Nonlinear electromagnetic waves in a stochastized electron gas with a nonquadratic dependence of energy on the momentum

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(Submitted 25 December 1990)
Zh. Eksp. Teor. Fiz. **99**, 1711–1717 (June 1991)

The interaction between nonlinear electromagnetic waves in an electron plasma with a charged particle dynamically stochastized by these waves is studied. It is shown that chaotization of the process inevitably accompanies propagation of the waves, which leads in turn to their slowing down due to the scattering by the stochastized group of particles.

Propagation of solitary electromagnetic waves and dynamic stochastization are being vigorously studied recently as two manifestations of the the essential nonlinearity of a physical system. This "essence" is often measured by whether one of these phenomena can be achieved. In a number of cases both phenomena could exist in the same system.

In this paper a model medium for a study of the interaction between propagating nonlinear waves and dynamical chaos of the charge carriers is considered. The nature of these nonlinear phenomena is quite different (corresponding to a complete integrability or nonintegrability of the evolution equations describing them). In our case this contradiction is eliminated by the fact that a nonlinear wave is a macroscopic phenomenon,[1] while dynamic stochastization is implemented on the microscopic level[2] in the dynamics of an individual conduction electron.

Consider an ideal electron gas with the dispersion law given in Ref. 3. Such a dispersion law of charge carriers occurs, e.g., in a homogeneous semiconducting superlattice

$$\varepsilon(\mathbf{p}) = p_\perp^2/2m - \Delta\cos(p_z/d\hbar), \tag{1}$$

where Δ is the halfwidth of the miniband, d is the period of the superlattice whose axis is denoted by z, and m is the effective mass of the charge carriers.

As it is shown in Ref. 1, in such a system the Maxwell equation for the z-component of the vector-potential of electromagnetic field A_z is the Sine–Gordon equation under the condition that the relaxation frequency ν of the electrons is substantially smaller than the characteristic frequency of the field:

$$\nu \ll (\partial A_z/\partial t)/A_z,$$

the characteristic spatial scale of which is substantially longer than the electron wavelength λ_c and the period d of the superlattice:

$$\frac{\partial^2 F}{\partial t^2} - c^2\Delta F + W^2\sin(F) = 0, \tag{2}$$

where $F = edA_z/c\hbar$, c is the speed of light in the medium, and W is the characteristic frequency.

The Sine–Gordon equation (2) possesses several essentially nonlinear solutions. In this paper, we employ a solution in the form of a cnoidal wave

$$F = 2\arcsin\left[\varkappa\,\mathrm{sn}\left(\frac{W(r-vt)}{(c^2-v^2)^{1/2}};\varkappa\right)\right] + \pi, \tag{3}$$

where v is the wave velocity, r is the coordinate along the direction of the propagation, and $0 \leqslant \varkappa \leqslant 1$ is a parameter which determines the degree of nonlinearity of the wave. Here for example the electric component of the field is of the form

$$E_z = 2\frac{\hbar\varkappa}{ed}\frac{vW}{(c^2-v^2)^{1/2}}\,\mathrm{sn}\left(\frac{W(r-vt)}{(c^2-v^2)^{1/2}};\varkappa\right). \tag{4}$$

On the other hand, Bass *et al.*[2,4] have shown that in the same system the charge motion is stochastized provided the system is located in a constant magnetic field H in the direction perpendicular to the axis of the superlattice (for example, along the x axis) and a variable electromagnetic field. In that case, at the top of the miniband a layer of dynamically stochastized charges is formed for arbitrarily small amplitudes of the electromagnetic wave (there are no random forces in the system). This is due to the instability of the separatrix of Larmor oscillations relative to the dynamic stochastization under the effect of periodic forces (cf. Ref. 5). The width of the stochastized layer is uniquely determined in this case by the parameters of the electromagnetic wave.[2]

