasicircular orbits in a potential approximating that produced by a central point mass, however, the $m = 1$ mode is immediately set apart (we will make this clear in Sec. 4).

3.2. Spherical systems

We can in fact reduce the stability problem for a spherical system to a simpler problem (and one already treated above), the stability problem for an appropriate cylindrical system, by writing the linearized kinetic equation in $r, \theta, \varphi, v_r, v_\perp$, and α (where r, θ, φ are spherical coordinates, v_r is the radial velocity of a star, $v_\perp^2 = v_\theta^2 + v_\varphi^2$, v_θ and v_φ are the velocity components in the direction of θ and φ, and $\alpha = \arctan(v_\varphi/v_\theta)$ that describes the perturbation of a sphere with distribution function $F_0(E,L)$ (see, e.g., Refs. 12 and 13):

$$\frac{\partial F_1}{\partial t} + \frac{v_\perp}{r}\hat{L}F_1 + \hat{D}F_1 = \frac{\partial \Phi_1}{\partial r}\frac{\partial F_0}{\partial E} v_r + \frac{1}{r}\hat{L}\Phi_1 \frac{\partial F_0}{\partial v_\perp}, \tag{19}$$

where we have introduced the operators

$$\hat{L} = \cos\alpha \frac{\partial}{\partial\theta} + \frac{\sin\alpha}{\sin\theta}\frac{\partial}{\partial\varphi} - \sin\alpha \operatorname{ctg}\theta \frac{\partial}{\partial\alpha},$$

$$\hat{D} = v_r \frac{\partial}{\partial r} - \frac{v_r v_\perp}{r} + \left(\frac{v_\perp^2}{r} - \frac{\partial\Phi_0}{\partial r}\right)\frac{\partial}{\partial v_r}.$$

The operator $\hat{L}$ has the standard form for the operator representing an infinitesimal rotation about the y axis[19] (expressed in terms of the Euler angles θ, φ, α).[3)]

In the present case of complete spherical symmetry, the angular part of the perturbation of the potential can be separated out in a form proportional to a particular spherical harmonic: $\Phi_1 \propto Y_l^m(\theta,\varphi)$. The kinetic equation can be solved in a natural way in a coordinate system in which $\hat{L}$ is diagonal, i.e., one corresponding to rotation about the z' axis of the rotating system. In that system, the perturbation of the distribution function can be expanded as

$$F_1 = \sum_s F_s(r, v_r, v_\perp) T_{ms}^l(\varphi', \theta', \alpha'), \tag{20}$$

where

$$T_{ms}^l(\varphi_1, \theta, \varphi_2) = \exp(-im\varphi_1 - is\varphi_2) P_{ms}^l(\cos\theta),$$

and $P_{ms}^l(\cos\theta)$ are triply indexed functions;[19] in particular, the $P_{m0}^l(\cos\theta)$ are, up to a constant, identical to the associated Legendre functions. The potential can also conveniently be written as

$$\Phi_1 = \chi(r,t) T_{m0}^l(\varphi, \theta, \alpha),$$

or in primed coordinates,

$$\Phi_1 = \chi(r,t) \sum_s \alpha_s^l T_{ms}^l(\varphi', \theta', \alpha'). \tag{21}$$

where the α_s^l are coefficients corresponding to the rotation that carries the y axis into the z axis. Thus, in the primed system, we have independent equations for each of the expansion functions (20):

$$\frac{\partial F_s}{\partial t} + \frac{v_\perp}{r} isF_s + \hat{D}F_s = \frac{\partial \Phi_s}{\partial r}\frac{\partial F_0}{\partial v_r} + is\frac{\Phi_s}{r}\frac{\partial F_0}{\partial v_\perp}, \tag{22}$$

with $\hat{L}F_s = isF_s$. Equation (22) is identical to the equation for the response of a cylindrical system to a flute perturbation of the potential in the form $\Phi(r,\varphi) = \chi(r)\exp(is\varphi)$, where r and φ are cylindrical coordinates. To solve the original "spherical" problem, it is thus necessary to find a solution of the "cylindrical" problem (22) with parity l for all s in the interval $(-l, l)$ (see below).

