Nonlinear waves with dispersion and nonlocal damping

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We use the Whitham averaging method to obtain a set of equations for the amplitude, phase, and velocity of an almost periodic wave which is a solution of the Korteweg–de Vries equation with an arbitrary linear dissipative term. We find the structure of the solution near the leading front of the wave. We study in detail the case, which corresponds to a plasma wave, of damping proportional to $|k|$. We construct a stationary solution for that case.

1. INTRODUCTION

Whitham[1] developed an efficient method for describing nonlinear waves in dispersive media. The idea of his method consists in first constructing a rigorous solution of the corresponding equation. Afterwards the arbitrary constants occurring in that solution are assumed to be slowly varying functions of the coordinates and the time. For those functions approximate equations are derived. In Refs. 2 and 3 the Whitham method was generalized to the case of weak dissipation and was used to describe the evolution of nearly periodic solutions of the Korteweg–de Vries—Burgers equation:

$$\partial u/\partial t+u(\partial u/\partial x)+\partial^3 u/\partial x^3=\nu(\partial^2 u/\partial x^2). \tag{1}$$

Equation (1) usually describes correctly weak nonlinearity and dispersion. However, in many cases of practical importance the damping is not described by the "Burgers" right-hand side of (1) even in the long-wavelength limit. In media with dispersion the weak dissipation is, in general, nonlocal. In the present paper we consider an equation of the form

$$\partial u/\partial t+u(\partial u/\partial x)+\partial^3 u/\partial x^3=-\nu\int L(x-x')u(x')dx' \tag{2}$$

with a real integral kernel L and a small coefficient ν. [When writing down (2) we assumed the medium to be uniform.]

In fact the kernel L is given by its Fourier transform $L(k)$:

$$L(k)=\int L(x)\exp(-ikx)\,dx, \tag{3}$$

which determines in the linear approximation the damping of a wave with wave vector k:

$$\gamma=-\mathrm{Im}\,\omega=\nu L(k).$$

Hence it follows that $\nu L(k)>0$ and $L(k)$ is an even function of k. Equation (1) corresponds to $L(k)=k^2$. For the important case of ion-acoustic waves in a rarefield plasma with different temperatures ($T_e \gg T_i$) when the dissipation is determined by the Landau damping by electrons we have

$$L(k)=|k|. \tag{4}$$

[Here $\nu=(z/M)(\pi m T_e/8)^{1/2}$, where z is the ion charge and M and m are the ion and electron masses, respectively.] The same k-dependence of the damping of sound occurs in dielectrics at low temperatures (see Ref. 4, § 73).

The nature of the equations depends significantly on the behavior of L for low k. If $L(k=0)\neq 0$ the motion at any scale is damped after a finite time. Such a situation occurs, for instance, for the motion of the ionized component in a partially ionized plasma when the ion density is small as compared to the neutral particle density. We shall assume that, as is the case for (1) and (4),

$$L(k=0)=0. \tag{5}$$

2. AVERAGED EQUATIONS

We recall briefly of some properties of the periodic solutions of the Korteweg–de Vries equation

$$\partial u/\partial t+u(\partial u/\partial x)+\partial^3 u/\partial x^3=0, \tag{6}$$

i.e., Eq. (2) without a right-hand side.

For functions u depending on x and t in the combination $x-Ut$ Eq. (6) reduces to an ordinary differential equation which has the first integral:

$$u_{xx}=6B+Uu-u^2/2, \tag{7}$$

which after multiplication with u_x and integration takes the form

$$u_x^2=-72A+12Bu+Uu^2-u^3/3\equiv{}^1/_3(6\alpha-u)(u-6\beta)(u-6\gamma), \tag{8}$$

where A, B, U, α, β, and γ are constants and

$$A=-\alpha\beta\gamma,\ B=-(\alpha\beta+\alpha\gamma+\beta\gamma),\ U=2(\alpha+\beta+\gamma),\ \alpha\geqslant\beta\geqslant\gamma.$$

Instead of α, β, and γ we shall use their combinations r_α:

$$r_1=3(\beta+\gamma),\ r_3=3(\alpha+\gamma),\ r_3=3(\alpha+\beta),\ r_3\geqslant r_2\geqslant r_1.$$