As it is well known, the electrons in the region of dynamic chaos are uniformly distributed in a region of the phase space of Larmor oscillations. This implies that the contribution of these carriers to the conductivity vanishes, since they form only nonthermal fluctuations. Their statistics is not determined by thermodynamic considerations (as in the case of equilibrium thermal noise), but solely by the parameters of the fields and the superlattice itself.[2]

Evidently, in the general case both groups of carriers ought to be taken into account. In that case, the propagation of an electromagnetic wave in a superlattice located in an external constant magnetic field leads on the microscopic level to a dynamic chaotization of the electronic plasma, while on the macroscopic level it leads to the formation of a cnoidal wave scattering off the electrons it has chaoticized itself. Both phenomena and their interactions are inevitable properties of the system under consideration. They arise for the same reason, nonlinearity of the dispersion law (1), and a consistent treatment of them does not involve any arbitrary parameters.

GENERAL APPROACH

To describe such an electron gas it is necessary to solve the system of equations which includes the kinetic equation

$$\frac{\partial}{\partial t}f + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}f + e\left(\mathbf{E} + \frac{[\mathbf{v}H]}{c}\right)\frac{\partial}{\partial \mathbf{p}}f = -\frac{f-f_0}{\tau} \tag{5}$$

and the Maxwell equation

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$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}-\Delta\mathbf{A}=\frac{4\pi}{c}\int_{-\Delta}^{\Delta} e v(\varepsilon) f(\varepsilon) g(\varepsilon) d\varepsilon, \qquad (6)$$

where $g(\varepsilon)$ is the density of states and the integration is carried over the whole miniband (1).

Here the distribution function [at least in the τ-approximation (5)] is constant in the region of dynamic chaos (since this type of motion is ergodic[5])

$$f\sim dI d\theta \sim d\varepsilon d\theta/\Omega(\varepsilon), \qquad (7)$$

so that an investigation of Eq. (5) is reduced to the determination of the boundaries of this region. For this purpose, it is sufficient to solve the corresponding system of equations for the characteristics which coincide with the trajectories of individual electrons in the real electromagnetic field, but without taking their scattering into account, i.e., it is sufficient to consider a conservative system with the Hamiltonian (1).

To derive the wave equation it is necessary to take into account that the distribution function consist of two qualitatively different parts (cf. Fig. 1): a regular part (following Ref. 1 we choose here the Boltzmann distribution) for $-\Delta\leqslant\varepsilon\leqslant\varepsilon^*$ and the stochastized part (7) for $\varepsilon^*\leqslant\varepsilon\leqslant\Delta$. Then

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}A_z-\frac{\partial^2}{\partial r^2}A_z=\frac{4\pi}{c}(j(\varepsilon_R^*(A_z);A_z)+j_S(\varepsilon^*(A_z);A_z)), \qquad (8)$$

where j_R and j_S describe the contributions to the conduction of the corresponding regular and stochastized groups of carriers.

Since the separatrix (in this case of the Larmor oscillations) is unstable against dynamic stochastization[5], the chaotized group of carriers can not be eliminated and only the width of the layer it occupies (region 1 in Fig. 1) depends on the specific parameters of the waves propagating in the medium. In the expression for $\varepsilon^*(A_z)$, the explicit dependence of the boundary of the stochastic layer on the parameters of the waves is identified.

Practically, this problem reduces to a solution of the self-consistent system (5), (8). In this paper we confine ourselves to the case of a weak stochastization $\Delta-\varepsilon^*\ll\Delta$, i.e., although the stochastization substantially changes the nature of the electron motion, the relative volume of the region of phase space occupied by it is small. Then the interaction between the cnoidal waves and the dynamic chaos can be viewed in terms of stochastic perturbation theory.[6,7 1)] We choose the cnoidal wave (4) as the unperturbed solution; the wave propagation along (x axis) or perpendicular (y axis) to the magnetic field. The period of such a wave is equal to

$$T=Q^{-1}=\frac{2K(\varkappa)}{\pi W}\frac{(c^2-v^2)^{1/2}}{v},$$

which for a typical superlattice constitutes $T\sim K(\varkappa)c/v\cdot 10^{-12}$ s.