The second part of the reduction procedure entails writing a prescription for calculating the perturbation of the density ρ_1 of the sphere. Assuming that Eq. (22) for F_s has been solved; to calculate ρ_1, we must transform back to the original unprimed coordinate system in (20), using

$$T_{ms}^{l}(\varphi',\theta',\alpha')=\sum_{s'} T_{ms'}^{l}(\varphi,\theta,\alpha)\,(\alpha_{s'}^{l})^{*}. \qquad (23)$$

Since the expression for the density perturbation, $\rho_1 = \int F_1 v_\perp dv_\perp dv_r d\alpha$, includes an integration over α, the net result is that only one term (corresponding to $s'=0$) remains in each of the sums in (23), and the desired expression is

$$\rho_l{}'=T_{m0}^{l}(\varphi,\theta,\alpha)\left[\sum_{s} \alpha_s{}^{l}\int F_s(r,v_r,v_\perp)v_\perp\, dv_\perp\, dv_r\right], \qquad (24)$$

where[20]

$$\alpha_s{}^{l}=\frac{(l-s)!}{(l+s)!}[P_l{}^{s}(0)]^2 = (l+s)!\,(l-s)!\Big/\left[\left(\frac{l+s}{2}\right)!\left(\frac{l-s}{2}\right)!\,2^l\right]^2,$$

for $l \pm s$ even and $\alpha_s^l = 0$ for $l \pm s$ odd.

We can apply this procedure to derive an integral equation resembling Eq. (8), which was derived earlier for a cylinder and disk. Introducing action-angle variables, as in Sec. 2, for the two-dimensional motion of a star in its orbital plane, we obtain instead of (2),

$$\frac{\partial f^{(s)}}{\partial t}+is\Omega_{pr}f^{(s)}+\Omega_1\frac{\partial f^{(s)}}{\partial w_1} =\frac{\partial f_0}{\partial I_1}\frac{\partial \Phi^{(s)}}{\partial w_1}+is\Phi^{(s)}\left(\frac{\partial f_0}{\partial I_2}-\frac{1}{2}\frac{\partial f_0}{\partial I_1}\right),$$

where

$$\Phi_s=\Phi^{(s)}\exp(im\bar{w}_2),\quad f_s=f^{(s)}\exp(is\bar{w}_2),$$

and the remaining notation is the same as in Sec. 2. The equation analogous to (3) is

$$\frac{\partial F^{(s)}}{\partial t}+is\Omega_{pr}F^{(s)}+\Omega_1\frac{\partial F^{(s)}}{\partial w_1}=\Omega_1\frac{\partial F_0}{\partial E}\frac{\partial \Phi^{(s)}}{\partial w_1} + is\Phi^{(s)}\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right). \qquad (25)$$

Proceeding now to a discussion of the low-frequency modes (with the same assumptions as in Sec. 2), we may put $F^{(s)} = F_1^{(s)} + F_2^{(s)}$, and in place of (5), we obtain for the "exchange" function $F_1^{(s)}$

$$-(\omega-s\Omega_{pr})F_1{}^{(s)}=s\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right)\frac{1}{2\pi}\int_0^{2\pi}\Phi^{(s)}\,dw_1. \qquad (26)$$

Accordingly, Eq. (24) gives the perturbed density ρ_1:

$$\rho_1 =-T_{m0}{}^{l}(\varphi,\theta,\alpha)\sum_{s=-l}^{l}{}'\alpha_s{}^{l}\int v_\perp\, dv_\perp\, dv_r s\left(\frac{\partial F_0}{\partial L}+\Omega_{pr}\frac{\partial F_0}{\partial E}\right)\overline{\Phi^{(s)}}\Big/ (\omega-s\Omega_{pr})\equiv-T_{m0}^{l}(\varphi,\theta,\alpha)B(r). \qquad (27)$$

It still remains to take advantage of Poisson's equation, which yields the desired integral equation,

$$\chi(r)=\frac{4\pi G}{2l+1}\int r'^2\,dr' F_l(r,r')B(r'), \qquad (28)$$

where we have already encountered $F_l(r,r')$ [see Eq. (13)].

$B(r)$ is defined in (27); it depends on

$$\overline{\Phi^{(s)}}=\int_0^{2\pi}\chi(r(\mathrm{I},w_1))\exp[is\delta(\mathrm{I},w_1)]\,dw_1,$$

i.e., we have in fact again derived an equation for an unknown function of one variable, $\chi = \chi(r)$.