The periodic solution of Eq. (6) has the form

$$u(x,\ t)=(2a/s^2)\mathrm{dn}^2(y,\ s)+U-(2a/3s^2)(2-s^2), \tag{9}$$

where $\mathrm{dn}(y,\ s)$ is a Jacobi function of modulus s, $y=(a/6s^2)^{1/2}(x-Ut)$, and the amplitude a, the phase velocity U, and the modulus s can be expressed in terms of the r_α as follows:

$$a=r_2-r_1,\ s^2=(r_2-r_1)/(r_3-r_1),\ U={}^1/_3(r_1+r_2+r_3).$$

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The role of the adiabatic invariant for Eq. (7) is played by the function

$$W=-\frac{1}{36}\oint u_x\,du.$$

The wavelength λ and the average values over a period of the oscillations can be expressed in terms of derivatives of W ($\varkappa\equiv 1/\lambda$ is the wave number):

$$\lambda=(\partial W/\partial A)_{B,U}\equiv W_A,\quad \bar{u}=-6\varkappa W_B,\quad \overline{u^2}=-72\varkappa W_U. \tag{10}$$

We also note a relation between the derivatives of W:

$$UW_B=BW_A+6W_U. \tag{11}$$

The Whitham method consists in averaging over the period of the waves of the three relations expressing the conservation laws for the initial equation. One of these relations is the initial equation (2) itself. However, if we average (2), the average of the right-hand side vanishes by virtue of (5) since the left-hand side of (5) is also such an average. (The "correction" appearing in the next approximation is small, since ν is small.) We therefore obtain as a result exactly the equation obtained in Ref. 1 for the Korteweg–de Vries equation, i.e., the first Eq. (14) from Ref. 2:

$$\partial\bar{u}/\partial t+{}^1/_2\,(\overline{\partial u^2/\partial x})=0. \tag{12}$$

To obtain the second equation we must multiply Eq. (2) by u and average. After simple calculations we find

$$\frac{\partial}{\partial t}\left(\frac{\overline{u^2}}{2}\right)+\frac{\partial}{\partial x}\left(\frac{\overline{u^2}}{3}-\overline{uu_{xx}}-\frac{\overline{u_x^2}}{2}\right)=36\nu\varkappa Z, \tag{13}$$

where we have introduced the notation

$$Z=-{}^1/_{36}\oint u(x)L(x-x')u(x')\,dx\,dx'. \tag{14}$$

We note that the quantity Z has the meaning of the dissipative function of the system.

Finally, we can choose as the last equation the "conservation law for the number of waves"[1] which is, of course, satisfied also when there is damping present:

$$\partial\varkappa/\partial t+\partial\varkappa U/\partial x=0. \tag{15}$$

Now expressing the average values in terms of W using (10) and (11) we are finally led to a set of equations which differ from the set (17) of Ref. 2 only by the substitution of W by Z:

$$\begin{gathered}DW_A/Dt-W_A(\partial U/\partial x)=0,\\ DW_B/Dt-W_A(\partial B/\partial x)=0,\\ DW_U/Dt-W_A(\partial A/\partial x)=-\nu Z.\end{gathered} \tag{16}$$

[The quantity Z is determined by Eq. (14), $D/Dt=\partial/\partial t+U(\partial/\partial x)$.]

For many applications we must change to the "Riemann variables" r_α. It is clear that the equations for the r_α differ from the corresponding Whitham equations only in the presence of right-hand sides:

$$\partial r_\alpha/\partial t+v_\alpha(\partial r_\alpha/\partial x)=\nu\rho_\alpha. \tag{17}$$

According to what we have said, we can obtain the expressions for the ρ_α from Eqs. (19) of Ref. 2 by the same substitution of W by Z:

$$\rho_1=-\frac{3Z}{2(K-E)}\left(\frac{6s^2}{a}\right)^{1/2},\quad \rho_2=\frac{3Z}{2(E-s'^2K)}\left(\frac{6s^2}{a}\right)^{1/2},$$
$$\rho_3=\frac{3Z}{2E}\left(\frac{6s^2}{a}\right)^{1/2}. \tag{18}$$

Here $K=K(s)$ and $E=E(s)$ are complete elliptical integrals; expressions for the group velocities v_α are given in Ref. 2, $s'^2=1-s^2$.

