Theory of two-dimensional superfluidity in a nonuniform external field

S. I. Shevchenko

Low-Temperature Physico-Technological Institute, Academy of Sciences of the Ukrainian SSR

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In this paper we show that the phenomenon of Bose-Einstein condensation can take place in a two-dimensional ideal Bose gas subjected to a nonuniform field that varies on a macroscopic scale. We calculate the Bose-Einstein condensation temperature T_0 for the class of fields that are power-law functions of coordinates. We prove that inclusion of interactions between the bosons prevents Bose condensation, but that for the case of a dilute gas T_0 remains a special temperature in whose neighborhood the system makes a transition to the superfluid state. We investigate the effect of a nonuniform field on the Berezinskii-Kosterlitz-Thouless phase transition.

The study of superfluidity in thin films has a long history (see, e.g., Ref. 1). In recent times, the problem of superfluidity in ultrathin (down to monolayer) films has attracted special interest. Thus, experiments[2] have shown that when ^{4}He is adsorbed on a surface as a thin film, some of the adsorbate forms a solid layer while the remaining ^{4}He behaves rather like a dilute surface Bose gas. At low temperatures this Bose gas enters the superfluid state; the fact that its phase transition temperature can be almost three orders of magnitude lower than that of bulk He-II suggests that the gas phase is extremely dilute.

In another group of experiments, the properties of spin-polarized atomic hydrogen H$\downarrow$ adsorbed by a surface covered with a film of helium are investigated.[3] In these experiments the film of hydrogen also forms a quasi-two-dimensional dilute gaslike phase. Although at this time superfluidity in films of spin-polarized hydrogen has yet to be observed experimentally (which is also the case for three-dimensional superfluidity of H$\downarrow$), there is no doubt that this type of superfluidity is possible in principle.

Two-dimensional superfluidity (more precisely, quasi-superfluidity) can occur in an exciton gas created at the surface of a semiconductor (see, e.g., Refs. 4 and 5). These ideas are also applicable to structures made up of layers with alternating electronic and hole conductivity, in which pairing of spatially separated electrons and holes can lead to boson formation.[6,7] (The authors of Ref. 8 report that this type of pairing has been observed experimentally.)

As a final example, we should mention another system—the system of Abrikosov vortices in a number of high-temperature superconductors. The authors of Ref. 9 showed that this system can be treated as a two-dimensional "Bose gas," in which we may expect two-dimensional "superfluidity." In Ref. 9, it was established that the partition function for the Abrikosov vortices is isomorphic to the partition function for a two-dimensional Bose gas. The isomorphism arises from the fact that we can identify the vortex lines with world lines of a two-dimensional boson.

From a theoretical point of view the possibility of superfluidity in a two-dimensional system was long considered a dubious proposition, until the work of Berezinskii[10] and Kosterlitz-Thouless[11] proved that two-dimensional superfluidity is hindered by the presence of vortices in the two-dimensional system, which diffuse transverse to the superfluid flux under the action of the Magnus force, as a result of which the flux attenuates with time. Below the critical temperature T_{BKT}, these vortices are bound in pairs with other vortices of opposite circulation, and the diffusion of vortices transverse to the flux is forbidden because the Magnus force acts on components of a pair in mutually opposite directions. This leads to dissipationless flow (i.e., to superfluidity) at temperatures below T_{BKT}. Later, in the work of Nelson and Kosterlitz[12] it was shown that at the temperature T_{BKT} the superfluid density ρ_s changes discontinuously from a value connected with T_{BKT} by the relation $T_{\mathrm{BKT}} = \pi\hbar^2\,\rho_s(T_{\mathrm{BKT}})/2m$ to zero.$^{1)}$ In this relation, ρ_s should be understood to mean the superfluid density renormalized by thermally excited vortices (see Sec. 3). The experiments of Bishop and Reppy[13] and a number of other authors (see, e.g., Ref. 14) on monolayer films of ^{4}He completely confirm these latter predictions of the theory.

The BKT theory describes a transition to the superfluid phase for a spatially uniform system. In this paper we construct a theory of two-dimensional superfluidity in a nonuniform external field. Interest in this problem is motivated by the following circumstance: in the three-dimensional case, the transition to the superfluid phase is accompanied by the phenomenon of Bose-Einstein condensation and by the appearance of long-range order in the system at the transition point. This allows us to describe the superfluid phase in terms of an order parameter. In two dimensions Bose condensation cannot occur at nonzero temperatures, either for an ideal Bose gas or for a Bose gas with interactions, and so an order parameter cannot be introduced.

However, the conclusion that there is no Bose condensation in a two-dimensional Bose gas is correct only for the spatially uniform case. In a paper by the author,[15] it was shown that an ideal two-dimensional Bose gas placed in a nonuniform external field exhibits Bose-Einstein condensation at a certain nonzero temperature T_0, just as in three-dimensional case. It must be emphasized that this result applies specifically to an ideal Bose gas: since the phenomenon of Bose condensation in a nonuniform field involves spatial redistribution of the bosons and occupation of the lowest energy state by a macroscopic number of particles, it is clear that the inclusion of interactions between the bosons, which inhibits the latter process, can radically change the situation. A further goal of this paper is to address a number of ques-

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tions related to the influence of a nonuniform field on the Bose condensate and superfluidity of a two-dimensional system of interacting bosons.

The fundamental result of this paper can be stated as follows: inclusion of interactions between the bosons eliminates Bose condensation and long-range order in a system of two-dimensional bosons placed in a nonuniform field. However, it is noteworthy that the Bose condensation temperature T_0 for a dilute Bose gas in a nonuniform field is a stable characteristic of the system. When interactions between the bosons are included, the phase transition associated with Bose condensation is replaced by a transition involving the appearance of superfluidity in the system, and the transition temperature T_c to the superfluid state differs only slightly from the temperature T_0. The more dilute the Bose gas is, the smaller is the correction to T_0. The diluteness parameter is $\ln^{-1}(1/na^2)$, where n is the two-dimensional boson density and a is the characteristic radius over which the repulsive potential between bosons acts. For an applied external field $u(\mathbf{r})$ that is a power-law function of one of the coordinates ($u \propto x^\alpha$), this correction depends considerably on the value of the exponent α. For $\alpha \lesssim 1$ the correction is proportional to the first power of the diluteness parameter, while for $\alpha \gtrsim 1$ the correction is small and is proportional to the αth root of the diluteness parameter. Since under the conditions of the experiments large values of $\ln(1/na^2)$ are difficult to achieve, let alone large values of the root of this logarithm, in practice T_c will be close to T_0 only for $\alpha \lesssim 1$. Because T_0 decreases as α decreases, the optimum value of α from the point of view of experiment is $\alpha \approx 1$.

The paper is set up in the following way. In Sec. 1 we show that the presence of a nonuniform field leads to the phenomenon of Bose condensation in a two-dimensional ideal Bose gas. Although some of the results of this section have already been published in a short communication,[15] here they are given a more comprehensive and rigorous proof. In Sec. 2 we set up the theory of a two-dimensional weakly nonideal Bose gas, and show that when this system is placed in a nonuniform field the temperature at which a nonzero superfluid density appears is close to the Bose condensation temperature. In Sec. 3 we discuss features of vortex formation in a nonuniform field and the influence of the nonuniform field on the BKT transition temperature. In the Conclusion we discuss the similarities and differences in the behavior of a Bose and a Fermi gas in a nonuniform field.

1. BOSE CONDENSATION IN A TWO-DIMENSIONAL IDEAL BOSE GAS

Let us first discuss the behavior of a two-dimensional ideal Bose gas in a nonuniform external field. Assume the characteristic distance over which the external field varies is on the order of the size of the system. Then in zero approximation a quasiclassical approach can be used to describe the behavior of the gas. If the potential energy of a boson in the external field is $u(\mathbf{r})$, the distribution function of the bosons in a state of thermodynamic equilibrium will have the form

$$n(\mathbf{p},\mathbf{r})=\left\{\exp\left[\left(\frac{p^2}{2m}+u(\mathbf{r})-\mu\right)\frac{1}{T}\right]-1\right\}. \tag{1}$$

Here $\mathbf{p}$ is the two-dimensional momentum, m is the mass of a boson, and μ is the chemical potential of the Bose gas. The chemical potential is found as usual from the normalization condition

$$N=\int\frac{d\mathbf{p}}{(2\pi\hbar)^2}\int d\mathbf{r}\,n(\mathbf{p},\mathbf{r}), \tag{2}$$

where N is the total number of particles in the system.

The most physically transparent results are obtained for the case of a field $u(\mathbf{r})$ that is a quadratic function of one of the coordinates, i.e.,

$$u(\mathbf{r})=-u_0[1-(x/L_x)^2], \tag{3}$$

where L_x is the size of the system in the direction of the x-axis.