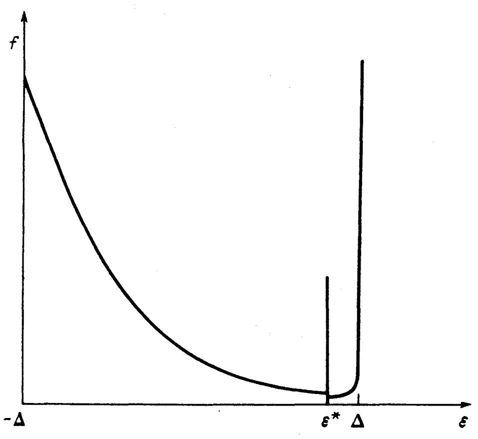

FIG. 1. Electron distribution function.

CHAOTIZATION OF THE DYNAMICS OF THE CHARGE CARRIERS

To determine the region of stochastization of the dynamic system with the Hamiltonian (1) in the external electromagnetic field (4),

$$H=\frac{p_x^2}{2m}+\frac{(p_y-eHz/c)^2}{2m}-\Delta\cos\left(\frac{p_z d}{\hbar}+F\right), \qquad (9)$$

we use the Chirikov method[5] valid for weakly perturbed systems. A meaningful criterion for the validity of the approximation is indicated below.

We take the overlapping of neighboring resonances of unperturbed Larmor oscillations and the electromagnetic wave (4) as the chaotization criterion. Expanding (9) into a Fourier series it is easy to verify[2] that the resonances of the dynamic system correspond to the fulfillment of the condition

$$r\Omega(\varepsilon)=Q(\varkappa)(2l+1), \qquad (10)$$

where r is odd (even) when the wave propagates along (across) the magnetic field, l is an integer and

$$\Omega(\varepsilon)=\pi\frac{eHd}{c\hbar}\left(\frac{\Delta}{m}\right)^{1/2}\left[2K\left(\frac{\Delta+\varepsilon}{\Delta}\right)^{1/2}\right]^{-1}$$

is the Larmor frequency of the unperturbed oscillations [$E_z=0$, Eq. (9)].

This resonance structure is quite complex. From all the possible overlapping conditions one should choose the least restrictive. It can be shown that this choice depends on the ratio of the frequencies $\delta=\Omega(\varepsilon)/Q(\varkappa)$ at the resonance point (10).

Using results of Ref. 2 it is easy to obtain the explicit form of the Chirikov overlap criterion.

When the wave moves along the x axis

$$\delta\gg 1:\ \left(\frac{\Delta-\varepsilon}{\varepsilon}\right)^{1/2}\ln\left(\frac{\Delta-\varepsilon}{32\varepsilon}\right)\leqslant U,$$

$$\delta\ll 1:\ \left(\frac{\Delta-\varepsilon}{\varepsilon}\right)^{1/2}\ln\left(\frac{\Delta-\varepsilon}{32\varepsilon}\right)\pi\frac{HK(\varkappa)}{nc}\leqslant U, \qquad (11)$$

$$U=\alpha 3\pi(2p+1)\left[1-\frac{p_{y0}}{mv_y}\right]\left[\operatorname{sech}\left(\alpha\frac{\pi(2p+1)K'(\varkappa)}{2K(\varkappa)}\right)\times\operatorname{sech}\left(\frac{\pi(2p+1)K'(k)}{2K(\varkappa)}\right)\right]^{1/2}.$$

When the wave moves along the y axis (and for arbitrary δ)