Essentially the only important kind of spherically symmetric system is one with radially elongated orbits. We therefore put $F_0 = F(L)\varphi_0(E)$, where $F(L)$ is close to $\delta(L^2)$; for example (for small L_T)

$$F(L)=\frac{1}{L_T{}^2}\exp\left(-\frac{L^2}{L_T{}^2}\right),\qquad \int F(L)\,dL^2=\int\delta(L^2)\,dL^2=1. \qquad (29)$$

Equating the integral equations obtained for a "cold" system ($L_T = 0$) at the limits of stability ($\omega = 0$, $L_T = (L_T)_{\min}$), we obtain expressions analogous to (17) and (18):

$$(L_T)_{min}=\frac{2^{1/2}\gamma}{\bar{A}[l(l+1)]^{1/2}},\qquad (\Omega_{pr})_T=\frac{2^{1/2}\gamma}{[l(l+1)]^{1/2}}. \qquad (30)$$

Note that in actuality, for the present case, requiring that the system be at the limit of stability determines the value of $F(L)$ at $L = 0$ [Eq. (30) has been derived for a Maxwell distribution (29)]. Since $\gamma(l)$ is essentially constant in a spherical system and odd-m modes are also suppressed, Eq. (30) shows that the mode $l = 2$, which corresponds to ellipsoidal deformation, is again special. This is the direct analog of a bar mode in a disk system. For stationary states, the stability requirement against such a deformation imposes constraints on the minimum possible dispersion among orbital precession rates. These constraints become quite stringent for systems in a self-consistent equilibrium state, in which the potential Φ_0 is determined by the density distribution of the very same stars affected by the perturbations—in other words, there is no massive halo.

The most natural form for a general stability criterion applicable to any system would be the ratio of that part of the kinetic energy (of the active subsystem of stars) attributable to orbital precession, $T_{\rm pr}$, to the magnitude of the gravitational energy of interaction among those stars, $|W_G|$. This ratio must be less than some critical value for a system to be stable. In light of the large scale of the dominant mode encompassing the system as a whole, which mode therefore does not depend on structural features, it would not be surprising if the actual value of the critical ratio were essentially the same from one system to another. This would then furnish a natural way to improve upon a previous stability criterion,[7,12] namely the minimum of $2T_r/T_\perp$, where T_r and $T_\perp$ are the total radial and transverse kinetic energies of stars in the system. For self-consistent equilibria, this ratio must obviously bear a simple relation to $T_\perp/|W_G|$. Under those ("normal") circumstances in which $\Omega_{\rm pr} \sim \Omega_1$ and $\Omega_{\rm pr} \simeq \Omega_2$, this improvement in the stability criterion is not essential. But there are examples of anomalously slow precession that are extremely important in practice. Above all, these include systems with quasiradial orbits.

We encounter this same anomaly, however, when most of the particles in a system have orbits that are almost closed. This is the case, for instance near the center of a stellar system in which the gravitational potential is approximately quadratic. This, incidentally, is why "rounded" orbits, in addition to quasiradial, can participate in the formation of the central bar of a spiral galaxy. Although the integral equations (8) and (9) that we introduced in Sec. 2 were used above (§3.1) only for highly eccentric orbits, they also apply to bar formation in the general case.

Orbits are also almost closed (i.e., they are approximately Keplerian ellipses) when a large mass is present at the center of a system. Specifically, this would be the case in the immediate vicinity of a massive object (a black hole, for example) at the center of a galaxy. Processes taking place in the neighborhood of such objects, which probably have a direct bearing on galactic nuclear activity, are a research topic in their own right. Other examples—thin planetary rings and ring galaxies—are briefly taken up in Sec. 4.

The most obvious (and most striking) manifestation of radial orbit instability in spherical systems is the formation of elliptical galaxies. The aggregate observational data on the latter can be simply and naturally accounted for[5] by imagining that they were produced in the collapse of a spherical star cluster (initially far from equilibrium) accompanied by radial orbit instability.

The basic facts are as follows. Above all else, the surface brightnesses $I(r)$ [and therefore, presumably, the surface densities $\sigma(r)$] of elliptical galaxies are described surprisingly well by a single function—one that depends on neither the size nor the mass of the galaxy:

$$I(r) \propto \exp[-7.67(r/r_e)^{1/4}],$$

where r_e is the radius containing half the emitted light. This is the well-known $r^{1/4}$ law of de Vaucouleurs.[21] Next, the distribution of galaxies as a function of the ratio of semiaxes c/a has a characteristic maximum near $c/a = 0.5$–0.6.[22] Elliptical galaxies, even those that are quite oblate, rotate very slowly, with the kinetic energy of rotation being only a small fraction of the potential (gravitational) energy.[23] Also worthy of note is the radially elongated nature of the dominant stellar orbits in these galaxies.