Equations (16) to (18) solve the problem we have posed.

We note that the averaged equations (16) and (17) can in principle be applied also for the description of the dynamics of growing waves in unstable media when $\nu L(k)>0$. However, in the present paper we shall not consider this case.

3. CALCULATION OF THE DISSIPATIVE FUNCTION Z

For the actual calculation of the quantity Z we must change in Eq. (14) to Fourier components. First of all we expand the periodic solution (9) of the Korteweg-de Vries equation in a Fourier series. This solution is a periodic function of y of period $\lambda=1/\varkappa=2\pi/k_1$. The wave vector k_1 of the basic period is equal to

$$k_1=\left(\frac{a}{6s^2}\right)^{1/2}\frac{\pi}{K(s)}. \tag{19}$$

The function (9) is an even real function of y and we can expand it in a Fourier series of the form

$$u(y)=\bar{u}+\frac{2a}{s^2}\sum_{n=1}A_n\cos(k_1ny) \tag{20}$$

($\bar{u}$ is the average value of u over a period). The problem is reduced to expanding the function $\mathrm{sn}^2(y)=[1-\mathrm{dn}^2(y)]/s^2$. Introducing a new variable $\Phi=k_1y$ we have:

$$\mathrm{sn}^2(\Phi/k_1)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\Phi), \tag{21}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{sn}^2(\Phi/k_1)\exp(in\Phi)\,d\Phi,$$
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{sn}^2(\Phi/k_1)\,d\Phi. \tag{22}$$

The calculations of the a_n coefficients are similar to those carried out for the expansions in Fourier series of the function $\mathrm{sn}(y)$ (see Ref. 5, § 22.6); we note that Eq. (8.146.26) for the series (21) in Ref. 6 is incorrect. For the calculation of the a_n we must consider an integral of the form (22) along a contour in the complex plane of the Φ variable formed by a parallelogram with vertices $(-\pi,\ \pi,\ 2i\tau,\ -2\pi+2i\tau)$. Here and in what follows

$$\tau=\pi K(s')/K(s). \tag{23}$$

The integrals over the lateral sides of the parallelogram cancel one another because of the periodicity of the integrand. The integral over the upper side is equal to the required integral over the lower side, multiplied by $[-\exp(-2n\tau)]$. On the other hand, the integral over the contour can be expressed in terms of the residue in the second-order pole at $\Phi=i\tau$ lying inside the contour. After some simple calculations we get

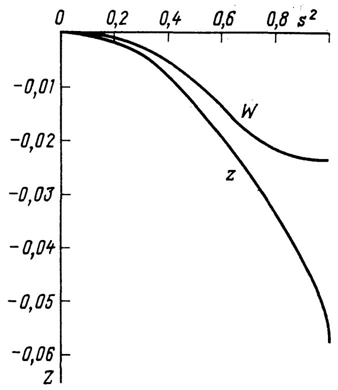

FIG. 1. The dissipative function.

$$a_n = -n(\pi/sK)^2/[\mathrm{sh}\,(n\tau)],$$
$$a_0 = (K-E)/s^2K. \tag{24}$$

Comparing this with (20) we find

$$A_n = n(\pi/K)^2/[\mathrm{sh}\,(n\tau)], \quad \bar{u} = \gamma + 2aE(s)/K(s)s^2. \tag{25}$$

We can now evaluate the function Z. We substitute the series (20) into (14) and use (3) and (5) and also the fact that $L(k)$ is an even function of its argument. As a result we get an expression for Z which is very convenient for numerical calculations:

$$Z = -4(a/6s^2)^{3/2}K(s)\sum_{n=1}^{\infty} A_n^2 L(nk_1). \tag{26}$$

4. STATIONARY SHOCK WAVE

Stationary solutions of Eqs. (16) which describe a shock wave in media with dispersion and weak damping were found in Refs. 2 and 3. (This kind of solution was first considered in Ref. 7.) The dissipation was considered there to be a "Burgers" one. The method developed here enables us to solve this problem for any form of weak dissipation.

