Interaction of solitons through phonons in the φ^4–φ^2 model

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The indirect interaction of solitons through phonons is analyzed in the one-dimensional φ^4–φ^2 model. The interaction of planar domain walls in a d-dimensional model is also analyzed. It is shown analytically and numerically that the indirect interaction is repulsive, strengthens with increasing temperature, and is of considerably longer range than the static interaction of solitons. The interaction of solitons in the case of a pinning of solitons is also studied.

1. INTRODUCTION

The primary reason for the unflagging interest in classical soliton solutions is that they are nonperturbative and are characterized by a certain topological index[1] which ensures their stability. These properties of a soliton invite one to associate a corresponding extended particle with it. If the solitons are sufficiently far apart, they can be treated as a system of widely separated particles, and one can ask about the potential energy of their interaction. This question bears directly on planar domain walls. A two-soliton solution is generally unstable, so some additional forces are introduced in order to hold the solitons in place and make the question meaningful.[1]

In the case of the one-dimensional (1D) model

$$H=-(m^2/2)\varphi^2+(\lambda/4)\varphi^4+{}^1/_2(d\varphi/dx)^2 \quad (1)$$

the single-soliton solution is

$$\varphi_1(x)=\varphi_0\,\mathrm{th}\,(mx/2^{1/2}),\quad \varphi_0^2=m^2/\lambda, \quad (2)$$

and we write the two-soliton solution in the form[2]

$$\varphi_2(x)=\varphi_0\,\mathrm{th}\,[m(x-R)/2^{1/2}]\,\mathrm{th}\,[m(x+R)/2^{1/2}], \quad (3)$$

where $2R$ is the distance between the soliton and the antisoliton. The static interaction potential is then[1,2]

$$V(R)=-2^{5/2}(m^3/\lambda)\exp(-2^{1/2}mR)=-6E_0\exp(-2^{1/2}mR), \quad (4)$$

where

$$E_0=(2^{3/2}/3)\,(m^3/\lambda) \quad (5)$$

is the energy of the soliton.

In this paper we consider a renormalization of soliton-soliton interaction (4) to allow for thermally excited phonons. In other words, we find the potential of the indirect interaction of solitons through harmonic excitations. Phonons are not scattered by solitons,[4] in contrast with, say, impurities, but when a phonon passes through a soliton it undergoes a phase shift, as we will see below. Correspondingly, when a phonon passes through two solitons the phase shift becomes a function of the distance between the solitons. This subtle effect means that phonons introduce some additional spatial correlations between the solitons. The corresponding renormalization of the interaction of two domain walls through phonons occurs in multidimensional systems, but the explicit expression for the potential of the indirect interaction of the walls through the phonons is determined by the dimensionality of the domain wall, as we will see below. Specifically, the interaction weakens with increasing dimensionality.

To hold the solitons in place, we impose a slight potential excitation on the field energy. This spatially localized excitation pins the solitons. Alternatively, instead of introducing an external pinning potential one could make use of the circumstance that the time over which the soliton and antisoliton approach each other is exponentially long, $\propto \exp(2\sqrt{2^{3/2}}mR)$, if the distance between solitons is much greater than their width. This process is then adiabatically slow compared with the processes in which the phonon propagates from one soliton to the other. As we will see below, the two approaches lead to the identical result. The potential of the indirect interaction of solitons depends on the distance between the solitons in a way quite different from that of a static potential. In addition, at nonzero temperature the attraction gives way to repulsion of the two solitons.

2. STATISTICAL FIELD FUNCTIONAL IN THE CASE OF TWO SOLITONS

The statistical functional of field theory is

$$Z=\int D\varphi(x)\exp(-\beta H). \quad (6)$$

The idea underlying the approximate treatment of (6) in the case of a single soliton is to first write[2,5]

$$\varphi(x)=\varphi_1(x)+\eta(x), \quad (7)$$

where $\varphi_1(x)$ is the single-soliton solution from (2), and to then find the statistical functional in (6) in the harmonic (i.e., Gaussian) approximation:

$$Z=\int Dx_0\exp(-\beta E_0)\int D\eta(x)\exp\left\{-\beta\int dx\left[\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2+\frac{m^2}{2}V_1(x)\eta^2(x)\right]\right\}. \quad (8)$$

