Sound amplification mechanism in a weakly ionized gas

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The universality of the mechanism for acoustic instability in a plasma which stems from reversal of the second viscosity is analyzed.

Aleksandrov *et al.*[1] have proposed a new mechanism for sound amplification in a beam-driven discharge. That mechanism involves friction between the electrons and the neutral gas. However, certain results of Ref. 1 require refinement, in our opinion.

As Ingard and Gentle have shown,[2] friction between electrons and neutral particles does not by itself lead to amplification or absorption of sound. Aleksandrov *et al.*[1] actually analyzed the amplification (or absorption) associated with the finite relaxation time of the plasma density ($\tau_r = \nu_r^{-1}$) and the finite length scale of the diffusive "relaxation" of the electron distribution ($L_E \sim D/v_{e0}$) associated with the departure from equilibrium in the medium caused by the propagation of a sound wave. In this paper we retain the notation of Ref. 1.

The first process leads to a temporal deviation from local behavior and gives rise to a second viscosity. When sound waves $\propto \exp(-i\omega t + ikr)$ propagate in a relaxing medium, the pressure perturbation P' differs from the equilibrium pressure P'_{eq}:

$$P' = P'_{\text{eq}} - \xi'(\omega)\operatorname{div}\mathbf{v}. \tag{1}$$

Here ξ' is the complex second-viscosity coefficient, whose imaginary part ($\operatorname{Im}\xi'$) determines the dispersion of the sound velocity, and whose real part ($\xi = \operatorname{Re}\xi'$) determines the absorption of the sound.

The second process is characterized by spatial relaxation, and it determines the spatial deviation of the medium from local behavior. For isotropic media with a dispersion which is linear in the harmonic approximation ($d^2\omega/dk^2 = 0$ or $\partial P'/\partial t = -\text{const}\,\partial P'/\partial x$), however, this process can also be reduced to a relation like (1) with a corresponding value of $\xi'(\omega)$. For acoustic perturbations it is thus appropriate to discuss this process in terms of the second viscosity, combining these effects under the common name of "relaxation viscosity," which characterizes dissipative processes which occur in a medium because of the finite time required for the pressure to become steady following compression or rarefaction in a sound wave. The sign of the coefficient is determined by the sign of the feedback between the density perturbations in the wave and the relaxing parameters of the system.[3,4]

By explicitly introducing the relaxation-viscosity coefficient, we can write the dissipation coefficient Γ in the simple standard form

$$\Gamma = -\Omega^2\xi/2c^2\rho, \tag{2}$$

where

$$\xi = -\frac{\rho}{\Omega}\operatorname{Im}\frac{\omega^2}{k^2}, \tag{3}$$

$$c = \Omega/k \tag{4}$$

is the sound velocity, and $\omega \equiv \Omega + i\Gamma$.

In the model of Ref. 1, for example, we find the following results from Eqs. (7)–(11)[1)] of Ref. 1 for transverse perturbations (we are ignoring diffusion terms):

$$\begin{gathered}\xi = -\alpha P_e/\Omega(4+\alpha^2),\\ \frac{c^2}{c_0^2} = 1 + \frac{P_e}{\gamma_a P}\left(\frac{2}{4+\alpha^2} - \hat{T}_{e0}\right),\\ \frac{\Gamma}{\Omega} = \frac{\alpha P_e}{2P\gamma_a(4+\alpha^2)}.\end{gathered} \tag{5}$$

The growth rate (5) found here agrees to within a factor of 1/2 with that given in Ref. 1 [see expression (12) of that paper].

Correspondingly, for longitudinal perturbations we find

$$\begin{gathered}\xi = -\frac{\alpha P_e(\hat{v}_{e0} - \hat{T}_{e0})}{\Omega(4+\alpha^2)\hat{v}_{e0}},\\ \frac{c^2}{c_0^2} = 1 + \frac{2P_e(\hat{v}_{e0} - \hat{T}_{e0})}{\gamma_a P(4+\alpha^2)\hat{v}_{e0}},\\ \frac{\Gamma}{\Omega} = \frac{P_e\alpha(\hat{v}_{e0} - \hat{T}_{e0})}{2\gamma_a P v_{e0}(4+\alpha^2)}.\end{gathered} \tag{6}$$

Expression (6) differs from the corresponding growth rate in Ref. 1 by a factor of 1/2 and in the appearance of $\hat{T}_{e0}$ in place of a one. These discrepancies can apparently be ascribed to misprints.