We first of all note that the general properties of the solution in an arbitrary case are close to the properties of the solution obtained in Refs. 2 and 3 for the averaged equation (1). Indeed, the stationary solution of Eqs. (16) depends on $x - Ut$ with a constant U. Since the first two equations (16) did not change their form, the quantity B also turns out to be constant. For a single discontinuity we have, as before,[2]

$$B = 0, \quad U = 1/2, \tag{27}$$

where the amplitude at the leading front is $a = \frac{3}{2}U = \frac{3}{4}$. Because of this the way the amplitude a of the wave, the wavenumber $\varkappa$, the average value $\bar{u}$, and the "dispersion" $S = \overline{u^2} - \bar{u}^2$ depend on the parameter s^2 is described as before by Eq. (27) from Ref. 2. (See also Fig. 1a from Ref. 2.)

The coordinate-dependence of the remaining parameter A on the other hand is given by the equation

$$\frac{dW}{dx} = \nu Z, \tag{28}$$

which requires numerical integration. We note that in the Burgers case $Z = W$ and (28) has a simple analytical solution [Eqs. (24) and (28) from Ref. 2].

The function $Z(s^2)$ for the case (4) and the analogous function $W(s^2)$ for the Burgers case (1) are shown in Fig. 1. When $s^2 \to 0$ both functions behave similarly:

$$W(s^2) \approx -\frac{\pi s^4}{64 \cdot 2^{1/2}}, \quad Z(s^2) \approx -\frac{\pi s^4}{64},$$

whereas when $s^2 \to 1$ they behave significantly differently: $Z(s^2)$ has as $s^2 \to 1$ a singularity in its derivative which is connected with the nonanalyticity of the function $L(k)$ as $k \to 0$.

We study the behavior of $Z(s^2)$ as $s^2 \to 1$. For the case (4) considered here Eq. (26) takes the form

$$Z(s^2) = -\frac{\pi^5}{16K^4(1-s^2-s^4)}\sum_{n=1}^{\infty}\frac{n^3}{\mathrm{sh}^2[n\pi K(s')/K(s)]},$$
$$s'^2 = 1 - s^2. \tag{29}$$

Using the Poisson formula we can transform the sum (29):

$$\sum_{n=1}^{\infty} F(n) = -1/2F(0) + \int_0^{\infty} F(x)\,dx + 2\sum_{l=1}^{\infty}\int_0^{\infty} F(x)\cos(2\pi lx)\,dx, \tag{30}$$

where in our case

$$F(n) = n^3/\mathrm{sh}^2\,(an), \quad a = \pi^2 K(s')/K(s) \approx \pi^2/\ln^2\,[16/(1-s^2)]. \tag{31}$$

Since the parameter $a \to 0$ as $s^2 \to 1$ we introduce instead of x the variable $w = ax$. We then have from (29) to (31)

$$Z(s^2) \approx -\frac{1}{\pi^3}\Big[\int_0^{\infty} w^3\,\mathrm{sh}^{-2}(w)\,dw + 2\sum_{l=1}^{\infty}\int_0^{\infty} w^3\,\mathrm{sh}^{-2}w\cos(2\pi lw/a)\,dw\Big]. \tag{32}$$

Since as $a \to 0$ the function $\cos(2\pi lw/a)$ oscillates rapidly, the main contribution to the integrals under the summation sign comes from small w. Therefore

$$J_l \equiv \int_0^{\infty} w^3\,\mathrm{sh}^{-2}w\cos(2\pi lw/a)\,dw \approx \int_0^{\infty} w\cos(2\pi lw/a)\,dw = -\frac{a^2}{4\pi^2 l^2}. \tag{33}$$

(To give a meaning to the integral we must represent the cosine as a sum of two exponents and carry out in each term the appropriate rotation of the contour in the complex plane.)

Summing now the series (32) we finally get as $s^2 \to 1$

$$Z(s^2) \approx -\frac{3}{2\pi^3}\zeta(3) + \frac{\pi}{12}\left(\ln\frac{16}{1-s^2}\right)^{-2}. \tag{34}$$

It is clear that the derivative of the function $Z(s^2)$ becomes infinite as $s^2 \to 1$.