The "experimental" density distributions obtained by numerically modeling the collapse of systems of gravitating particles conform quite well to the de Vaucouleurs law, and they thus correspond to reality only when the initial state of the collapsing system is sufficiently "cold," i.e., the virial ratio $V = 2T/|W| \ll 1$ (T is the initial kinetic energy of the system and W is its initial potential energy; at equilibrium, $V = 1$).[8,9] This is a highly ambiguous result, as it is precisely in cold collapse that radial orbit instability should develop most naturally. The fact that it does develop in such numerical experiments has been specially checked and confirmed. This was demonstrated directly in Ref. 24: the rms particle angular momentum increased, the degree of anisotropy in the system decreased, etc.

Furthermore, computer modeling has shown[6] that $c/a = 0.5$–0.6 in typical systems obtained in collapse from a cold initial state, which exactly corresponds to the location of the maximum in the distribution of the apparent flattening of elliptical galaxies.[22] Upon more detailed comparison, good qualitative agreement has also been found[25] between the distribution of c/a in real elliptical galaxies and in systems obtained via numerical modeling. The slow rotation and the radially elongated dominant orbits are also a completely natural outcome of collapse from an almost spherical initial configuration. We note that previously proposed scenarios for the formation of elliptical galaxies (after their negligible rotation had been discovered[23]) were based on the assumption that they retained some residual anisotropy, left over from their original much more substantial (and scarcely realistic) anisotropy. What makes the present scheme so attractive is that anisotropy develops in and of itself during contraction and as stars acquire radial motion, which in turn leads to large-scale instability—i.e., the development of an essentially ellipsoidal triaxial system from an initially spherical cluster.

4. LOW-FREQUENCY MODES IN RING AND DISK SYSTEMS WITH QUASICIRCULAR ORBITS

The integral equations (8) and (9) are suitable for describing low-frequency modes (with $\overline{\omega} = \omega - m\overline{\Omega}_{pr} \sim \omega_G$, $\Delta\Omega_{pr}$) in systems with quasicircular orbits[4)] under two conditions (see Sec. 2): $\omega_G \ll \Omega_1$, and $\Delta\Omega_{pr} \ll \Omega_1$. Those conditions are obviously satisfied in the thin rings encircling the giant planets (Saturn, Uranus), for example. What is perhaps more surprising is that they are also met rather accurately by normal spiral galaxies. The first of the two conditions corresponds to the fact that only a relatively small fraction of the total mass of a spiral galaxy plays an active role in perturbations—namely, matter concentrated in the flattest and coldest subsystems (and here following quasicircular orbits). The precession rate of such quasicircular orbits is given by

$$\Omega_{pr}(r) = \Omega_2 - \Omega_1/2 \approx \Omega(r) - \varkappa(r)/2,$$

where $\Omega(r)$ is the angular velocity of a star in a circular orbit at a distance r from the center, and $\varkappa(r) = [4\Omega^2 + rd\Omega^2/dr]^{1/2}$ is the epicyclic frequency. A spiral galaxy undergoes rigid-body rotation near the center, and far from the center it is most often $\Omega(r) \propto 1/r$ (although almost-rigid-body rotation does extend out to the observable limits for many galaxies). For approximately rigid-body rotation, the second condition is a trivial consequence, since then not only do we have $\Omega_{pr} \ll \Omega_1$, but even $\Omega_{pr} \ll \Omega_1$ for the rate itself. If $\Omega \propto 1/r$ far from the center, then $\Omega_{pr} \approx (1 - 2^{-1/2})\Omega$, i.e., $\Omega_{pr} \approx 0.3\Omega_2 \approx 0.2\Omega_1$, so that even the more restrictive form of condition 2 can be assumed to be approximately satisfied (when $\Delta\Omega_{pr}$ is replaced by Ω_{pr}).

For the spread in stellar precession rates, which make the most important contribution to the perturbation, the accuracy with which condition 2 is satisfied can be improved. There is no need for remote orbits to satisfy it, in general, at least in calculating the eigenfrequencies of spiral modes, if the spirals themselves are merely a response of the outlying regions of the galaxy to a self-sustaining bar mode at the center. All previous theories of spiral structure have yielded galactic spiral arms that rotate anomalously slowly. For the Milky Way, for example, all Ω_p fall in the range ≈ 11 to ≈ 25 km/s·kpc. Even the latter (highest) estimate is manifestly lower than the typical rotation rates of stars, $\Omega \approx \Omega_2$, and furthermore, than their radial frequencies $\varkappa = \Omega_1 > \Omega$, with $\Omega(r = \overline{r} \approx 5\text{ kpc}) \approx 45$ km/s·kpc. In return, the values of Ω_p in the indicated range fit the typical orbital values Ω_{pr} quite well.