The important point is that growth rates of the form (5) and (6) were found in Ref. 1 in an inappropriate model: The primary friction effect—the entrainment of the plasma by the gas—was not considered. Working from the transport equation for the ion momentum,

$$M\frac{d\mathbf{v}_i}{dt} = e\mathbf{E} - \nu_{in}M(\mathbf{v}_i - \mathbf{v}) - \nabla P_i,$$

we find

$$\mathbf{v}_i = \mu_i\mathbf{E} + \mathbf{v}. \tag{7}$$

This expression must be taken into account in the balance equation for the ion density [Eq. (5) in Ref. 1]. As we will see below, when this expression is now written in the form

$$\frac{\partial n_i}{\partial t} + \operatorname{div}(\mu_i E + \mathbf{v})n_i = q_0 N - \nu_r n_i \tag{8}$$

it gives rise to a change in the relationship between the per-

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turbations $\tilde{n}$ and the quantities $\widetilde{N}$ and $\widetilde{E}$. It also changes the sign of the dissipative coefficients (5) and (6).

For a quasineutral plasma ($\nu_u \tau_M \ll 1$), the relation written in Ref. 1 (the ion current and $\hat{v}_r$ are being ignored),

$$\frac{\tilde{n}}{n}=\frac{\tilde{N}}{N}\left(2-i\frac{\omega}{\nu_r}\right)^{-1},$$

is replaced by the following relation, which we find from (8):

$$\frac{\tilde{n}}{n}=\frac{\tilde{N}}{N}\left(1-i\frac{\omega}{\nu_r}\right)\left(2-i\frac{\omega}{\nu_r}\right)^{-1}. \tag{9}$$

Using (9), we can write the dispersion relation for an arbitrary propagation direction of the sound wave as follows (in the limit $k\lambda_u \ll 1$, $\Gamma \ll \nu_r$, Ω, which was used in Ref. 1):

$$\frac{M}{T_0}\frac{\omega^2}{k^2}=\frac{C_{P0}-iC_{P1}\Omega\tau_r-iC_{P2}\Omega\tau_E-C_{P3}\Omega^2\tau_r\tau_E}{C_{V0}-iC_{V1}\Omega\tau_r-iC_{V2}\Omega\tau_E-C_{V3}\Omega^2\tau_r\tau_E}. \tag{10}$$

Table I shows values of the relaxation specific heats at constant pressure and constant volume, C_{Pj} and C_{Vj} ($j=0$–3), for the cases in which the sound is propagating across ($\eta=0$) and along ($\eta=1$) the electric field. The quantity $\tau_E = D/v_{e0}C_0$ in (10) is the diffusion time scale, where

$$D=\begin{cases} D_T, & \eta=0 \\ |D_E|, & \eta=1 \end{cases}.$$

The dispersion relation (10) has the standard form for a medium with two coupled relaxation processes. In particular, with the appropriate changes in notation it applies to the case of a medium with a vibrational-rotational or intermode vibrational molecular relaxation. The relaxation-viscosity coefficient (3) and the sound velocity (4) were found in Ref. 5 for a dispersion relation like that in (10); these expressions are

$$\xi=(T_0/M)\rho[\tau_r(C_{V0}C_{P1}-C_{P0}C_{V1}) +\tau_E(C_{V0}C_{P2}-C_{P0}C_{V2})+\Omega^2\tau_r^2\tau_E(C_{V3}C_{P1}-C_{P3}C_{V1})]/ [(C_{V0}-C_{V3}\Omega^2\tau_r\tau_E)^2+(C_{V1}\Omega\tau_r+C_{V2}\Omega\tau_E)^2], \tag{11}$$

$$c^2=(T_0/M)[(C_{P0}-C_{P3}\Omega^2\tau_r\tau_E)(C_{V0}-C_{V3}\Omega^2\tau_r\tau_E) +(C_{P1}\Omega\tau_r+C_{P2}\Omega\tau_E)(C_{V1}\Omega\tau_r+C_{V2}\Omega\tau_E)]/[(C_{V0}-C_{V3}\Omega^2\tau_r\tau_E)^2 +(C_{V1}\Omega\tau_r+C_{V2}\Omega\tau_E)^2]. \tag{12}$$

The dissipation coefficient is expressed in terms of the second-viscosity coefficient and the velocity of sound in accordance with (2).