In the Burgers case [Eq. (1)] there occur the quantities n^4 and w^4 in the sum (26) and correspondingly in the integral (27). Performing the same calculations we find as $s^2 \to 1$

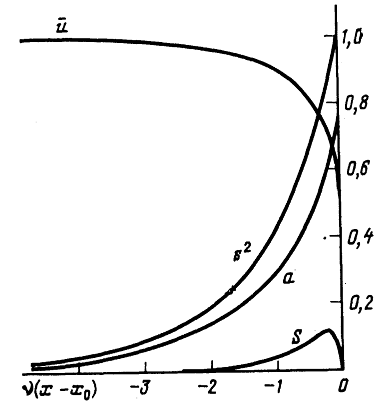

FIG. 2. The distribution of averages in a stationary shock front.

$$W(s^2)\approx-\frac{1}{30\cdot 2^{1/2}}\left[1-\frac{15}{32}(1-s^2)^2\ln\frac{16}{1-s^2}-\frac{31}{64}(1-s^2)^2\right].$$

(35)

In that case the singularity turns out to be weaker and the derivative remains finite. One checks easily that this is a manifestation of a general statement: the derivative of the function $Z(s^2)$ remains finite if $L(k)\propto k^{2n}$ for small k and becomes infinite when $L(k)\propto k^{2n+1}$.

We now use the dissipative function found here for the case (4). The result of integrating Eq. (28) which determines the change of the parameter s^2 in the shock wave as a function of $\nu(x-x_0)$ is shown in Fig. 2. We also show there the behavior of other average quantities. In Fig. 3 we show