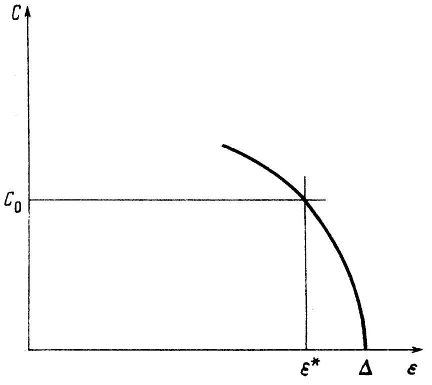

FIG. 2. Chaotization criterion.

$$\left(\frac{\Delta-\varepsilon}{\Delta}\right)^{1/2}\ln\left(\frac{\Delta-\varepsilon}{\varepsilon}\right)\leqslant\alpha\frac{3\pi(2p+1)}{2}\times\left[\operatorname{cosech}\left(\alpha\frac{\pi(2p+1)K'(\varkappa)}{2K(\varkappa)}\right)\times\operatorname{sech}\left(\frac{\pi(2p+1)K'(k)}{2K(\varkappa)}\right)\right]^{1/2}\left\{\left|1-\frac{p_y}{mv_y}\right|^{1/2}+\left|\frac{\Delta}{mv_y^2}\right|^{1/4}\right\}. \tag{12}$$

The Chirikov criterion is presented graphically in Fig. 2 where $C(\varepsilon)$ and C_0 are respectively the left and the right parts of the criteria (11) and (12). One can observe that the dynamic chaos region does exist for any values of the parameters of the electromagnetic field (4).

To obtain an analytic relationship between the location of the boundary of the layer ε^* and the parameters of the wave, we assume equality in (11) and (12). This also determines the physical meaning of the condition that the weak-stochastization approximation be well-posed: $\Delta-\varepsilon^*\ll\Delta$.

DYNAMICS OF A NONLINEAR WAVE IN A SUPERLATTICE

Utilizing the Boltzmann distribution in the regular region $-\Delta\leqslant\varepsilon\leqslant\varepsilon^*$ and the stochastized distribution (7) in the region $\varepsilon^*\leqslant\varepsilon\leqslant\Delta$, we arrive at the wave equation (8) explicitly:

$$\frac{\partial^2}{\partial t^2}F-c^{-2}\frac{\partial^2}{\partial r^2}F+W_0^2\sin F=j_s, \tag{13}$$

$$W_0^2=4\pi\frac{e^2n\,\Delta d^2}{\hbar^2}\frac{I_1(\Delta/kT)}{I_0(\Delta/kT)},$$

$$\langle j_s\rangle=0,$$

$$\langle j_s^2\rangle=\left(\frac{kT}{\Delta}\right)^{1/2}\left[\frac{e^2n\,\Delta d^2}{\hbar^2}\right]^2\cdot\frac{128}{3}\frac{\exp(-\Delta/kT)}{I_0(\Delta/kT)}\pi^2\Phi, \tag{14}$$

$$\Phi=E_i\left(\frac{\Delta-\varepsilon}{\varepsilon}\right)+\ln\frac{32\Delta}{\Delta-\varepsilon}\exp\frac{\Delta}{kT},$$

where $\langle\ \rangle$ indicates an average over the realizations, I_i is the Bessel function of imaginary argument and E_i is the integral exponential function.

Since the time required for phases of the Larmor oscillations to become decorrelated (of the order of their period) is substantially smaller than the typical time scales of a cnoidal wave (for typical superlattices, e.g., with parameters $d=100$ Å, $m=10^{-28}$ G, $n=10^{13}$ cm^{-3}, $c=10^9$ cm/s, $H\sim10^2$ Hs and $v\ll c$) the random process j_s can be viewed as δ-correlated. The spatial-correlation radius r_{corr} in the case of a propagation along the x axis is unbounded since the Larmor oscillations are not connected with the motion of the particles along the magnetic field. In the case of propagation parallel to the superlattice layers, r_{corr} may reach the value of the characteristic spatial scale Tv of the wave and even exceed it:

$$r_{\text{corr}}/Tv=\frac{\pi^2(\Delta n)^{1/2}}{HK(\varkappa)}\sim\frac{10\text{ G}}{HK(\varkappa)}. \tag{15}$$