For $\tau_r = 0$ or $\tau_E = 0$, we find from (11) and (12) the second-viscosity coefficient and the sound velocity in the ordinary form for the single-relaxation model:[5]

$$\xi=\frac{2\rho\mu_{10}^{r}C_{V0}^2}{C_{V0}^2+\Omega^2\tau_r^2C_{V1}^2}, \tag{13}$$

$$c^2=\frac{C_{V0}^2u_0^2+\Omega^2\tau_r^2u_1^2C_{V1}^2}{C_{V0}^2+\Omega^2\tau_r^2C_{V1}^2} \tag{14}$$

or

$$\xi=\frac{2\rho\mu_{20}^{E}C_{V0}^2}{C_{V0}^2+\Omega^2\tau_E^2C_{V2}^2}, \tag{15}$$

$$c^2=\frac{C_{V0}^2u_0^2+\Omega^2\tau_E^2u_2^2C_{V2}^2}{C_{V0}^2+\Omega^2\tau_E^2C_{V2}^2}, \tag{16}$$

where $\mu_{i,l}^{k} = \tau_k(u_i^2-u_l^2)C_{Vi}/2C_{Vl}$ is the low-frequency viscosity coefficient ($\Omega\tau_k \ll C_{Vl}/C_{Vi}$), $u_i = (C_{Pi}T_0/C_{Vi}M)^{1/2}$ and $u_l = (C_{Pl}T_0/C_{Vl}M)^{1/2}$ are the velocities of low-frequency and high-frequency sound, and $i, l = 0$–3 and $k = r, E$.

It follows from (13), (14), and (2) that if the diffusion terms are ignored the dissipative coefficients differ from those in (5) and (6):

$$\frac{\Gamma}{\Omega}=-\frac{P_e\alpha}{2\gamma_a P(4+\alpha^2)},\quad \eta=0, \tag{17}$$

and

$$\frac{\Gamma}{\Omega}=-\frac{P_e(\hat{v}_{e0}-\hat{T}_{e0})}{2\gamma_a P\hat{v}_{e0}}\frac{\alpha}{(4+\alpha^2)},\quad \eta=1. \tag{18}$$

We thus see that perturbations which are transverse to the field are not amplified as in Ref. 1 and are instead absorbed, according to the model of this paper, with (9). According to (18), the condition under which longitudinal waves are amplified is $\hat{v}_{e0} < \hat{T}_{e0}$, i.e., the direct opposite of the condition found from (6).

Aleksandrov *et al.*[1] studied the case $\alpha < 1$, $\hat{v}_{e0} = 0$,

TABLE I.

C	$\eta = 0$	$\eta = 1$
C_{P0}	$2C_{P\infty}+\frac{P_e}{P}C_{V\infty}(1-2\hat{T}_{e0})$	$2\hat{v}_{e0}C_{P\infty}+\frac{P_e}{P}(\hat{v}_{e0}-\hat{T}_{e0})C_{V\infty}$
C_{P1}	$C_{P\infty}+\frac{P_e}{P}C_{V\infty}(1-\hat{T}_{e0})$	$C_{P\infty}\hat{v}_{e0}+\frac{P_e}{P}(\hat{v}_{e0}-\hat{T}_{e0})C_{V\infty}$
C_{P2}	$2\hat{D}_TC_{P\infty}+\frac{P_e}{P}C_{V\infty}(\hat{D}_T-\hat{T}_{e0})$	$\frac{D_E}{\lvert D_E\rvert}\left\{2C_{P\infty}+\frac{P_e}{P}\left[1-\frac{\hat{T}_{e0}D_L}{D_E}-\frac{\varphi_1 v_{e0}}{D_E}\hat{v}_{e0}+\frac{(\varphi_2+\varphi_3)}{D_E}v_{e0}\right]C_{V\infty}\right\}$
C_{P3}	$C_{P\infty}\hat{D}_T+\frac{P_e}{P}C_{V\infty}(\hat{D}_T-\hat{T}_{e0})$	$C_{P2}-C_{P\infty}DE/\lvert D_E\rvert$
C_{V0}	$2C_{V\infty}$	$2\hat{v}_{e0}C_{V\infty}$
C_{V1}	$C_{V\infty}$	$\hat{v}_{e0}C_{V\infty}$
C_{V2}	$2C_{V\infty}\hat{D}_T$	$2C_{V\infty}D_E/\lvert D_E\rvert$
C_{V3}	$C_{V\infty}\hat{D}_T$	$C_{V\infty}D_E/\lvert D_E\rvert$