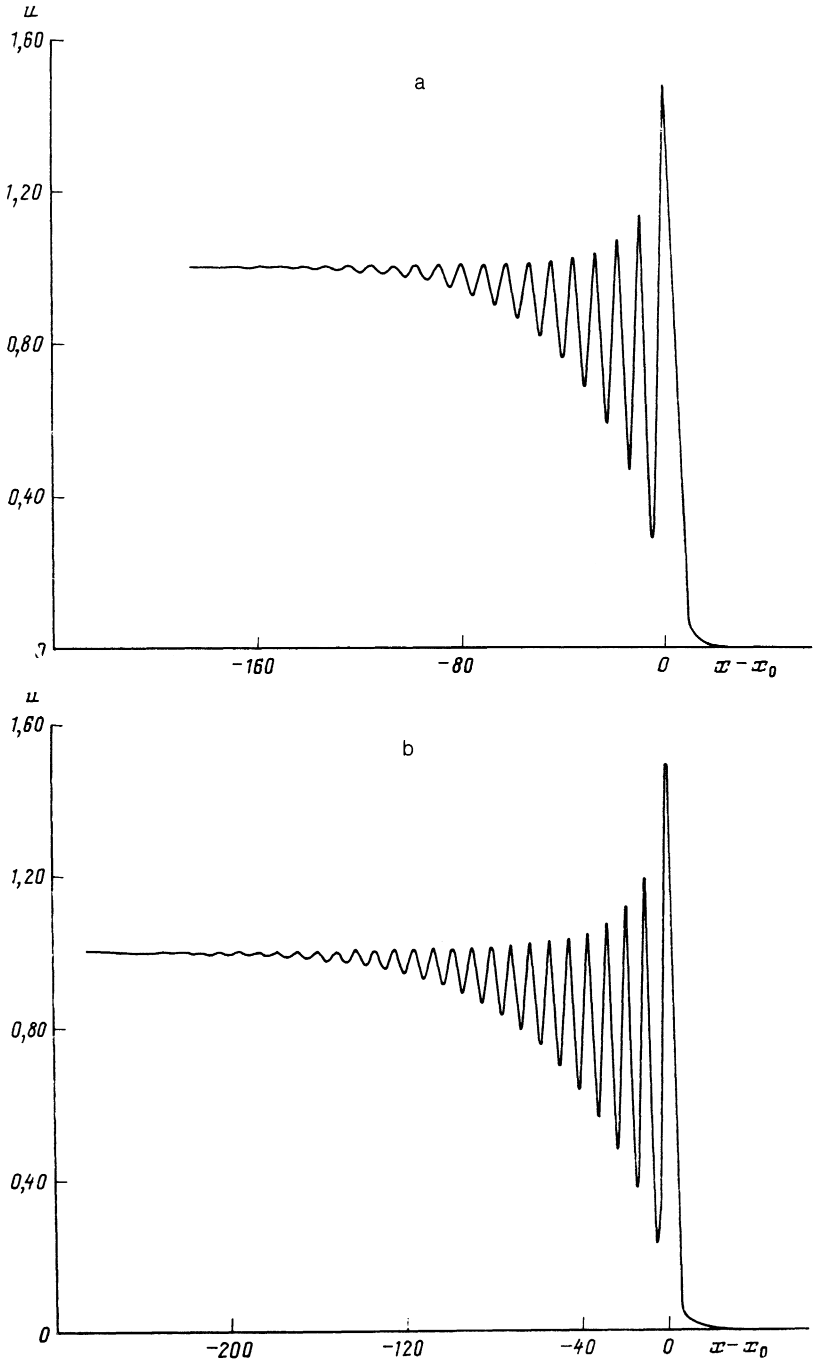

FIG. 3. Oscillatory structure of a stationary shock wave for $\nu=0.05$. a: for the case (6); b: for the case (1).

the structure of the shock wave. The oscillations at the leading front have the shape of solitons of height $\frac{3}{2}$, which corresponds to the already mentioned equation $a = \frac{3}{4}$. This is a general result which is independent under condition (5) of the actual form of the dissipative term. The solitons are sufficiently sharply pronounced for small values of $\nu \leqslant 10^{-4}$. The distance between the first and the second soliton is $\lambda_0 \approx (3/2^{1/2})(\ln(\nu^{-1}) + 4)$. When we go away from the leading front the oscillations diminish in amplitude and become sinusoidal in character. Their amplitude then decreases exponentially:

$$a = C_0 \exp[-\nu(x-x_0)/2^{1/2}], \quad C_0 \sim 1. \tag{36}$$

It is clear that the argument of the exponential asymptotically differs by a factor $2^{1/2}$ from the Burgers case. This leads (for the same ν) to a faster damping of the oscillations. The wavelength, however, remains the same $\lambda_0 = 2^{1/2}\pi$.

The problem of the establishment of the stationary solution constructed here is of interest. If the shock wave is formed as the result of the decay of an initial jump in the density (see Ref. 8), the reaching of the stationary picture starts from the leading edge where the dissipation is particularly important. With increasing time the region of the stationary solution propagates ever further backwards along the oscillating region, but sufficiently close to the trailing edge there is always a nonstationary region where the dissipation is unimportant.

5. BEHAVIOR OF THE SOLUTION NEAR THE LEADING FRONT

An important feature of Eqs. (17) for the quantities r_α is the presence in their solutions of singular points—the leading front and trailing edge,[8] which bound the oscillation region. At the trailing front the amplitude of the oscillations becomes zero. In that point the conservation law (9) for the number of waves is violated; it is a source for new waves (see Refs. 2 and 10). Near the trailing edge, though, the dissipation is unimportant. Its structure is described therefore by the formulae obtained in Ref. 8.

On the leading front $s^2 \to 1$ and the waves split up into separate solitons. The structure of the leading front depends on the damping. The way s becomes unity for Eq. (1) was elucidated in Ref. 2. We determine this behavior for the general Eq. (2).

First of all we note, by somewhat generalizing the derivation of Eq. (35) of Ref. 2, that the behavior of s near the leading-front point $x = x^+(t)$ is given by the formula $(s'^2 = 1 - s^2)$

$$s'^4[\ln(16/s'^2) + {}^1/_2] = -(12/a^2)[(dr^+/dt) - \nu\rho_3^+](x - x^+),$$
$$\rho_3^+ = \rho_3 \ (s=1). \tag{37}$$

The quantity ρ_3^+ can in turn be expressed in terms of the quantity $Z^+ = Z(s^2 = 1)$ at the leading front. Using (18) we have

$$\rho_3^+ = {}^3/_2 (6/a)^{1/2} Z^+. \tag{38}$$

Using Eq. (34) for $Z(s^2 \to 1)$ we get finally:

$$\rho_3^+ = -(144/\pi^3)(a/6)^{3/2}\zeta(3), \quad \zeta(3) \approx 1{,}202. \tag{39}$$

In the case (1), however, when $L(k) = k^2$ we can easily obtain Eq. (34) of Ref. 3.

In conclusion we note that the asymptotic expression (37) is necessary as the natural boundary condition for the numerical integration of the Whitham equations (see Refs. 3 and 9).

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