The one-soliton potential is

$$m^2V_1(x)=\frac{3}{2}\lambda\varphi_1^2(x)-\frac{m^2}{2}=\frac{m^2}{2}[3\,\mathrm{th}^2(mx/2^{1/2})-1]. \quad (9)$$

The integral over $\eta(x)$ in (8) can be evaluated easily if we know the eigenvalues of the Hamiltonian

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+m^2V_1(x). \quad (10)$$

In the case (9), the problem can be solved exactly.[1,6] There are two discrete energy levels,

$$\omega_0^2=0,\quad \eta_0(z)=\mathrm{ch}^{-2}z,$$
$$\omega_1^2={}^3/_2m^2,\quad \eta_1(z)=\mathrm{sh}z\cdot\mathrm{ch}^{-2}z, \quad (11)$$

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where $z = mx/2^{1/2}$ is a dimensionless constant. There is also a continuum of levels,

$$\omega_k^2=m^2(2+{}^1/_2k^2), \quad \eta_k(z)=e^{ikz}[3\text{th}^2z-1-k^2-3ik\text{th}z]. \quad (12)$$

The energy level $\omega_0^2 = 0$ here describes a free displacement of the soliton, while the other energy levels are vibrational modes. Phonons correspond to the solutions (12). Substituting (11) and (12) into (8), we find[3,5]

$$Z=\frac{C}{\beta\omega_1^2}\exp(-\beta E_0)\prod_k(1/\beta\omega_k^2), \quad (13)$$

where C is the contribution from the translational mode with $\omega^2 = 0$.

We turn now to the two-soliton solution (2). The equation for the eigenfunctions and eigenvalues becomes

$$\left[-\frac{1}{2}\frac{d^2}{dz^2}+V_2(z,a)\right]\psi_n(z,a)=\lambda_n(a)\psi_n(z,a), \quad (14)$$

where V_2 is the two-soliton potential

$$V_2(z,a)=3\text{th}^2(z+a)\,\text{th}^2(z-a)-1,$$
$$a=mR/2^{1/2}, \quad \lambda_n=\omega_n^2/m^2. \quad (15)$$

For $a = 1$, the distance between solitons is equal to the size of one soliton, as can be seen from (1). If we wish to ignore the distortion of one soliton caused by the other, we must set $a \gg 1$.

By analogy with (8)–(13), we construct a statistical functional for two solitons:[2]

$$Z=\int da\int dx\exp[-2\beta E_0-\beta V(a)]\prod_n(1/\beta\lambda_n m^2)$$
$$=JC\exp(-2\beta E_0)\int da\exp[-\beta V_{eff}(a)], \quad (16)$$

where

$$V_{eff}(a)=V(a)+\tilde{V}(a)=V(a)+kT\sum_n\ln(\beta\lambda_n m^2). \quad (17)$$

It can be seen from this expression that the potential of the indirect interaction of solitons through phonons, $\tilde{V}(a)$, arises as the phonon vibrational modes become thermally populated.

There is no difficulty in generalizing (17) to the d-dimensional problem, in which the solitons are $(d-1)$-dimensional planar domain walls oriented perpendicular to the x axis. The potential of the indirect interaction per unit area of a domain wall is

$$\tilde{V}(a)=\frac{kT}{(2\pi)^{d-1}}\sum_n\int d^{d-1}\mathbf{k}_\perp\ln\left\{\beta\bar{m}^2\left[\lambda_n(a)+\frac{1}{2}k_\perp^2\right]\right\}, \quad (18)$$

where $\mathbf{k}_\perp$ is a wave vector which lies in the plane of the domain wall.