Substituting (3) into (2) and replacing the integration variable x by $p_zL_x/(2mu_0)^{1/2}$, we are led to the following equation for the chemical potential (we assume that $u_0 \gg T$)

$$\frac{N}{L_xL_y}=\frac{1}{(2\pi\hbar)^2(2mu_0)^{1/2}}\int d\mathbf{p}\,dp_z\left\{\exp\left(\frac{p_x^2+p_y^2+p_z^2}{2mT}-\frac{u_0+\mu}{T}\right)-1\right\}^{-1}. \tag{4}$$

The integral on the right-hand side of this equation precisely coincides with the integral that arises in the corresponding equation for the chemical potential of a three-dimensional ideal Bose gas. However, it is well known that the latter case exhibits the phenomenon of Bose-Einstein (BE) condensation, i.e., the accumulation of a macroscopic number of particles in the state with energy $\varepsilon = 0$. The phenomenon of BE condensation is associated with the convergence of the integral (4) over three-dimensional momentum space for $\mu + u_0 = 0$. Because this same integral over p appears in the present case, it is obvious that BE condensation occurs in a two-dimensional ideal gas placed in a field $u(\mathbf{r})$ of form (3), just as for the three-dimensional Bose gas. We can find the temperature T_0 for the transition to a state with a Bose condensate from (4) by setting $\mu + u_0 = 0$. As a result we obtain

$$T_0^{3/2}=\frac{\hbar^2n_0}{m}\frac{(4\pi u_0)^{1/2}}{\zeta(3/2)}, \tag{5}$$

where $n_0 = N/L_xL_y$ is the boson density in the system in the absence of the nonuniform field, and $\zeta(3/2)$ is the Riemann zeta function.

Let us turn now to the case of an arbitrary potential field $u(\mathbf{r})$. Carrying out the integration of (2) over momentum $\mathbf{p}$, we are led to the equation

$$N=\frac{mT}{2\pi\hbar^2}\int d\mathbf{r}\left\{-\ln\left[1-\exp\left(-\frac{u(\mathbf{r})-\mu}{T}\right)\right]\right\}. \tag{6}$$

Let us assume that $u(\mathbf{r})$ has the form of a potential well whose center is located at the point $x = y = 0$; $u(\mathbf{r})$ could also be a one-dimensional potential well, in which case we will assume that the minimum of the potential $u(x)$ is located at $x = 0$.

The presence or absence of a Bose condensate is determined by the convergence of the integral in (6) when $\mu + u_0 = 0$, where u_0 is the value of the potential at the minimum point. If $u(\mathbf{r})$ increases with increasing x and y faster than $\ln r$, and if the condition $u(\infty) \gg T$ holds at the edge of the sample, i.e., when $x \approx L_x$ and $y \approx L_y$, then the integral in (6) obviously converges for large x and y.

For small $u(\mathbf{r})-\mu$, or more precisely when $u(\mathbf{r})-\mu \ll T$, the expression under the integral sign has the form $\ln[u(\mathbf{r})-\mu]/T$; therefore, when $u(\mathbf{r})$ depends only on a single coordinate, the integral $\int dx \ln[u(x)-\mu]/T$ converges for small x as long as $u(x)$ goes to zero in this limit more slowly than $\exp(-L_x/|x|)$. Thus, when u has a power-law dependence on x, the integral in (6) converges and the phenomenon of BE condensation can occur. For a two-dimensional well BE condensation can occur only if $u(\mathbf{r})$ goes to zero more slowly than $\exp(-L^2/r^2)$, because when $u[\mathbf{r} \propto \exp(-L^2/r^{2+\varepsilon})]$ the integral $\int dr r/r^{2+\varepsilon}$ diverges.

For potential wells $u(x)$ of the form

$$u(x)=-u_0[1-|x/L_x|^\alpha] \tag{7}$$

the integration over x in (6) is easily carried out and we find that the temperature of Bose condensation equals

$$T_0=\left[\frac{\hbar^2 n_0}{m} u_0^{1/\alpha} \frac{\pi}{\Gamma(1+1/\alpha)\zeta(1+1/\alpha)}\right]^{\alpha/(\alpha+1)}, \tag{8}$$

where $\Gamma(x)$ is the gamma function.

This result for the transition temperature can be obtained from simple physical considerations. The Bose condensation temperature T_0 coincides in order of magnitude with the degeneracy temperature, i.e., the temperature at which the average distance between bosons becomes comparable to the thermal de Broglie wavelength. Thus, in the two-dimensional case the degeneracy temperature is of order

$$\hbar^2 \bar{n}/m \approx T_0. \tag{9}$$

For the spatially inhomogeneous systems under discussion here, $\bar{n}$ in this expression should be understood to mean the average boson density in the system, i.e.,

$$\bar{n}=\frac{N}{L_y L_x(T)} \equiv n_0 \frac{L_x}{L_x(T)}, \tag{10}$$

where $L_x(T)$ is that characteristic length along the x-axis over which the Bose gas is concentrated for a given temperature T. Since an ideal Bose gas is localized in regions of space where its energy $u(x)$ satisfies the condition $u(x)<T$, the length $L_x(T)$ is found from the relation

$$u_0[L_x(T)/L_x]^\alpha \approx T. \tag{11}$$

Substituting this expression for $L_x(T)$ into (9) and (10), we obtain

$$T_0 \approx \left(\frac{\hbar^2}{m}\frac{N}{L_x L_y} u_0^{1/\alpha}\right)^{\alpha/(\alpha+1)}. \tag{12}$$

For $\alpha \gtrsim 1$ this relation for T_0 coincides with the result (8).

These considerations are based on the quasiclassical approach, which is valid provided that the de Broglie wavelengths of all the particles are small compared to distances over which the external field varies. However, particles with small momenta, which are important in the analysis of the problem of Bose condensation, have wavelengths that are commensurate with the wavelength over which the external field varies. Therefore, a more careful investigation of the problem is required. In a field of the form (3) we can obtain an exact solution.

In the external field (3) the state of the bosons is described by a wave function ψ which satisfies the Schroedinger equation

$$\left\{-\frac{\hbar^2}{2m}\nabla^2-u_0\left[1-\left(\frac{x}{L_x}\right)^2\right]\right\}\psi_n=\varepsilon_n\psi_n. \tag{13}$$

The eigenvalue spectrum of this equation is the sum of the eigenvalues for free motion with momentum $\hbar\varkappa$ along the y-axis and the eigenvalues of a harmonic oscillator

$$\begin{gathered}\varepsilon_n=\frac{\hbar^2 k^2}{2m}+n\hbar\omega+\varepsilon_0,\\ \omega=\left(\frac{2u_0}{m}\right)^{1/2}\frac{1}{L_x}, \qquad \varepsilon_0=\frac{1}{2}\hbar\omega-u_0,\end{gathered} \tag{14}$$

where n is a nonnegative integer.

Now the normalization condition, which determines the chemical potential μ, has the form $(\tilde{u}=u-\varepsilon_0)$

$$N=\int_{-\infty}^{\infty}\frac{L_y}{2\pi}dk\sum_n\left\{\exp\left[\left(\frac{\hbar^2k^2}{2m}+n\hbar\omega-\tilde{\mu}\right)\frac{1}{T}\right]-1\right\}^{-1}. \tag{15}$$

Making use of the Poisson summation formula, we rewrite this equation in the following form:

$$\begin{aligned}N=\frac{L_xL_y}{2\pi}\frac{m^{1/2}}{\hbar(2u_0)^{1/2}}\int_{-\infty}^{\infty}dk\int_0^{\infty}d\varepsilon\left\{\exp\left[\left(\frac{\hbar^2k^2}{2m}+\varepsilon-\tilde{\mu}\right)\frac{1}{T}\right]\right.\\ \left.-1\right\}^{-1}\left[1+2\sum_{s=1}^{\infty}\cos\left(2\pi s\frac{m^{1/2}L_x}{\hbar(2u_0)^{1/2}}\varepsilon\right)\right].\end{aligned} \tag{16}$$

The terms in (16) that contain a cosine will oscillate rapidly as $L_x \to \infty$. However, due to the presence of an infinite number of terms it is not *a priori* obvious that the sum over s will be small as $L_x \to \infty$. We will show that in fact the sum over s vanishes as $L_x \to \infty$. Making use of the expansion $[\exp x-1]^{-1}=\Sigma_{n=1}^{\infty}\exp(-nx)$ and carrying out the integration in (16) with respect to k and ε, we obtain the following expression for the sum over s [we will omit the factor in front of the integral in (16) for the time being]:

$$\frac{1}{\varkappa^2}\sum_{n=1}^{\infty}\left[\operatorname{cth}\left(\frac{\pi n}{\varkappa}\right)-\frac{\varkappa}{\pi n}\right]\frac{\pi\varkappa}{2n}n^{1/2}\exp\left(\frac{\tilde{\mu}}{T}n\right). \tag{17}$$

Here we have introduced the notation $\varkappa=2\pi m^{1/2}TL_x/\hbar(2u_0)^{1/2}$.