1) *Note.* Here $C_{P\infty}$ and $C_{V\infty}$ are the specific heats corresponding to the translational degrees of freedom at constant pressure and constant volume. The D_T, D_E, and D_L were introduced in Ref. 1.

$\eta = 1$ separately. In the limit $\alpha \to 0$ we find from (2), (15), and (16)

$$\frac{\Gamma}{\Omega} = -\frac{P_e \hat{T}_{e0} |D_E|}{4\gamma_a P \Omega \tau_E D_E}. \qquad (19)$$

The coefficient (19), unlike the dissipative coefficient associated with the relaxation of the degree of ionization, agrees with the corresponding result of Ref. 1, since the entrainment of the plasma by the gas, described by expression (7), is unimportant in this limit.

Formally we find $\Gamma/\Omega \to \infty$ as $\Omega\tau_E \to 0$ from (19). However, at least three limitations are imposed on the parameter $\Omega\tau_E$.

In the first place, expression (19) was derived in the approximation $\Gamma \ll \Omega$, so we must require

$$\Omega\tau_E \gg \frac{P_e \hat{T}_{e0}}{4\gamma_a P}.$$

Second, the model used in Ref. 1 assumes that the electron temperature is quasisteady. With $\hat{v}_{e0} = 0$, that assumption is valid if

$$\Omega\tau_e \ll \Omega\tau_E, \qquad (20)$$

where $\tau_e = (\delta\nu_{en})^{-1}$, δ is an accommodation coefficient, and ν_{en} is the electron-neutral collision rate. This limitation must be taken into consideration in (for example) a comparison of (19) with the thermal (in the terminology of Ref. 1) amplification mechanism, with the growth rate

$$\nu_T \sim \frac{P}{P}\delta_T \delta\nu_{en} \sim \frac{P_e}{P}\frac{\delta_T}{\tau_e}.$$

Since we have[6] $\lambda_u \sim v_{e0}\tau_e$, we similarly find the following condition from (20):

$$\Omega\tau_E \gg k\lambda_u c_0 / v_{e0}. \qquad (21)$$

Third and finally, in real physical problems it is unlikely that the equality $\hat{v}_{e0} = 0$ will be satisfied exactly. For $\hat{v}_{e0} \ll 1$, according to (15), relation (19) is valid if

$$\hat{v}_{e0} = \frac{C_{V0}}{G_{V2}} \ll \frac{kD_E}{v_{e0}}. \qquad (22)$$

Using $k\lambda_u \ll 1$, we see that the example

$$\frac{kD_E}{v_{e0}} \sim \frac{k\lambda_u}{10}, \quad v_{e0} \approx 2{,}4\cdot 10^6 \text{ cm/s}, \qquad (23)$$

discussed in Ref. 1, shows just how small $\hat{v}_{e0}$ must be if we are to observe amplification (or absorption) with the rate (19). Example (23) satisfies condition (21).

It can be seen from (2), (11), and (12) that if there are several relaxation processes ($\tau_r \neq 0$, $\tau_E \neq 0$) it will not be possible to identify dissipation coefficients which are linked with one specific process. This can be done only in the limits in which the relaxation of the corresponding degree of freedom is fast or slow (in comparison with the wave period).