3. ENERGY LEVELS OF A TWO-SOLITON POTENTIAL

Since problem (14) is identical to the Schrödinger equation for a particle in a double potential well, it is clear that the two discrete energy levels (11), which characterize the one-soliton potential, split in two, because of tunneling between the potential wells (Fig. 1). The continuous spectrum of energy levels in (12) undergoes no changes, because of the spatial localization of potential (15) (Ref. 7). Since we do not know the exact solutions of Eq. (14), we will use some approximate methods. For the discrete energy levels in the case $a \gg 1$ we can use the familiar WKB result for a symmetric double potential well:[6]

$$\lambda_{1,2}=\mp m^2\eta_0(a)\eta_0'(a),$$
$$\lambda_{3,4}=m^2[{}^3/_2\pm\eta_1(a)\eta_1'(a)].$$

Substituting the functions η_0 and η_1 from (11), we find

$$\lambda_{1,2}=\mp m^2\text{th}a/\text{ch}^4a\approx\mp16e^{-4a},\ \lambda_{3,4}$$
$$=m^2[{}^3/_2\mp\text{th}a(1-2\text{th}^2a)\text{ch}^{-2}a]$$
$$\approx m^2[{}^3/_2\mp4e^{-2a}\pm48e^{-4a}]. \quad (19)$$

We recall that the static interaction potential of solitons in terms of the dimensionless unit a [expression (15)] is

$$V(a)=-6E_0\exp(-4a). \quad (20)$$

It can be seen from (19) that the first eigenvalue λ_1, which determines the square of the frequency of the fundamental mode, is negative. The reason is an instability of the given soliton-antisoliton pair with respect to the attractive potential (20). This is a purely relaxational mode, and the corresponding annihilation time is $\tau = |\lambda_1|^{-1/2} = \frac{1}{4}\exp(2a)$. In the approximation $a \gg 1$ this turns out to be a very long time. Accordingly, as we mentioned in the Introduction, we can treat the process in which the solitons approach each other as being adiabatically slow, and we can independently examine the dynamic processes of the propa-

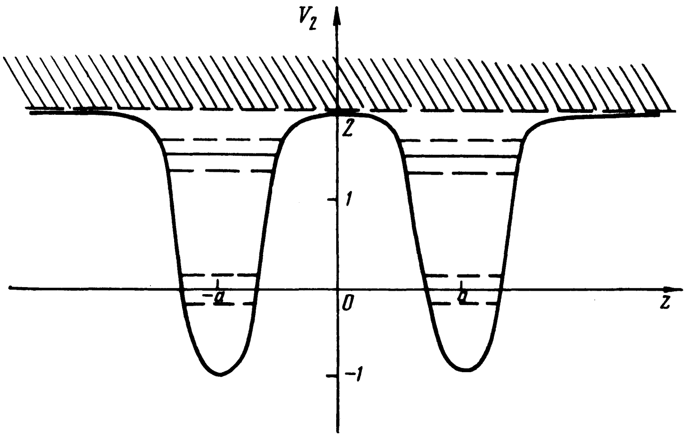

FIG. 1. Splitting of the discrete energy levels of the one-soliton problem (solid lines) in a two-soliton potential (dashed lines). The hatched region corresponds to the continuous spectrum of phonon excitations.

gation of phonons from one soliton to another, with time scales $|\lambda_n|^{-1/2} \sim 1$. Their contribution is represented in (17) by the other energy levels. A different approach, which is taken in Sec. 5 of this paper, is to add a potential perturbation to the Hamiltonian of the two-soliton problem to cancel the negative energy of the ground state, $\lambda_1 < 0$.

4. INDIRECT INTERACTION OF SOLITONS

In the semiclassical approximation the next energy level turns out to be positive. Actually, the translational invariance of the system of two solitons as a whole means that the exact value of λ_2 must be zero. The accuracy of the WKB approach for the lower pair of energy levels is therefore $\pm\exp(-4a)$. The second pair of energy levels in (19) must accordingly be written with the same accuracy:

$$\lambda_{3,4} \approx m^2[{}^3/_2 \pm 4\exp(-4a)].$$

Substituting this result into (17), we find that the contribution of the discrete energy levels is zero in the WKB approximation. If we substitute (19) into (17) and make use of the above comments regarding the first pair of energy levels, we find the following contribution from the discrete energy levels:

$$2kT\ln(3\beta m^2/2) - (64/9)kT\exp(-4a). \quad (21)$$

Let us analyze the continuum contribution to $V_{\rm eff}$. As we mentioned in the Introduction, the dispersion relation between the phonon frequencies and the wave vector determined by (12) remains the same in the two-soliton problem. It would thus appear at first glance that this contribution vanishes. However, when we sum over k in (17) we need to know the permissible values of k, which are determined by the periodic boundary conditions on the eigenfunctions $\eta_k(z)$. As was shown in detail by Rajaraman,[1] this procedure leads us to the formula

$$\sum_k \ln(\beta\lambda_k) = \frac{1}{2\pi}\int_{-\infty} dk\left[\frac{mh}{2^{1/2}} + \frac{\partial}{\partial k}\delta_2(k,a)\right]\ln\left\{\beta m^2\left(\frac{1}{2}k^2+2\right)\right\}, \quad (22)$$

where $\delta_2(k,a)$ is the phase shift of the phonon after it passes through the two-soliton potential from left to right. This shift evidently depends on the distance between solitons. This shift is known exactly for one-soliton potential (9).

In the two-soliton case we assume that the phase shift is caused exclusively by the soliton on the left if the phonon lies to the left of the origin; correspondingly, for $x > 0$, the phase shift of the phonon is caused exclusively by the soliton on the right. As we will see below in a comparison with the results of a numerical analysis at low temperatures, these assumptions give a good description of the soliton interaction problem. Since the phase-shift problem is equivalent to the problem of a one-dimensional scattering of a free particle (phonon) by a soliton potential,[7] the phonon wave function can be written

$$\psi_k(z) = \begin{cases} \exp(ikz) + R\exp(-ikz), & z \ll -a, \\ S_1\exp(ikz) + R_1\exp(-ikz), & z \approx 0, \\ S\exp(ikz), & z \gg a. \end{cases} \quad (23)$$

The transmission coefficient S determines the phase shift in which we are interested, $\delta_2(k,a)$.

We now make use of the known fact[4,8] that the reflection coefficient R of a one-soliton potential is zero. That this is true can be seen easily from the solutions (12) of the one-soliton problem. We can then replace (23) by

$$\psi_k(z,a) = \begin{cases} \exp(ikz), & z \ll -a, \\ \exp[ikz + i\delta_1(k,a)], & z \approx 0, \\ \exp[ikz + 2i\delta_1(k,a)], & z \gg a, \end{cases}$$

where, according to (12),

$$\delta_1(k,a) = -\operatorname{arctg}\left[\frac{3k\,\mathrm{th}\,a}{3\,\mathrm{th}^2 a - 1 - k^2}\right] - \operatorname{arctg}\left[\frac{3k}{2-k^2}\right]. \quad (24)$$

The expression $\delta_2(k,a) = 2\delta_1(k,a)$ determines the resultant phase shift of the phonon after it passes through the two-soliton potential well.

Substituting (24) into (22), we find the contribution of the continuum to the potential of the indirect interaction:

$$kT\exp(-2a)\frac{12}{\pi}\int_0^\infty \frac{k^2-1}{(k^2+1)^2}\ln\left[\beta m^2\left(\frac{1}{2}k^2+2\right)\right]dk. \quad (25)$$

We have omitted terms which do not depend on the distance a.

Finally substituting (25) into (22), then substituting into (17), and recalling that in the approximation $\exp(-2a)$ the discrete energy levels of localized vibrational excitations $\lambda_{1,2,3,4}$ do not contribute to $\tilde{V}(a)$, we find the following expression for the interaction potential of two solitons at a nonzero temperature:

$$V_{eff}(a) = -6E_0e^{-4a} + \frac{12}{\pi}kTe^{-2a}[\alpha + \gamma\ln(\beta E_0)], \quad (26)$$

where

$$\alpha = \int_0^\infty \frac{k^2-1}{(k^2+1)^2}\ln(2+k^2/2)\,dk = 0{,}985,$$

$$\gamma = \int_0^\infty \frac{k^2-1}{(k^2+1)^2}\,dk = -0{,}006.$$

Figure 2 shows plots of the effective interaction potential $V_{\rm eff}(a)$ and of the indirect interaction potential $\tilde{V}(a)$. These results will be discussed and compared with the numerical results in Sec. 6.