For small $\tilde{\mu}/T$ it is the large n that are important in the sum over n, and we can sum the series by replacing it with an integration. As a result (17) can be written

$$\frac{\pi^{1/2}}{2\varkappa^{1/2}}\int_0^{\infty}\left(\operatorname{cth}x-\frac{1}{x}\right)\exp\left(\frac{\tilde{\mu}\varkappa x}{\pi T}\right)\frac{dx}{x^{1/2}}. \tag{17a}$$

In what follows we will verify that as $\varkappa \to \infty$ in the neighborhood of T_0 and below the quantity $\delta \equiv |\tilde{\mu}|\varkappa/\pi T \to 0$. Therefore, the integral in (17a) can be replaced by $\int_0^{\infty}\exp(-\delta x)(dx/x^{1/2})=(\pi/\delta)^{1/2}$.

Returning to (16), we are led to the following equation for the chemical potential

$$N=\frac{S}{\pi}\frac{mT^{3/2}}{\hbar^2u_0^{1/2}}\left\{\int_0^{\infty}\frac{x^{1/2}dx}{\exp(x-\tilde{\mu}/T)-1}+\frac{\pi}{4}\frac{\hbar(2u_0)^{1/2}}{TL_xm^{1/2}}\left(\frac{T}{|\tilde{\mu}|}\right)^{1/2}\right\}. \tag{18}$$

Some uncomplicated analysis of this equation shows that

$$\left(\frac{|\tilde{\mu}|}{T}\right)^{1/2}=\begin{cases}[1-(T_0/T)^{3/2}]\zeta(3/2)/2\pi^{1/2}, & T\gtrsim T_0\\ \left(\frac{\pi u_0}{2m}\right)^{1/2}\frac{-\hbar}{TL_x\zeta(3/2)[1-(T_0/T)^{3/2}]}, & T\lesssim T_0\end{cases}. \tag{19}$$

The interval of temperatures over which a transition takes place from one value of the chemical potential to another is of order $[(u_0/m)^{1/2}\hbar T_0/L_x]^{1/2}$. As L_x increases, the width of this temperature interval shrinks, and as $L_x\to\infty$ a phase transition occurs. The phase transition temperature T_0 coincides with the value obtained in the quasiclassical approximation.

For the case where the potential of the external field is a quadratic function of coordinates [i.e., has the form (3)], it can be shown that the transition to a state with a Bose condensate is a third-order phase transition, i.e., at the temperature T_0 the specific heat of the two-dimensional ideal Bose gas has a kink. We will omit the details of the calculation, since both the order of the phase transition and the character of the order that appears in the system change when interactions between the bosons are included. However, the transition temperature T_0 is stable with respect to weak interactions between the particles of the gas.

2. SUPERFLUID DENSITY OF A WEAKLY NONIDEAL BOSE GAS

Let us turn to an investigation of a weakly nonideal Bose gas. A weakly nonideal Bose gas placed in a nonuniform external field (a potential well) will accumulate in regions where the potential satisfies $u(\mathbf{r})<T$; as the temperature decreases, these regions occupy a smaller and smaller volume. This being the case, the density of the gas will naturally increase, reaching its maximum value at the bottom of the potential well. If the potential $u(\mathbf{r})$ varies over macroscopic distances, then at each point in space we may assume that the system is quasiuniform, and use the known results from the theory of a uniform Bose gas for its description. As we have already noted in the introduction, for the uniform case the transition to the superfluid state in a two-dimensional Bose system takes place at a temperature $T_{\rm BKT}$; this temperature is connected with the superfluid density ρ_s by the relation $T_{\rm BKT}=(\pi\hbar^2/2m)\rho_s(T_{\rm BKT})$. In a nonuniform external field the superfluid phase appears primarily at the bottom of the potential well as the temperature decreases, and the temperature of the superfluid transition [when the potential $u(\mathbf{r})$ is slowly varying] is determined from the relation $T_c=(\pi\hbar^2/2m)\rho_s(0,T_C)$, where $\rho_s(0,T_c)$ is the superfluid density at the bottom of the well.

It should be emphasized that although the superfluid density ρ_s will initially be nonzero only at the bottom of the potential well, the corresponding phase transition temperature is determined by the properties of the entire system. The reason for this is that both the total density $\rho(\mathbf{r})$ and the superfluid density $\rho_s(\mathbf{r})$ [and, in particular, $\rho_s(0)$] are functions of the chemical potential. The latter, however, is found from the normalization condition, which integrates contributions from all regions of the system. As we will show below, for the case of a weakly nonideal Bose gas the primary contribution to the normalization condition for $T\sim T_c$ comes from regions in which the interaction of the bosons can be completely neglected. This is why the temperatures T_c and T_0 differ only by a small correction (which measures the extent to which the Bose gas is nonideal). Thus, for a weakly interacting Bose gas in a nonuniform field the superfluid transition, like the three-dimensional case, turns out to be tied to the statistical properties of an ideal Bose gas.

Turning now to calculations, we must include the fact that the bare interaction of two bosons in the two-dimensional case is strongly renormalized by many-body effects. As a result, if a is the characteristic radius over which the repulsive potential acts, the effective interaction of two bosons in a spatially uniform medium takes the form (see Refs. 16 and 17)

$$u(\mathbf{r}_1-\mathbf{r}_2)=\frac{4\pi\hbar^2}{m}\ln^{-1}\left(\frac{1}{na^2}\right)\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{20}$$

In what follows we will denote the coefficient in front of the δ function by γ for brevity, i.e.,

$$\gamma=\frac{4\pi\hbar^2}{m}\ln^{-1}\left(\frac{1}{na^2}\right). \tag{21}$$

Because the kinetic energy of a "typical" boson is of order $\hbar^2n/m$, while the interaction energy of a single boson is of order γn, the kinetic energy is large compared to the interaction energy when the condition $\ln(1/na^2)\gg1$ holds. Thus, a dilute two-dimensional Bose gas is weakly nonideal in the same sense as a dilute three-dimensional Bose gas. However, in the two-dimensional case it is considerably more difficult to make the gas dilute. This is because the condition $na^3\ll1$, which is sufficient for diluteness in the three-dimensional gas, is replaced by the considerably more restrictive condition $\ln^{-1}(1/na^2)\ll1$.

After renormalization of the interaction the Hamiltonian of a weakly nonideal Bose gas can be written in the form

$$\hat{H}=\int d\mathbf{r}\left[\frac{\hbar^2}{2m}\nabla\hat{\psi}^+\nabla\hat{\psi}+u(\mathbf{r})\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})\right]+\frac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'\,\hat{\psi}^+(\mathbf{r})\hat{\psi}^+(\mathbf{r}')\gamma\delta(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}). \tag{22}$$

Despite the weakness of the interactions between bosons, the potential energy cannot be neglected, i.e., treated within perturbation theory, because for small momenta the kinetic energy is smaller than the potential energy. The most systematic way to solve this problem is to divide the operator $\hat{\psi}(\mathbf{r})$ into two parts, corresponding to small and large boson momenta respectively. In this case it is convenient to reformulate the part corresponding to small momenta in terms of new "density-phase" variables. For weakly interacting bosons it is possible to choose a momentum p_0 that separates the "fast" and "slow" parts of $\hat{\psi}(\mathbf{r})$ in such a way that we can use ordinary perturbation theory to describe the fast particles (i.e., the potential energy is small compared to the kinetic energy) while at the same time treating the density fluctuations of the slow particles as small (for details see Refs. 17 and 18).

We can, however, use an approach that is less rigorous but considerably simpler to obtain the same results.

Let us turn to the interaction representation. Then for the operator

$$\hat{\psi}(\mathbf{r}, \tau) \equiv \exp[(\hat{H}-\mu\hat{N})\tau]\hat{\psi}(\mathbf{r}) \exp[-(\hat{H}-\mu\hat{N})\tau] \quad (23)$$

we can use (22) to obtain an equation of motion

$$-\frac{\partial\hat{\psi}}{\partial\tau} = -\frac{\hbar^2}{2m}\nabla^2\hat{\psi} + [u(\mathbf{r})-\mu]\hat{\psi} + \gamma\hat{\psi}^+\hat{\psi}\hat{\psi}. \quad (24)$$

If the oscillations of the boson density are small, it is convenient to introduce the "density-phase" variables:

$$\hat{\psi}(\mathbf{r}) = \exp[i\hat{\varphi}(\mathbf{r})][\rho(\mathbf{r})+\delta\hat{\rho}(\mathbf{r})]^{1/2}. \quad (25)$$

Here the phase operator $\hat{\varphi}(\mathbf{r})$ and the operator correction $\delta\hat{\rho}(\mathbf{r})$ satisfy the commutation relations

$$\delta\hat{\rho}(\mathbf{r})\hat{\varphi}(\mathbf{r}') - \hat{\varphi}(\mathbf{r}')\delta\hat{\rho}(\mathbf{r}) = i\delta(\mathbf{r}-\mathbf{r}'), \quad (26)$$

where $\rho(\mathbf{r})$ is the average boson density at the point $\mathbf{r}$:

$$\rho(\mathbf{r}) = \mathrm{Sp}\,\{\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r}) \exp(-\beta\hat{H})\}/\mathrm{Sp}\,\{\exp(-\beta\hat{H})\}$$
$$\equiv \langle\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle. \quad (27)$$

The representation (25) is useful only under conditions where the operator correction $\delta\hat{\rho}(\mathbf{r})$ is small compared to $\rho(\mathbf{r})$. More precisely, the following inequality must hold

$$\rho^2(\mathbf{r}) \gg \langle[\delta\hat{\rho}]^2\rangle. \quad (28)$$

It can be verified that the expression on the right side of this inequality diverges due to the zero-point oscillations at large wave vectors. The need to separate $\hat{\psi}(\mathbf{r})$ into "fast" and "slow" parts mentioned above is associated with just this problem. A systematic approach eliminates these divergences, and the contribution from the zero-point oscillations becomes finite. We assume that condition (28) is fulfilled and expand the density operator in terms of the small fluctuation correction $\delta\hat{\rho}(\mathbf{r})$.