For example,

$$\Gamma = \begin{cases} \Gamma_{10}^{r,0} + \Gamma_{20}^{E,0}, & c = u_0 \quad \text{for} \quad \Omega\tau_r \ll \frac{C_{V0}}{C_{V1}}, \frac{C_{P0}}{C_{P1}}, \\ & \qquad\qquad\quad \Omega\tau_E \ll \frac{C_{V0}}{C_{V2}}, \frac{C_{P0}}{C_{P2}} \\ \Gamma_{10}^{r,\infty} + \Gamma_{31}^{E,0}, & c = u_1 \quad \text{for} \quad \Omega\tau_r \gg \frac{C_{V0}}{C_{V1}}, \frac{C_{P0}}{C_{P1}}. \\ & \qquad\qquad\quad \Omega\tau_E \ll \frac{C_{V1}}{C_{V3}}, \frac{C_{P1}}{C_{P3}} \end{cases}$$

Here

$$\Gamma_{il}^{k,0} = -\frac{\mu_{il}{}^k \Omega^2}{u_l^2}, \quad \Gamma_{il}^{k,\infty} = -\frac{\mu_{il}{}^k C_{Vl}{}^2}{\tau_k{}^2 C_{Vi}{}^2 u_i^2}$$

are the low-frequency and high-frequency dissipation coefficients,[7,5] each of which is expressed in terms of a relaxation-viscosity coefficient associated with a specific relaxation process.

In conclusion we wish to stress that the relaxation-viscosity formalism describes the acoustics of equilibrium and nonequilibrium media under the condition that the absorption over a wavelength is small ($\Gamma \ll \Omega$), regardless of the particular parameter which is undergoing relaxation. For an isotropic medium with linear dispersion, both spatially local processes with a finite relaxation time (second viscosity and radiative thermal conductivity)[7] and such nonlocal processes as diffusion or radiationless thermal conductivity contribute to the relaxation viscosity. The radiationless thermal conductivity also leads to an equation of state like (1) (see Ref. 8, for example), with a coefficient $\xi_\varkappa = \varkappa(1/C_{V_\infty} - 1/C_{P_\infty})$. This coefficient can be written in the standard form for a low-frequency relaxation-viscosity coefficient ($\xi_\varkappa = 2\rho\mu$):

$$\mu = \frac{\tau_\varkappa (u_N^2 - u_L^2) c_{VN}}{2C_{V\infty}},$$

where $\tau_\varkappa = \varkappa / u_L^2 \rho_0$ is the time scale of the thermal conductivity, $u_L = (C_{P\infty} T_0 / C_{V\infty} M)^{1/2}$ is the Laplace velocity of the low-frequency (adiabatic) sound ($\omega\tau_\varkappa \ll 1$), $u_N = (C_{PN} T_0 / C_{VN} M)^{1/2}$ is the Newtonian velocity of high-frequency (isothermal) sound ($\omega\tau_\varkappa \gg 1$), and $C_{PN} = C_{VN} = -1$.

The second-viscosity formalism, which makes possible a unified analysis of the stability of sound in a medium with relaxing parameters, shows its worth in an analysis of nonlinear acoustic dynamics. Under the condition $\Gamma \ll \Omega$, which we have already mentioned, the propagation of waves of small but finite amplitude (the second order of an amplitude perturbation theory) is described by a linear second viscosity.[3] In higher orders, it is modified,[9] but we retain the transparent physical picture of the evolution of the acoustic waves in a nonequilibrium relaxing medium.

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[1)] Using (7), we see that the last of Eqs. (11) in Ref. 1 should be $(i\eta v_{e0} + kD_T + \varkappa_1 k\eta^2)\tilde{n}/n + [i\eta v_{e0}(v_{e0} - 1) + kD_0 + \varkappa_3 k\eta^2]\tilde{\gamma}/\gamma - (i\eta v_{e0} + \varkappa_2 k\eta^2)\tilde{N}/N + \varkappa_2 k(\mathrm{e}\tilde{\mathbf{E}}/E) + i(v_{e0}/k)(\mathbf{k}\tilde{\mathbf{E}}/E) = 0.$

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