5. SOLITON PINNING

As we mentioned in the Introduction, in order to eliminate the problem of the instability of the ground state of two solitons—an instability which is manifested by a negative value of the eigenvalue λ_1 for a translational mode—we need to hold the solitons in place. In other words, we need to break the continuous translational symmetry of the Hamiltonian. We can do this by (for example) using a lattice version of the field.[9] Alternatively, we could add to (1) a perturbation

$$h(x)\varphi^2(x), \quad (27)$$

which holds the soliton at the point at which $h(x)$ is at a maximum. Using (27), we can write the Euler-Lagrange equation in the form

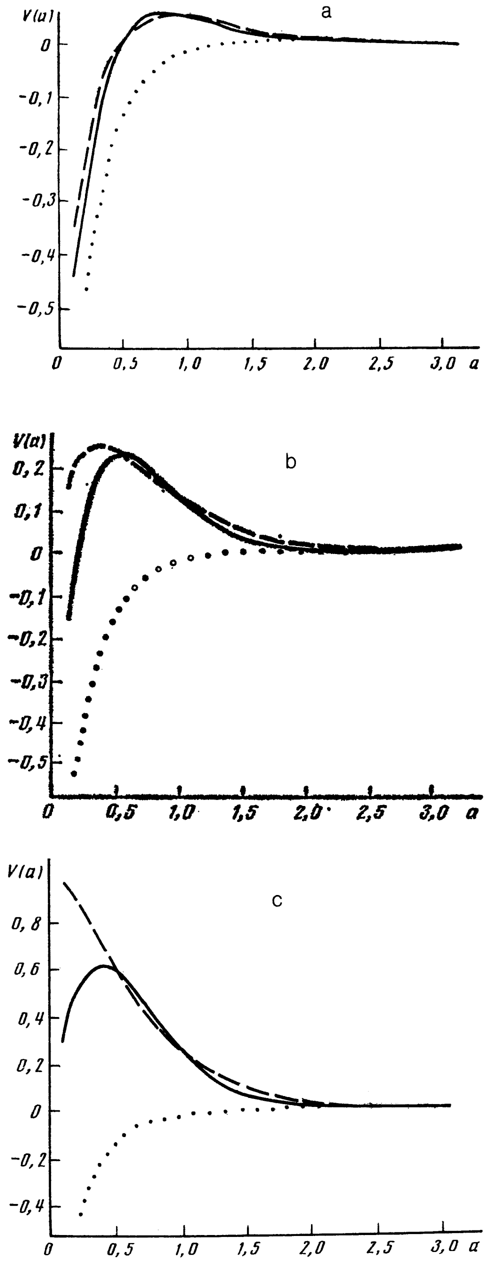

FIG. 2. Potential of the indirect interaction between solitons through phonons as a function of the distance a and of the temperature $t = E_0/kT$, where E_0 is the soliton energy. Solid line—Numerical results [expression (38)]; dashed line—analytic results [expression (26)]; dotted line—static potential [expression (20)].

$$d^2\varphi/dx^2-\lambda\varphi^3+m^2\varphi-h(x)\varphi=0. \tag{28}$$

Making use of the circumstance that $h(x)$ is small, we write the one-soliton solution (28) in the form

$$\tilde\varphi_1(x)=\varphi_1(x)+\vartheta(x), \tag{29}$$

where φ_1 is the soliton solution (2) without pinning, and ϑ is the correction for (27). To first order in the perturbation (27), the equation for ϑ becomes

$$-d^2\vartheta/dx^2-m^2[1-3\,\text{th}^2(mx/2^{1/2})]\vartheta=-h(x)\varphi_1(x), \tag{30}$$

The solution of this equation can be written in the form

$$\vartheta(x)=\sum_k\frac{\eta_k(x)}{\lambda_k(x)}a_k, \tag{31}$$

where

$$a_k=-\int_{-\infty}^{\infty}dx\eta_k(x)h(x)\varphi_1(x),$$

and $\eta_k(x)$ are the eigenvalues of one-soliton problem (11), (12).