By substituting (25) into Eq. (24), we easily obtain equations of motion for the phase $\hat{\varphi}(\mathbf{r})$ and the correction operator $\delta\hat{\rho}(\mathbf{r})$. If we retain only terms linear in $\delta\hat{\rho}$ and $\hat{\varphi}$ in these equations, we can find the spectrum of elementary excitations. It has the Bogolyubov form

$$E(k) = \{\varepsilon(k)[\varepsilon(k)+2\gamma\rho]\}^{1/2}, \quad \varepsilon(k) = \hbar^2k^2/2m. \quad (29)$$

Note that we are talking about spatially uniform systems, or a quasi-uniform description of spatially nonuniform systems, in which, e.g., the density $\rho(\mathbf{r})$ is treated as a constant in which the $\mathbf{r}$ dependence is purely parametric.

We then average both parts of the equation of motion for the phase $\hat{\varphi}$ [i.e., we multiply this equation by $\exp(-\beta\hat{H})$ and take the trace], which leads to a relation between the boson density $\rho(\mathbf{r})$ and the applied field $u(\mathbf{r})$:

$$\rho(\mathbf{r}) = \frac{m}{4\pi\hbar^2}\ln\left(\frac{1}{\rho(\mathbf{r})a^2}\right)[\mu-u(\mathbf{r})] - \int\frac{d^2k}{(2\pi)^2}\frac{\varepsilon(k)}{E(k)}n(E_k). \quad (30)$$

Here $n(E)$ is the Bose-Einstein distribution function. In obtaining (30) we have retained terms no higher than second order in $\delta\hat{\rho}(\mathbf{r})$ in the corresponding equation of motion, which is correct for small fluctuations of the boson density. Furthermore, we have omitted the contribution to (30) from zero-point oscillations (which diverge within our method of calculations). A more precise calculation (compare with Ref. 17) shows that the contribution from zero-point oscillations is smaller than that from the terms omitted in (30).

Let us now note that inequality (28), which was a presupposition in deriving (30), is equivalent to the condition $\rho \gg \rho_n$, where ρ_n is the normal density of a Bose gas. However, Eq. (30) also remains valid when the inequality $\rho \gg \rho_n$ is violated if in the expression for the energy of an elementary excitation we replace $\gamma\rho$ by $\gamma\rho_s$. As was shown in Ref. 19, this replacement can be justified for temperatures T that satisfy the inequality $T \ll \hbar^2\rho_s/m$. In what follows we will be interested in the high-temperature region ($T \gg \gamma\rho_s$), which is compatible with this inequality by virtue of the weakly nonideal nature of the Bose gas ($\ln(1/\rho a^2) \gg 1$).

Evaluating the integral over momentum in (30) and replacing $\gamma\rho$ by $\gamma\rho_s$ in the result, in the high-temperature region we obtain

$$\rho(\mathbf{r}) = \frac{m[\mu-u(\mathbf{r})]}{4\pi\hbar^2}\ln\frac{1}{\rho(\mathbf{r})a^2} - \frac{mT}{2\pi\hbar^2}\ln\frac{T}{2\gamma\rho_s(\mathbf{r})}. \quad (31)$$

This equation determines the total density $\rho(\mathbf{r})$ as a function of temperature, external field, and chemical potential, an expression for which will be obtained below. It is also easy to find the normal density $\rho_n(\mathbf{r})$.

The normal density is given by the well-known Landau expression

$$\rho_n = -\frac{1}{2m}\int\frac{d^2p}{(2\pi\hbar)^2}p^2\frac{\partial n(E_p)}{\partial E_p}. \quad (32)$$

Because ρ_n in (32) is understood to be the particle density and not the mass density, the mass m appears in the denominator of the factor in front of the integral. In the high-temperature region, for all momenta p that satisfy the condition $p^2/2m > \gamma\rho_s$ we may assume that $E(p) \approx p^2/2m$. Because it is just these momenta that give the primary contribution to the integral, we have

$$\rho_n \approx -\frac{m}{2\pi\hbar^2}\int_{\gamma\rho_s}^{\infty} dE\, E\frac{\partial n}{\partial E} = \frac{mT}{2\pi\hbar^2}\ln\frac{T}{\gamma\rho_s(\mathbf{r})}. \quad (33)$$

The difference $\rho - \rho_n$ is, according to Landau, the superfluid density in the system. We will denote this density by ρ_s^0, thereby differentiating it from the total superfluid density ρ_s, which also includes a contribution (negative) from thermally excited vortices as well as from rotons and phonons. In contrast to the density ρ_s, which undergoes a discontinuity at the temperature of the superfluid transition from some finite value to zero, the density ρ_s^0 is a smooth function of temperature and does not equal zero even above T_c. The density ρ_s^0 will reduce to zero at a certain temperature $T_c^0 > T_c$. Although no phase transition takes place in the system at the temperature T_c^0, this temperature is nevertheless a special one.

The point is that the unbound thermally-excited vortices that exist in the system for $T > T_c$ and give rise to dissipation of a steady-state current retain a certain degree of correlation in their motion. In particular, if the distance between vortices and antivortices is smaller than a certain coherence length $\xi_+(T)$ (see, e.g., Ref. 20), a vortex and an antivortex behave like a bound pair. This fact, and also the fact that the density ρ_s^0 is nonzero for $T > T_c$, makes possible the propagation of density oscillations of the superfluid component in the film at these temperatures, i.e., third sound.[21] Furthermore, the propagation of third sound is assisted by pinning of vortices at defects and at the rough substrate. Thus, at the temperature T_c^0 new collective effects appear which are completely accessible to measurement.

It is obvious that as the temperature decreases the value of ρ_s^0 first becomes nonzero at the bottom of the potential well $u(\mathbf{r})$. Therefore, the equation we use to find T_c^0 takes the form $\rho_s^0(0,T_c^0) = 0$, which is equivalent to the equation $\rho_n(0,T_c^0) = \rho(0,T_c^0)$.

If we use Eqs. (31) and (33) to find ρ and ρ_n respectively, then at any point where the equation $\rho_n(0,T_c^0) = \rho(0,T_c^0)$ holds we encounter terms that diverge logarithmically, because at these points $\rho_s = 0$. The problem here is that in reality neither Eq. (31) nor Eq. (33) can be used in a region where $\rho_s = 0$. As we mentioned above, our approach to the description of a weakly interacting Bose gas is correct for $\hbar^2\rho_s/m \gg T$. Since it is the logarithm of ρ_s that enters into (31) and (33), we can obtain a result that is correct to first approximation by replacing the zero value of ρ_s by $mT/\hbar^2$ at the point of where the potential $u(\mathbf{r})$ has a minimum. Then from the equation $\rho_n(0,T_c^0) = \rho(0,T_c^0)$ it follows that

$$(\mu+u_0)\ln\frac{1}{\rho(0)a^2} = 4T_c{}^0\ln\frac{\hbar^2}{\gamma m}. \tag{34}$$

Relation (34) specifies $\mu + u_0$ as a function of the temperature T_c^0. On the other hand, the chemical potential can be found in the usual way from the requirement that the total number of bosons in the system equal the specified value N. Since there is still no superfluid phase in this system at $T = T_c^0$, it is necessary to find the relation between $\mu + u_0$ and T in the normal phase. This second relation between μ and T, combined with (34), determines the temperature T_c^0 we are looking for.