Using the statistical properties of j_s (at least in the limiting cases of homogeneous and δ-correlated random fields j_s) we can easily establish the behavior of arbitrary nonlinear waves in the medium by using the results of statistical perturbation theory. For example, for a sufficiently disperse wave (a chain of non-overlapping solitons) its interaction with the deterministic chaos results in slowing down[6,7] without a change in the locations of the centers of gravity of the bumps and in radiation of a continuous spectrum[7].

The average value of the speed of propagation varies over small time periods

$$t\ll t_1\frac{128}{13}\left(\frac{\Delta}{kT}\right)^{1/2}\frac{I_0(\Delta/kT)\exp(-\Delta/kT)}{I_1^2(\Delta/kT)}\times\left\{E_i\left(\frac{\Delta-\varepsilon}{\varepsilon}\right)+\ln\frac{32\Delta}{\Delta-\varepsilon}\exp\frac{\Delta}{kT}\right\}$$

according to the laws (cf. Ref. 6)

$$\langle v\rangle\approx v_0-\frac{3}{2}\pi^2v_0\frac{t}{t_1}$$

for a homogeneous fields j_s and

$$\langle v\rangle\approx v_0-v_0\frac{t}{t_1}$$

for a δ-correlated field.
The dispersion of the speed varies according to

$$\sigma\approx\frac{\pi^2t}{2t_1}$$

for a homogeneous field j_s and

$$\sigma\approx\frac{t}{2t_1}$$

for a δ-correlated field. Part of the energy of the nonlinear wave is transferred to the continuous spectrum

$$\langle p(\lambda)\rangle=\frac{\Omega(\lambda)}{\pi^2(\lambda^2+{}^1/_4)^2\lambda}\left[\delta(U(\lambda))\left(\frac{\lambda^2-{}^1/_4}{U(\lambda)}\right)^2+4\pi^2\lambda^2\operatorname{sh}^{-2}\left(\frac{\pi U(\lambda)}{2}\right)\right]W_0t_1$$

for a homogeneous field j_s and

$$\langle p(\lambda)\rangle=\frac{16\Omega(\lambda)}{\pi(\lambda^2+{}^1/_4)}W_0t_1$$

for a δ-correlated field where $p(\lambda)$ is the spectral density, and its parameter λ determines the radiation frequency $\Omega(\lambda)=\lambda+1/4$ and the wave vector $U(\lambda)=1-1/4\lambda^{-1}$ of the electromagnetic waves.

DISCUSSION

As was shown above, the propagation of nonlinear electromagnetic waves in an electromagnetic gas with a nonquadratic dispersion law (1) is inevitably accompanied by stochastization of part of the conduction electrons. This, in turn, affects the dynamics of the waves: they slow down because they scatter off the chaotized particles. From the point of view of the wave propagation proper, this is a new damping mechanism.

On the other hand, the efficiency with which these two qualitatively distinct, essentially nonlinear, phenomena interact does not involve any free parameters and is determined only by the characteristics of the medium and the wave velocity. A group of chaotized particles always exist (cf. Fig. 2) so that the interaction under consideration can not be avoided and a large number of harmonics of an arbitrary strictly non-linear wave can result in dynamical chaos and in a weakly nonlinear subsystem of quasiparticles. It would therefore seem that the interaction between the nonlinear electromagnetic waves and the dynamical chaos they produce in the form of quasiparticles on the microlevel in the medium has a wider physical significance.

[1)] To isolate in "the pure" form the interaction under consideration, we neglect in this paper the well-known additive contribution of the regular group of electrons to the random fluctuations of the stream.

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Translated by Samuel Kotz