We adopt the pinning potential

$$h(x)=\alpha\exp(-|x|/x_0),\quad x_0\ll a. \tag{32}$$

Using (11) and (12), we find that the solution (31) becomes

$$\vartheta(z)=-(4\alpha m^2x_0/3\lambda^{1/2})\,\text{sh}\,z/\text{ch}^2 z. \tag{33}$$

With pinning taken into account in first-order perturbation theory, the energy of the ground state is

$$\tilde E_1=\int dx\eta_0^2(x)[3\tilde\varphi_1^2(x)-1]\approx E_1-6\int dx\eta_0^2(x)\varphi_1(x)\vartheta(x). \tag{34}$$

Substituting (33) into this expression, we find

$$\tilde E_1=E_1+\alpha\pi m x_0^3/2^{1/2}.$$

Assuming $\tilde E_1\geqslant 0$, we find a condition on the parameters of the potential (32) (primarily on the parameter α) which pins the soliton at the point $x=0$.

Correspondingly, we can write the pinning potential for two solitons at the points $\pm a$:

$$h(x)=\alpha[\exp(-|x-a|/x_0)+\exp(-|x+a|/x_0)]. \tag{35}$$

In the case of the one-soliton potential, for which we know both the eigenstates $\eta_k(x)$ and the spectrum of eigenvalues λ_k, we can calculate the correction $\vartheta(x)$ in (33). In the two-soliton case, in contrast, this correction is unknown. Accordingly, in the numerical simulation we found the value for the parameter α which caused the ground-state energy to vanish for each value of a.

6. NUMERICAL RESULTS

For the computer analysis we adopted a one-dimensional chain of N particles,

$$H=a_0\sum_f\left[-\frac{m}{2}\varphi_f^2+\frac{\lambda}{2}\varphi_f^4+\frac{1}{2a_0^2}(\varphi_f-\varphi_{f+1})^2\right] \tag{36}$$

with the boundary conditions $\varphi_0=\varphi_{N+1}=0$. To simulate the continuum problem, we assigned the lattice constant a_0 values of 0.01 and 0.02. Correspondingly, the number of particles was assumed to be 1000 and 2000. The width of the soliton was estimated to be 100 and 200 particles in these terms. The eigenvalue equation (14) for the chain (36) becomes

$$[2-2a_0^2+6a_0^2\text{th}^2(a_0f-a)\text{th}^2(a_0f+a)-2a_0^2\lambda_n]\psi_f=\psi_{f+1}+\psi_{f-1}. \tag{37}$$

For each value of a we found N eigenvalues λ_n, which we then substituted into (17) after subtracting the first eigenvalue, $\lambda_1<0$. We carried out a numerical summation over all the eigenvalues. As a result we found the indirect-interaction potential $\tilde V(a)$ which is plotted in Fig. 2. Analytically, we can approximate $\tilde V(a)$ by

$$\beta\tilde V(a)=(2{,}00\pm0{,}01)\exp[-(0{,}33\pm0{,}01)(2a)^2]. \tag{38}$$

A change in the number of particles from $N=1000$ to

TABLE I.

n	$a=1$	$a=2$	n	$a=1$	$a=2$
1	−0,0003870	−0,0000089	11	0,0024990	0,0024005
2	0,0004952	0,0000037	12	0,0027163	0,0026157
3	0,0013425	0,0011118	13	0,0029544	0,0028527
4	0,0016259	0,0012740	14	0,0032142	0,0031110
5	0,0016666	0,0016255	15	0,0034935	0,0033901
6	0,0017485	0,0016820	16	0,0037945	0,0036898
7	0,0018494	0,0017705	17	0,0041146	0,0040101
8	0,0019778	0,0018924	18	0,0044561	0,0043506
9	0,0021295	0,0020370	19	0,0048167	0,0047114
10	0,0023028	0,0022072	20	0,0051982	0,0050922

$N=2000$ or a change in the lattice constant from $a_0=0.02$ to $a_0=0.01$ does not cause any changes in (38) which go beyond the indicated error in the approximation of $\widetilde{V}(a)$.