A major difference between the superfluid and normal phases is clearly the fact that $\rho_s^0 \neq 0$ in the superfluid. However, they also differ in the small-momentum form of their excitation spectra: for the superfluid phase the spectrum is phonon-like, i.e., $E = cp$, where c is the velocity of sound, whereas for normal phase we have $E = p^2/2m$ for small p. A proper description of the normal phase must take this fact into account, which implies that different forms of perturbation theory must be used above and below T_c^0. The form which describes the superfluid phase was illustrated at the beginning of this section, while for $T > T_c^0$ the interaction between bosons can be described within the self-consistent field approximation. In this case the self-consistent interaction potential of the bosons $\Lambda(r)$ satisfies the following equation (compare with Ref. 17) ($\varepsilon_a \equiv \hbar^2/2ma^2$):

$$\Lambda(\mathbf{r}) = \mu - u(\mathbf{r}) - \left[\pi m \ln\frac{\varepsilon_a}{|\Lambda(\mathbf{r})|}\right]^{-1} \times \int \frac{d^2p}{\exp\{\beta[p^2/2m - \Lambda(\mathbf{r})]\} - 1}. \tag{35}$$

The integral over momentum in (35) gives the boson density at the point $\mathbf{r}$; therefore, the last term in (35) describes the interaction energy of the bosons with the mean field. The chemical potential μ is found from the normalization condition

$$N = \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi\hbar)^2}\left\{\exp\left[\beta\left(\frac{p^2}{2m} - \Lambda(\mathbf{r})\right)\right] - 1\right\}^{-1}. \tag{36}$$

A solution to Eq. (35) cannot be obtained for arbitrary values of μ and T, so that in general it is impossible to carry out the integration in expression (36). However, if we assume that the inequality $\ln\ln(\beta_0\varepsilon_a) \gg 1$ holds (unfortunately, this "large" parameter of the theory cannot be made large under real conditions), it is possible to find an exact expression for the chemical potential in the range of temperatures where $\beta[u(\mathbf{r}) - \mu] \gg \ln\ln(\beta\varepsilon_a)/\ln(\beta\varepsilon_a)$. When this latter inequality holds, the solution of (35) is (also assuming that $1 \gg \beta[u(\mathbf{r}) - \mu]$):

$$|\Lambda(\mathbf{r})| = u(\mathbf{r}) - \mu + \frac{4T\ln\{\beta[u(\mathbf{r}) - \mu]\}}{\ln(\beta\varepsilon_a)}. \tag{37}$$

By substituting this expression into the equation for the chemical potential, we find that when the applied external field is a power-law function of the coordinate x with arbitrary exponent α the answer depends in a significant way on the magnitude of α. For $\alpha \lesssim 1$ the relation between temperature and chemical potential which follows from (36) and (37) has the form

$$\frac{\pi\hbar^2 n_0}{mT}\left(\frac{u_0}{T}\right)^{1/\alpha} = \Gamma\left(1+\frac{1}{\alpha}\right)\left\{\zeta\left(1+\frac{1}{\alpha}\right) - \frac{|\mu|}{T}\zeta\left(\frac{1}{\alpha}\right) - \frac{4}{\ln(\beta\varepsilon_a)}\left[\alpha\sum_{m=1}^{\infty}\frac{1}{m}\zeta\left(\frac{1}{\alpha}, m+1\right) + \zeta\left(1+\frac{1}{\alpha}\right) - \zeta\left(\frac{1}{\alpha}\right)\right]\right\}. \tag{38a}$$

If, however, $\alpha \gtrsim 1$ holds, then from (36) and (37) we have

$$\frac{\pi\hbar^2 n_0}{mT}\left(\frac{u_0}{T}\right)^{1/\alpha} = \Gamma\left(1+\frac{1}{\alpha}\right)\left[\zeta\left(1+\frac{1}{\alpha}\right) - \left[\frac{|\mu|}{T} + \frac{4\ln(T/|\mu|)}{\ln(\beta\varepsilon_a)}\right]^{1/\alpha}\alpha\Gamma\left(1-\frac{1}{\alpha}\right)\right]. \tag{38b}$$

In (38a) and (38b) we have assumed that $|1-\alpha| \gg \ln^{-1}(T/|\mu|)$.

Although, strictly speaking, Eqs. (38a) and (38b) are correct only when $|\mu|/T \gg \ln\ln(\beta\varepsilon_a)/\ln(\beta\varepsilon_a)$, they also give answers that are order-of-magnitude correct for $|\mu|/T \sim \ln\ln(\beta_0\varepsilon_a)/\ln(\beta_0\varepsilon_a)$ as well, i.e., for values of μ/T at which a nonzero superfluid density appears in the system, as follows from (34). Therefore, by replacing μ in (38a), (38b) with its value from (34), we obtain the temperature T_c^0 at which the density ρ_s^0 becomes nonzero at the bottom of the potential well. (More precisely, we obtain an order-of-magnitude estimate of the difference $T_0 - T_c^0$.) From (38a) and (38b), and also from the definition of the Bose condensation temperature T_0 [see (8)] we find that the difference $T_0 - T_c^0$ is small whenever the quantity $\ln^{-1}(1/\bar{n}a^2)$ (i.e., the degree of diluteness of the Bose gas) is small. It is important to note, however, that for $\alpha \lesssim 1$ the correction contains the small parameter to the first power, while for $\alpha \gtrsim 1$ it is proportional to the αth root of the small parameter. This circumstance has important implications for the experimental measurement of how close T_c^0 is to T_0 in a dilute two-dimensional gas. Because under experimental conditions it is difficult to produce values of $\ln(1/\bar{n}a^2)$ that exceed a hundred, the difference between T_0 and T_c^0 can be sizable $[(T_0 - T_c^0)/T_0 \sim 1/3]$ even for the case of a quadratic potential (i.e., for $\alpha = 2$).

We now note that the condition $\ln(1/\bar{n}a^2) \gg 1$ for weak nonideality of the Bose gas can be rewritten using (12) in the form

$$\ln\left[\frac{\varepsilon_a}{u_0}\left(\frac{1}{n_0a^2}\right)^\alpha\right] \gg \alpha+1. \tag{39}$$

This expression contains the depth of the potential well u_0 and the boson density n_0 in the absence of an external field. These quantities are parameters that can be varied independently in the experiments in order to fulfill the inequality (39).

As we have noted, no phase transition takes place in the system at the temperature T_c^0, because vortices created by thermal fluctuations cause a steady-state current to decay even for $\rho_s^0 \neq 0$. The transition temperature to the superfluid state (with undamped stationary current) is found from the relation $T_c = (\pi\hbar^2/2m)\rho_s(0,T_c)$ in the quasi-uniform approximation. However, at the phase transition point the renormalized superfluid density due to vortices ρ_s is the same in order of magnitude as the unrenormalized density ρ_s^0. Therefore the value of T_c can be found from the relation $T_c = C[\hbar^2\rho_s^0(0,T_c)/m]$, where C is a numerical factor of order unity. Using this relation and (38), in which T is replaced by T_c, it is not difficult to verify that $T_c = T_0 - O[1/\ln(\beta_0\varepsilon_a)]$ for $\alpha \leqslant 1$. Thus, for the two-dimensional dilute Bose gas in a nonuniform external field the superfluid transition temperature T_c practically coincides with the Bose condensation temperature T_0 of the ideal gas.

3. VORTICES IN A NONUNIFORM FIELD

In the previous section we completely neglected the possibility that quantized vortices could appear in our formulation of the thermodynamics of a weakly nonideal Bose gas. However, it is well-known that thermally excited vortices play an important role in two-dimensional systems at sufficiently high temperatures. These vortices determine the details of the phase transition that takes a two-dimensional Bose system from its normal state to the superfluid state in the spatially uniform case. As we mentioned in the introduction, this phase transition, the so-called Berezinskii-Kosterlitz-Thouless (BKT) transition, consists of the dissociation of vortex pairs at the temperature T_{BKT}. Below T_{BKT} all vortices with opposite circulation are bound in pairs, while above T_{BKT} there are unbound vortices in the system which also lead to dissipation of the superfluid flow.

In this section we will clarify how a nonuniform field changes the law of interaction between vortices and how this is related to the BKT transition. The goal of the calculations that follow is to find the energy of an isolated vortex and the energy of a system of vortices in the presence of a nonuniform external field.

The velocity field $\mathbf{v}_s(\mathbf{r})$ created by vortices whose centers are located at points $\mathbf{r}_i$ satisfies the equation

$$\mathrm{rot}_z\,\mathbf{v}_s = \frac{\hbar}{m}2\pi\sum_i n_i\delta(\mathbf{r}-\mathbf{r}_i). \tag{40}$$

Here n_i can take on the values ± 1.

Equation (40) determines the transverse part of the velocity $\mathbf{v}_s$. Its longitudinal part can be found from the equation of continuity, which in the general case is nonlinear in $\mathbf{v}_s$ because the superfluid density $\rho_s^0(\mathbf{r})$ depends on $\mathbf{v}_s$ (more precisely, on v_s^2). However, the dependence of ρ_s^0 on $\mathbf{v}_s$ is significant only in the neighborhood of the vortex core, whose size is on the order of the coherence length ξ. The length ξ is microscopically small, and in our calculations we will assume that the vortex core is pointlike. In fact, the assumption of vortex cores with zero size is already implied by the presence of δ functions on the right-handed side of Eq. (40). In what follows we will neglect the dependence of ρ_s^0 on $\mathbf{v}_s$.