To close, let us deal with the separability of the $m = 1$ mode in the thin rings of planets or ring galaxies with massive nuclei. We can calculate $\Pi(\mathbf{I},\mathbf{I}')$ from (8′), making use of the orbital equations in the epicyclic approximation:

$$r=R+\rho\sin w_1\equiv R+\delta_1,$$
$$\varphi=w_2+(2\Omega/\varkappa R)\rho\cos w_1\equiv w_2+\delta_2,$$

where $\rho=[2I_1/\varkappa(R)]^{1/2}$ is the size of an epicycle (the analog of the Larmor radius for a plasma in a magnetic field), and R is the radius at which its center lies ($\rho\ll R$). Since

$$\delta w_2\equiv w_2'-w_2=\delta\varphi-(\delta_2'-\delta_2),\quad \delta\varphi\equiv\varphi'-\varphi,$$

we have

$$\Pi(\mathbf{I},\mathbf{I}')$$
$$=2\pi\int dw_1\,dw_1'\exp[-imw_1'+imw_1-im(\delta_2'-\delta_2)]J_m(r,r'),$$

where $J_m(r,r')$ has been defined in accordance with (10′) using the Green's function Γ for a disk, i.e., (7″). Expanding $J_m(R+\delta_1,R'+\delta_1')$ and $\exp[-im(\delta_2'-\delta_2)]$ as power series in δ_1, δ_1', δ_2, and δ_2' out to second order, it can easily be shown that the lowest order in ρ for the function Π corresponds precisely to the $m=1$ mode, with

$$\Pi\approx(2\pi)^3\rho\rho'\left[\frac{1}{4}\frac{\partial^2 J_m(R,R')}{\partial R\,\partial R'}+\frac{\Omega'}{2\varkappa'R'}\frac{\partial J_m(R,R')}{\partial R}+\frac{\Omega}{2\varkappa R}\frac{\partial J_m(R,R')}{\partial R'}+\frac{\Omega\Omega' J_m}{\varkappa\varkappa' RR'}\right],\qquad(31)$$

where

$$\rho'=[2I_1'/\varkappa(R')]^{1/2},\ \Omega'\equiv\Omega(R'),\ \varkappa'\equiv\varkappa(R').$$

This expression holds not only for a thin disk, but also for a disk in the field of a massive central object, for example. In the present case, however, there is a simple analytic expression for the function $J_m\approx\ln|R-R'|/2\pi R$, and the leading term in square brackets on the right-hand side of Eq. (31) is the dominant one. Because of the logarithmic singularity of $J_m(R,R')$ at $R\approx R'$, a region near $R'=R$ of width $\Delta R'=N(\rho+\rho')$, $N\gg1$ must be excised when integrating over $dI_2'=dR'dI_2(R')/dR'$, which makes the derivations somewhat more difficult (for example, Eq. (31) can only be used for $|R'-R|>\Delta R'$). We have demonstrated above only that the $m=1$ mode is special for the case under consideration. This may relate to the advent of just this mode during formation of the thin elliptical ringlets around Saturn and Uranus that were discovered by the two Voyager spacecraft. Similar phenomena are encountered in ring galaxies as well: noncentral nuclei, and instances in which the nucleus touches the ring. The latter represent the final state resulting from the development of the $m=1$ unstable mode.

1) For systems consisting of general orbits that share similar values of Ω_{pr} (somewhat of an artificial situation, apart from a system with almost radial orbits), it is nevertheless possible to give both a criterion for instability, $(\partial/\partial L+\Omega_{pr}\partial/\partial E)\Omega_{pr}>0$), and a simplified version of the integral equation (8) in the form (9), which is only slightly different from (12).

2) For example, we have $\gamma(m)\propto m^{1/2}$ in a disk with $m\gg1$, and for a cylinder there is almost no m dependence at all. Equation (10) for the function Π simplifies when $m\gg1$, since then $J_m\approx2\delta(r-r')/m^2$ for a cylinder and $J_m\approx\delta(r-r')/m$ for a disk. Thus, one of the integrations in (10) can be dispensed with. The asymptotic expressions for J_m are most conveniently derived directly from Poisson's equation, assuming that $m^2\Phi_1/r^2\gg|d^2\Phi_1/dr^2|$, $|r^{-1}d\Phi_1/dr|$ in the latter.

3) In the present context, this was apparently first pointed out in Ref. 20.

4) We point out that these equations can also be reconciled with odd-m perturbations of systems that have nonsingular $\Phi_0(r)$ if we redefine the function $\chi\equiv\overline{\Phi}$ to be $\int_0^{4\pi}\Phi dw_1/4\pi$, and take the integral over w_1 in (8′) from 0 to 4π while multiplying the right-hand side of (8) by 1/2.

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