We thus see that the phonons have a fundamental effect on the soliton interaction potential. This effect occurs at temperatures above absolute zero. In the first place, the interaction through phonons is repulsive, so at low temperatures we find both a region in which solitons attract (by virtue of the static interaction potential) and a region of repulsion. Breakup of the soliton–antisoliton pair requires the surmounting of an energy barrier. Second, the indirect interaction $\widetilde{V}(a)$ is of much longer range than the static potential (20). The latter has a cutoff radius of 1/4, while that of the interaction (26) is 1/2. The numerical result in (38) yields $3^{1/2}/2\approx 0.87$.

Moreover, as can be seen from Fig. 2, there is no significant difference between the results of the numerical and analytic approaches at low temperatures. The difference does becomes noticeable at higher temperatures. Accordingly, although the original assumption that the contributions of the solitons to the resultant phonon phase shift $\delta_2(k,a)$ are independent is not exact, it does cover the basic features of the soliton-soliton interaction through phonons.

In the numerical calculations we also found the potentials of the indirect interaction of planar domain walls in the 2D and 3D cases, working from (18). Switching from sums over the transverse wave numbers in (18) to integrals, we find

$$d=2:\ \widetilde{V}(a)=\frac{kT}{2\pi}\sum_n\{(\pi^2/2+\lambda_n)[\ln(\lambda_n+\pi^2/2)-1]-\lambda_n\ln(\lambda_n+\pi^2/2)\},\qquad(39)$$

$$d=3:\ \widetilde{V}(a)=kT\sum_n\left[\frac{1}{2}\ln(\lambda_n+\pi^2/2)+2^{1/2}\lambda_n\,\mathrm{arctg}\,(\pi/2^{1/2}\lambda_n)\right].\qquad(40)$$

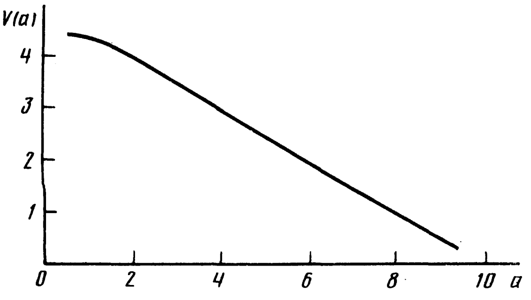

FIG. 3. Interaction potential of the solitons with pinning for $t=0.25$.

We recall that λ_n is the dimensionless eigenvalue spectrum of the two-soliton problem (37). Substituting the numerical values of $\lambda_n(a)$ into (39) and (40), and summing over these values, we find results for the indirect interaction for domain walls. We write these results as the approximate expressions

$$d=2:\ \widetilde{V}(a)=kT(0{,}40\pm0{,}01)\exp[-(0{,}33\pm0{,}01)(2a)^2],\qquad(41)$$

$$d=3:\ \widetilde{V}(a)=kT(0{,}0163\pm0{,}0010)\exp[-(0{,}33\pm0{,}01)(2a)^2]+kT(0{,}001\pm0{,}001)\exp(-2a).\qquad(42)$$

We first note that the decay of the interaction potential with increasing distance between the domain walls is described by $\exp(-4a^2/3)$, regardless of the dimensionality of the domain wall. The potential amplitude, on the other hand, falls off substantially as the dimensionality d of the space increases.

The result that the domain walls repel each other through the phonons can be understood easily by comparing the eigenvalues λ_n of the Schrödinger equation (37), which determine the squares of the phonon frequencies, for two values of the distance between solitons. Table I shows the first 20 values of $2a_0^2\lambda_n$ for $a=1$ and $a=2$, where $a_0=0.02$.

We see from this table that with increasing distance between the solitons all the eigenvalues λ_n of the Schrödinger equation (37) become lower, except λ_1.

We conclude with the results of a numerical analysis of $\widetilde{V}(a)$ with the pinning potential (35) in Eq. (37). These results are shown in Fig. 3 for the dimensionless temperature $t=\beta E_0=0.25$. We recall that for each value of a the pinning potential α in (35) was chosen so that the minimum eigenvalue λ_1 would be equal to zero (to within 10^{-12}). We see from Fig. 3 that the pinning substantially alters the shape of the indirect-interaction potential. The reason is that the pinning potential, to which the soliton is rigidly tied, leads to a direct interaction of the phonon with the soliton, which is absent, at an accuracy level $\exp(-4a)$, from the two-soliton problem without pinning.

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