From the equation of continuity written in the form

$$\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}=0, \tag{41}$$

it is obvious that if we introduce a stream function $\psi(\mathbf{r})$ such that

$$\begin{aligned} j_x &= \rho_s^0(\mathbf{r})\,v_{sx}(\mathbf{r}) = \partial\psi/\partial y, \\ j_y &= \rho_s^0(\mathbf{r})\,v_{sy}(\mathbf{r}) = -\partial\psi/\partial x, \end{aligned} \tag{42}$$

the equation of continuity will be satisfied identically. The equation for $\psi(\mathbf{r})$ may be obtained by substituting into (40) the expressions for the components of the velocity $\mathbf{v}_s$ from (42). As a result we find

$$\frac{1}{\rho_s^{02}}\nabla\rho_s^0\nabla\psi - \frac{1}{\rho_s^0}\nabla^2\psi = \frac{\hbar}{m}2\pi\sum_i n_i\delta(\mathbf{r}-\mathbf{r}_i). \tag{43}$$

Let us show that we can choose the condition $\psi(\mathbf{r})|_{\text{boundary}} = 0$ as a boundary condition for Eq. (43).

Actually, the definition of ψ and relation (42) imply that the relation $\psi(\mathbf{r}) = \text{const}$ defines a flux line, i.e., a line the direction of whose tangent coincides with the direction of the current at any point. However, if the current is not to intersect the boundary (i.e., there is no component to the current normal to the boundary), the boundary curve must coincide with one of the flux lines. From this it follows that $\psi(\mathbf{r}) = \text{const}$ on the boundary. Since the current is the derivative of ψ, without loss of generality we can set this constant equal to zero.

The solution to Eq. (43) is conveniently written in the form

$$\psi(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')Q(\mathbf{r}')d\mathbf{r}', \tag{44}$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function of Eq. (43), i.e., the solution to the equation

$$\frac{1}{\rho_s^{02}}\nabla\rho_s^0\nabla G(\mathbf{r},\mathbf{r}') - \frac{1}{\rho_s^0}\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{45}$$

where $Q(\mathbf{r})$ is the expression standing on the right side of (43).

If the Green's function $G(\mathbf{r},\mathbf{r}')$ is known, then it is easy to find the energy of a single vortex or a system of vortices. The energy of the vortex is a sum of the energy of the vortex core E_0 and the kinetic energy of the superfluid flow connected with the vortex, i.e.,

$$E=\sum_{i=1}^{n}E_0+\int\frac{m\rho_s{}^0v_s{}^2}{2}d\mathbf{r}. \tag{46}$$

We recall that ρ_s^0 is the superfluid particle density and not the mass density; therefore, the mass of a boson stands in front of ρ_s^0.

Replacing $\mathbf{v}_s$ in (46) by its value from (42), we obtain an expression for the kinetic energy [we have omitted the mass m in (47)]

$$\int\frac{1}{2\rho_s{}^0}(\nabla\psi)^2\,d\mathbf{r}=-\int\psi\,\mathrm{div}\left(\frac{1}{2\rho_s{}^0}\nabla\psi\right)d\mathbf{r}+\int\psi\frac{1}{2\rho_s{}^0}\nabla\psi\,d\mathbf{l}. \tag{47}$$

The second term in (47), which involves integration along the boundary, reduces to zero because $\psi=0$ on the boundary. Taking into account that $\psi(\mathbf{r})$ satisfies Eq. (43), and also the representation (44) for $\psi(\mathbf{r})$ in terms of the Green's function G, we can write the first term in (47) in the form

$$\frac{m}{2}\int Q(\mathbf{r})G(\mathbf{r},\mathbf{r}')Q(\mathbf{r}')\,d\mathbf{r}\,d\mathbf{r}'. \tag{48}$$

By substituting the explicit expression for $Q(\mathbf{r})$ from (44) into this expression, we finally obtain an expression for the energy of a system of vortices:

$$E=\sum_{i=1}^{n}E_0+\frac{m}{2}\sum_{i,j}\left(\frac{2\pi\hbar}{m}\right)^2 n_iG(\mathbf{r}_i,\mathbf{r}_j)n_j. \tag{49}$$

The problem is to find the Green's function $G(\mathbf{r},\mathbf{r}')$. Of course, the solution to Eq. (45) [i.e., $G(\mathbf{r},\mathbf{r}')$] cannot be found for arbitrary dependence of $\rho_s^0(\mathbf{r})$ on $\mathbf{r}$. However, if ρ_s^0 is a slowly varying function of the coordinate $\mathbf{r}$, the Green's function can be obtained in a very general form.

Let us assume first of all that $\rho_s^0=\text{const}$. Then it is not difficult to show that for a film occupying the strip $d\geqslant x\geqslant 0$ and $+\infty>y>-\infty$ the Green's function that satisfies the boundary condition $G|_{\text{boundary}}=0$ has the form

$$-\frac{\rho_s{}^0}{4\pi}\ln\left[\frac{\mathrm{ch}[\pi(y-y')/d]-\cos[\pi(x+x')/d]}{\mathrm{ch}[\pi(y-y')/d]-\cos[\pi(x-x')/d]}\right]. \tag{50}$$

Now let ρ_s^0 be a slowly varying function of x. Let us write the Green's function $G(\mathbf{r},\mathbf{r}')$ in the form of a sum

$$G(\mathbf{r},\mathbf{r}')=G_0(\mathbf{r},\mathbf{r}')+G_1(\mathbf{r},\mathbf{r}'). \tag{51}$$

Here G_0 is the primary part of the Green's function, while G_1 is a small correction which reduces to zero as the characteristic length L_ρ over which $\rho_s(\mathbf{r})$ varies goes to infinity. Including the result (50) and the fact that $G_0(\mathbf{r},\mathbf{r}')$ must be a symmetric function of its arguments, we may choose for $G_0(\mathbf{r},\mathbf{r}')$ the expression

$$G_0(\mathbf{r},\mathbf{r}')=-\frac{[\rho_s{}^0(\mathbf{r})\rho_s{}^0(\mathbf{r}')]^{1/2}}{4\pi}\times\ln\left[\frac{\mathrm{ch}[\pi(y-y')/d]-\cos[\pi(x+x')/d]}{\mathrm{ch}[\pi(y-y')/d]-\cos[\pi(x-x')/d]}\right]. \tag{52}$$

By direct substitution of $G_0(\mathbf{r},\mathbf{r}')$ into (45) it is easy to verify that (52) satisfies Eq. (45) to first order in the gradient $\nabla\rho_s^0$. The function G_1 is proportional to terms of second order in the gradient $\nabla\rho_s^0$ and satisfies the equation

$$\nabla^2G_1-\frac{1}{\rho_s{}^0}\nabla\rho_s{}^0\nabla G_1=-\frac{1}{2\rho_s{}^0}\left[\frac{3}{2}\frac{(\nabla\rho_s{}^0)^2}{\rho_s{}^0}-\nabla^2\rho_s{}^0\right]G_0(\mathbf{r},\mathbf{r}'). \tag{53}$$

For $\rho_s^0=x^{-2}$ the expression in the square brackets of (53) reduces to zero and $G_1\equiv 0$. For $\rho_s^0=x^{-2}$ the function (52) is an exact solution to Eq. (45). In the general case the relation between G_1 and G_0 is determined by the squared ratio of the length over which G_0 varies to the length over which ρ_s^0 varies. For $|\mathbf{r}-\mathbf{r}'|\ll d$ the characteristic length over which G_0 varies is exactly $|\mathbf{r}-\mathbf{r}'|$. Therefore, when $L_\rho\gtrsim d\gg|\mathbf{r}-\mathbf{r}'|$, the correction G_1 is actually small, and in place of the function G in Eq. (49) for the energy we may substitute G_0 from (52).

Rather than carrying out this substitution, we turn our attention to the fact that for $\mathbf{r}'\to\mathbf{r}$ (i.e., for $x'\to x$ and $y'\to y$ simultaneously) the function G_0 diverges logarithmically, which is similar to what happens in the spatially uniform case. However, it is noteworthy that regions with dimensions on the order of the coherence length ξ must be excluded from consideration for the reasons described at the beginning of this section. This implies that the points $\mathbf{r}$ and $\mathbf{r}+\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is an arbitrary vector whose length is on the order of the coherence length, are indistinguishable to this degree of accuracy. We will therefore replace the Green's function $G(\mathbf{r}_i,\mathbf{r}_i)$ entering into (49) with coinciding arguments by $G(\mathbf{r}_i,\mathbf{r}_i+\boldsymbol{\xi})$.

Carrying out this replacement, and keeping in mind that $G(\mathbf{r},\mathbf{r}')\approx G_0(\mathbf{r},\mathbf{r}')$, we find for the energy of the vortex system

$$E=\sum_{i=1}^{n}\left\{E_0+\frac{\pi\hbar^2\rho_s{}^0(\mathbf{r}_i)}{m}\ln\left(\frac{d}{\xi}\frac{2}{\pi}\sin\frac{\pi x_i}{d}\right)\right\}+\frac{\pi\hbar^2}{m}\sum_{\langle i,j\rangle}n_in_j[\rho_s{}^0(\mathbf{r}_i)\rho_s{}^0(\mathbf{r}_j)]^{1/2}\times\ln\left\{\frac{\mathrm{ch}[\pi(y_i-y_j)/d]-\cos[\pi(x_i+x_j)/d]}{\mathrm{ch}[\pi(y_i-y_j)/d]-\cos[\pi(x_i-x_j)/d]}\right\}. \tag{54}$$

This expression for the vortex energy in a nonuniform field in which the superfluid density ρ_s^0 is a slowly varying function of coordinates is the primary result of this section. Using this expression, it is easy to obtain the energy of a single vortex and a vortex pair with opposite circulations.

The expression for the energy of a single vortex can be immediately simplified if we take into account that d is a macroscopic length, while the quantity ξ is microscopically small and, consequently, $d/\xi\gg 1$. Therefore, the energy of a vortex will be finite either for $x/d\ll 1$ or for $(d-x)/d\ll 1$, i.e., in both cases the sine in the argument of the logarithm is small. Thus, the energy of a single vortex is

$$E_1=E_0+\frac{\pi\hbar^2\rho_s{}^0(\mathbf{r})}{m}\begin{cases}\ln(x/\xi), & x\ll d\\ \ln[(d-x)/\xi], & d-x\ll d\end{cases}. \tag{55}$$

For a pair of vortices with opposite circulation ($n_2=-n_1$) the expression for the energy depends considerably on the quantity $|y_1 - y_2|$. For $|y_1 - y_2| \gg d$ the interaction between them is exponentially small, i.e., $\propto \exp(-\pi|y_1 - y_2|/d)$, so that they reduce to two isolated vortices whose interaction can be neglected. The energy of this pair equals the sum of the energies (55) for each of the vortices.

For $|y_1 - y_2| \ll d$ and $x_1, x_2 \sim d/2$ the energy of a vortex pair equals

$$E_2=2E_0+\frac{\pi\hbar^2}{m}\left\{[(\rho_s{}^0(\mathbf{r}_1))^{1/2}-(\rho_s{}^0(\mathbf{r}_2))^{1/2}]^2\ln\frac{d}{\xi}\right.$$
$$\left.+2[\rho_s{}^0(\mathbf{r}_1)\rho_s{}^0(\mathbf{r}_2)]^{1/2}\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{\xi}\right\}. \tag{56}$$

We see that in the spatially nonuniform gas the character of the dependence of the interaction energy on the distance between vortices can differ considerably from this dependence in a spatially uniform gas. Let us pause to discuss the reasons for this disagreement in more detail.

In the case where ρ_s^0 does not depend on the coordinates the energy of a pair of vortices with opposite circulations is proportional to $\ln[|\mathbf{r}_1 - \mathbf{r}_2|/\xi]$ [see (56)]. This interaction law is a consequence of the slow decay of the velocity field of a vortex with increasing distance from its core ($v_s \propto 1/|\mathbf{r} - \mathbf{r}_i|$). This leads to a logarithmic dependence of the energy of a single vortex on the size of the container. For a pair of vortices with opposite circulations, due to partial cancellation of the velocities $\mathbf{v}_s$ from the different vortices, the total velocity field (at distances large compared to the size of the vortex pair) falls off more rapidly than r^{-1}. As a result the logarithmic dependence of the energy on the size of the container is replaced by a logarithmic dependence on the size of the vortex pair. This situation changes when ρ_s^0 begins to depend on the position.

If ρ_s^0 is a function of position, this cancellation of the velocity $\mathbf{v}_s$ from vortices with opposite circulation does not occur, which is easy to see from expression (42), (44), and (52). As a consequence of this, the energy of a vortex pair, like the energy of a single vortex, depends logarithmically on the container size. The smoother the spatial dependence, the smaller is the lack of cancellation of velocities $\mathbf{v}_s$ and the smaller the energy of the vortex pair. Therefore, it is natural for the coefficient in front of the logarithm in the first term of (56) to depend on the difference in the densities ρ_s^0 at points where the vortices are located. Because the energy should not depend on the sign of this difference (since positive and negative vortices are completely equivalent), it is also natural for the difference of the densities to appear raised to an even power. Finally, in view of the slowness of the variation of ρ_s^0 this power should be as small as possible, i.e., it should equal two.

Let us now discuss the question of what the BKT transition should look like in a system of vortices whose energy is given by the expression (54). For this we introduce a convention regarding the superfluid density ρ_s renormalized by vortices. In order to define ρ_s we write the total superfluid current $\mathbf{j}_s(\mathbf{r})$ in the form of the sum of the two terms [compare with (20)]:

$$\mathbf{j}_s(\mathbf{r})=\mathbf{j}_{\|}(\mathbf{r})+\mathbf{j}_v(\mathbf{r}), \tag{57}$$

where the current $\mathbf{j}_{\|}$ has zero circulation and is given by the expression

$$\mathbf{j}_{\|}(\mathbf{r})=\rho_s{}^0(\mathbf{r})\frac{\hbar}{m}\nabla\varphi=\rho_s{}^0(\mathbf{r})\mathbf{v}_s, \tag{58}$$

while $\mathbf{j}_v$ has zero divergence and is the contribution to $\mathbf{j}_s$ from the vortices. From (42) and (44) it follows that

$$\mathbf{j}_v(\mathbf{r})=[\nabla\times\mathbf{z}]\sum_i G(\mathbf{r}-\mathbf{r}_i)\frac{2\pi\hbar}{m}n_i. \tag{59}$$

The superfluid density ρ_s determines the magnitude of the total current flowing in the system at a given velocity $\mathbf{v}_s$, and is found from the relation

$$S\rho_s\mathbf{v}_s=\int d\mathbf{r}(\mathbf{j}_{\|}+\langle\mathbf{j}_v\rangle)$$
$$=S\rho_s{}^0\mathbf{v}_s+\int d\mathbf{r}\sum_i[\nabla\times\mathbf{z}]G(\mathbf{r}-\mathbf{r}_i)\frac{2\pi\hbar}{m}\langle n_i\rangle. \tag{60}$$

Here S is the area of the system, and the average $\langle n_i\rangle$ equals

$$\langle n_i\rangle=\sum_{\{n_k\}}n_i\exp[-\beta E_v(\{n_k\})]\Big/\sum_{\{n_k\}}\exp[-\beta E_v(\{n_k\})], \tag{61}$$

where $E_v(\{n_k\})$, the energy of the vortices, is a sum of two terms: the energy E from (54) and a correction E' related to flow along the flux system

$$E'=-m\mathbf{v}_s\int\mathbf{j}_v\,d\mathbf{r}. \tag{62}$$

Including the correction E' by perturbation theory and using the fact that for $\mathbf{v}_s = 0$ the average $\langle n_i\rangle = 0$, we arrive after a number of calculations at the following expression for the density:

$$\frac{\rho_s}{\rho_s{}^0}=1+\frac{\hbar^2\rho_s{}^0}{mT}\frac{(2\pi)^2}{S}\sum_{i,k}\langle n_in_k\rangle_0(\mathbf{r}_i\mathbf{r}_k). \tag{63}$$

Here $\langle\ldots\rangle_0$ denotes an average in which we set $\mathbf{v}_s = 0$. In obtaining (63) we have omitted terms that contain the derivative $\partial\rho_s^0(x)/\partial x$ in view of the slowness of the coordinate variation of $\rho_s^0(x)$.

In order to proceed further it is necessary to find $\langle n_in_k\rangle_0$. This is possible for the case of a dilute gas of vortices, i.e., in the situation where the temperature T is small compared to the energy of the vortex core E_0—more precisely, for $\exp(-E_0/T) \ll 1$. In this case it is sufficient to include only the interaction of vortices at points i and k in calculating $\langle n_in_k\rangle_0$, while neglecting their interaction with the remaining vortices. Taking into account that vortices at the points i and k have opposite circulations by virtue of the condition of "electrical neutrality" $\Sigma n_i = 0$, and using expression (56) for the energy of this vortex pair, we obtain from (63)

$$\frac{\rho_s}{\rho_s{}^0}=1-\frac{\hbar^2\rho_s{}^0}{mT}2\pi^2\exp(-2\beta E_0)\sum_{\mathbf{r}}\left(\frac{r}{\xi}\right)^2$$
$$\times\exp\left[-\frac{2\pi\hbar^2\rho_s{}^0}{mT}\left(\ln\frac{r}{\xi}+\frac{1}{2}\frac{x^2}{d^2}\ln\frac{d}{\xi}\right)\right]. \tag{64}$$

In writing this expression we have replaced the difference $[(\rho_s^0(x_1))^{1/2} - \rho_s^0(x_2))^{1/2}]^2$ entering into (56) by $\rho_s^0[(x_1 - x_2)/d]^2$. This replacement is valid because ρ_s^0 varies over distances on the order of d, while the important differences $x_1 - x_2$ are those that are small compared to x_1 and x_2 (for vortices whose centers lie in the central portion of the film).

Rather than carrying out the summation over **r** in (64), let us turn our attention to the following fact. Although we previously assumed that the superfluid density ρ_s^0 varied over distances d on the order of the width of the film, we can verify that expression (56) for the energy of a vortex pair [and together with it expression (64)] is also correct in situations where d is much smaller than the width of the film, provided that d is large compared to the average distance between vortices. Therefore, in the sum over **r** we can assume that d is large $(d \gg \xi)$ but still finite.

Let us convert the sum in (64) to an integral $(\Sigma_{111} \to \int d\mathbf{r}/\xi^2{}_{111})$, and take into account that in a self-consistent calculation we should replace ρ_s^0 by ρ_s in the exponent. If $d/\xi \to \infty$, then the equation we obtain for ρ_s has a solution for $\pi\hbar^2\rho_s/2mT \gg 1$. The BKT phase transition point corresponds to the equal sign in this condition, i.e.,

$$T_{\mathrm{BKT}} = \frac{\pi}{2}\frac{\hbar^2\rho_s(T_{\mathrm{BKT}})}{m}. \tag{65}$$

If d/ξ is a finite quantity, while d is large compared to the average distance between vortices, then the equation for ρ_s also has a solution for $T > T_{\mathrm{BKT}}$. Assuming that this equation is satisfied, after integrating over r in (64) (from ξ to ∞) we are led to the equation

$$\frac{\rho_s}{\rho_s{}^0} = 1 - \frac{4\pi^2\hbar^2\rho_s{}^0}{mT}\exp(-2\beta E_0)\int_0^{\pi/2} d\theta \times\left(\frac{mT}{\pi\hbar^2\rho_s}\frac{d^2}{\xi^2}\ln^{-1}\left(\frac{d}{\xi}\right)\frac{1}{\cos^2\theta}\right)^{\nu}\Gamma(\nu). \tag{66}$$

Here $\Gamma(\nu)$ is the gamma function, while

$$\nu = 2(1 - T_{\mathrm{BKT}}/T). \tag{67}$$

It is obvious that the integral over θ converges at the upper limit under the condition $\nu < 1/2$, which corresponds to the following limitation on the temperature: $T \leqslant \frac{4}{3}T_{\mathrm{BKT}}$. The value $T_c = \frac{4}{3}T_{\mathrm{BKT}}$ is the upper limit of temperatures for which it is meaningful to assign a local superfluid density ρ_s to a spatially nonuniform Bose gas. This temperature is reached in the limit $\exp(-2\beta E_0) \to 0$. For finite values of $\exp(-2\beta E_0)$ the limiting temperature for which ρ_s exists is smaller than $\frac{4}{3}T_{\mathrm{BKT}}$, and is determined by the characteristic scale d over which the nonuniform external field varies. However, for any value of d the density ρ_s changes discontinuously at the transition point from some finite value to zero. This assertion follows from the fact that on the left side of Eq. (66) ρ_s appears in the numerator, while on the right side it appears in the denominator.

Thus, we have shown that for the case where the nonuniform external field varies over distances on the order of the size of the system the relation between the superfluid transition temperature and the density ρ_S at the transition point is given by the Kosterlitz-Thouless relation (65), and in this case the transition temperature must in practice be found from the equation $T_c = \pi\hbar^2\rho_s(0,T_c)/2m$, as was done in the previous section. If, however, the field varies on scales that are small compared to the size of the system but large compared to the distance between vortices, then the relation between T_c and $\rho_S(T_c)$ is more complicated and can be obtained from (66). To sum up we are led to an expression of Kosterlitz-Thouless type, in which a numerical coefficient of order unity will appear in place of the factor $\pi/2$ there. Therefore, in both cases the temperature of the superfluid transition must be found from an equation of the form $T_c = C\hbar^2\rho_s(0,T_c)/m$, where $C \sim 1$. This implies (see the previous section) that $T_c \simeq T_0 - O(1/\ln(\beta_0\varepsilon_a))$.

CONCLUSION

In Sec. 2 we used the quasi-uniform approximation to describe the behavior of a weakly nonideal Bose gas in a nonuniform external field. The accuracy of this description is sufficient for us to calculate the local characteristics of the gas (e.g., the density at a point **r**). If, however, we return to the question of long-range order in this system, i.e., to the calculation of correlators of type $\langle\hat{A}(\mathbf{r})\hat{B}(0)\rangle$ as $r \to \infty$, then we must go beyond the framework of the quasi-uniform approximation. In Ref. 22 an approach was developed that led to an exact description of the system in a nonuniform field. Within the framework of this approach it is possible to solve the problem exactly for the case where the applied field is a linear function of coordinates. The spectrum and eigenfunctions of this system were found, as well as the correlator $\langle\hat{\psi}^+(x,y)\hat{\psi}(x,0)\rangle$, and it was established that a nonuniform field does not lead to the appearance of a Bose condensate or long-range order in a weakly nonideal gas.

With regard to the absence of a condensate in a nonideal gas, what is astonishing is the fact that the superfluid transition temperature T_c for such a gas in a nonuniform field practically coincides with the temperature T_0 for Bose condensation of an ideal gas. The reason for this involves a combination of two phenomena. For temperatures close to T_0 a compression of the Bose gas occurs at the bottom of the potential well. In an ideal gas this compression occurs as a phase transition and is accompanied by Bose condensation. In a weakly nonideal gas the interactions prevent unlimited compression and Bose condensation; however, the density of the gas at the bottom of the well increases significantly over a narrow region of temperatures. Although there is no phase transition at this stage, there is one in the next stage as the compressed fraction of gas becomes superfluid via the BKT mechanism. One can say that the statistics and high diluteness of the gas are responsible for the closeness of T_c and T_0.

The importance of both factors is apparent from a comparison of the behavior of a weakly ideal Bose gas in a potential well with the behavior of a Fermi gas in the same well. Within the framework of the self-consistent field approximation it is not difficult to show that in the neighborhood of the temperature T_0 when the inequalities $1 \gg \beta_0[u(x) - \mu] \gg 1/\ln(\beta_0\varepsilon_a)$ hold the density of a Bose gas equals

$$\rho(x) = \frac{mT}{2\pi\hbar^2}\ln\frac{T}{u(x) - \mu}, \tag{68}$$

while for $1 \gg \beta_0[\mu - u(x)] \gg 1/\ln(\beta_0\varepsilon_a)$ it equals

$$\rho(x) = \frac{m}{4\pi\hbar^2}[\mu - u(x)]\ln(\beta_0\varepsilon_a) = \frac{\mu - u(x)}{\gamma}. \tag{69}$$

Thus, for a dilute Bose gas in the neighborhood of T_0 the density of the gas varies considerably over a narrow interval of temperatures. In this case, below the temperature for Bose condensation of an ideal gas the density remains finite only by virtue of the repulsive force ($\rho \to \infty$ for $\gamma \to 0$).

For the case of a Fermi gas, the Pauli principle ensures a finite value of the density for any temperatures, even in the absence of an interaction. For a field $u(\mathbf{r})$ that varies over macroscopic distances the density of a two-dimensional ideal Fermi gas equals

$$\rho(\mathbf{r}) = \frac{mT}{2\pi\hbar^2} \ln\{1+\exp[\beta(\mu-u(\mathbf{r}))]\}. \tag{70}$$

The expression on the right side of (70) varies smoothly in the neighborhood of the degeneracy temperature of the Fermi gas in an external field. Near the potential minimum $u(\mathbf{r})$, at temperatures that are low compared to the degeneracy temperature (i.e., for $\beta[\mu - u(\mathbf{r})] \gg 1$), the density of fermions equals

$$\rho(\mathbf{r}) = \frac{m}{2\pi\hbar^2}[\mu-u(\mathbf{r})]. \tag{71}$$

Here, by setting $\mathbf{r}$ equal to zero we find the maximum value of the density of an ideal Fermi gas that is achievable by the compressive action of the nonuniform field. For the case where $u(\mathbf{r})$ has the form (3), by making the change of variable x to $p_z L_x/(2mu_0)^{1/2}$ we are led to an effective three-dimensional Fermi gas whose chemical potential at $T=0$ is easy to find. As a result, we have $\rho(0) = (m/2\pi\hbar^2)[(3\pi\hbar^2/2m)n_0 u_0^{1/2}]^{2/3}$.

Using expression (69), which is also valid for $T=0$, and substituting in the value of the chemical potential μ, which at $T=0$ equals the quantity $(\frac{3}{4}\gamma n_0)^{2/3} u_0{}^{1/3}$, we find that for a quadratic potential $u(\mathbf{r})$ the boson density is $\rho(0) = (\frac{3}{4}n_0)^{2/3}[(mu_0/2\pi\hbar^2)\ln(1/a^2\rho(0)]^{1/3}$. To within a factor of order unity, the ratio of the boson to the fermion density equals

$$\left[\ln\frac{4\pi\varepsilon_a}{u_0}\left(\frac{1}{a^2 n_0}\right)^2\right]^{1/3}. \tag{72}$$

Under the conditions of the experiments this quantity is of order unity, and consequently the density of bosons differs only weakly from the density of fermions (with the same mass). The difference in the densities will increase as the exponent α in the power law decreases, where α determines the smoothness of variation of the field $u(x)$ with coordinates. Thus, we have verified that the degree of diluteness of the gas $\ln^{-1}(1/a^2\bar{n})$ and the exponent α are the two fundamental parameters that determine the properties of a Bose gas in a nonuniform field.

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