Wave propagation and scattering in media with anomalously large-scale random inhomogeneities

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Distinctive features of the propagation and scattering of waves in a randomly inhomogeneous medium are analyzed for the case in which anomalously large-scale fluctuations in the medium have a correlation function with a singular spectrum $\propto 1/q^{\gamma}$ in the limit $\mathbf{q}\to 0$. The exactly solvable model of an instantaneously homogeneous medium is analyzed as a limiting case. The asymptotic behavior of the mean field in this case is a superexponential attenuation $\propto \exp(-\text{const}\cdot r^2)$. A simple method is proposed for calculating the asymptotic behavior of the mean field. This method generalizes the Bourret approximation to the case of singular correlation functions, for which the total scattering cross section diverges in the Born approximation. It leads to a superexponential attenuation at $\gamma \geqslant 2$. Particular features of the angular distribution of the scattering intensity which stem from multiple scattering by anomalously large-scale inhomogeneities are discussed. The scattering function is studied as a function of the path length traversed by the wave in a random medium. A new method is proposed for calculating the intensity of multiple small-angle scattering. This method is free of ultraviolet divergences. An expression derived here for the total cross section for multiple scattering is valid in the case of a superexponential attenuation of the mean field.

The propagation and scattering of waves in randomly inhomogeneous media have been studied quite thoroughly.[1] In particular, it has been shown by field-diagram methods that, under some fairly general assumptions, the mean field falls off exponentially, $\propto \exp(-\tau r)$, at large distances r because the deterministic component becomes random. In the case of weak scattering the corresponding attenuation index τ (the extinction coefficient) can be calculated in the lowest, single-loop approximation (the Bourret approximation[2]). On the other hand, there are systems in which the scattering is weak but the extinction coefficient calculated in this manner turns out to be infinite.[3] This is true in particular of all degenerate systems which have Goldstone fluctuations (with a singular spectrum $\propto 1/q^2$; Ref. 4). This "infrared" divergence of the extinction coefficient for correlation functions of the Goldstone type was recognized a long time ago (see, for example, the first edition of Landau and Lifshitz' book[3]) in the particular case of the scattering of light at the point of a second-order phase transition, in which case the correlation radius is infinite. An attempt[5] to eliminate this divergence through the use of the nonlinear Kraichnan approximation[6] turned out to be unsatisfactory:[7] A systematic analysis shows that the divergence exists even in this approximation. A summation of the infrared divergences of the diagrams showed[8] that the asymptotic behavior of the mean field in this case is superexponential: $\propto \exp[-\tau r \ln(\text{const}\cdot r)]$. This result was subsequently derived[9] by a simpler method: through the use of various types of perturbation theories for small- and large-scale inhomogeneities (a mode separation method).

We now know of a number of systems in which the fluctuation spectrum has a $1/q^{\gamma}$ singularity in the limit $\mathbf{q}\to 0$. The case $\gamma = 2$, for example, is realized for transverse fluctuations in nematic liquid crystals[10] and magnetic materials,[3] while the case $\gamma = 1$ is realized for longitudinal fluctuations in these systems.[4] The case $\gamma = 2 - \eta$, where the Fisher index η depends on the dimensionality of the space and the number of components in the order parameter, is typical of fluctuations in the order parameter at the critical point. The relation $\eta > 0$ holds for most known systems, but cases with $\eta < 0$ are also being examined.

In the present paper we take a look at the particular features of the propagation and scattering of waves in randomly inhomogeneous media with anomalously large-scale fluctuations whose spectrum has a $1/q^{\gamma}$ singularity with $\gamma < 3$ in the limit $\mathbf{q}\to 0$. To analyze the effect of a singularity in the correlation function, we consider an exactly solvable model: wave propagation in a medium with extremely large-scale inhomogeneities [with a correlation function $\propto \delta(\mathbf{q})$]. The mean field in a medium of this sort decays in a superexponential fashion, $\propto \exp[-(\tau r)^2]$ (the Fourier spectrum of the exact propagator has an essential singularity at infinity but no other singularities in the plane of the complex variable q; i.e., it is an entire function). We propose a method for calculating the mean field in randomly inhomogeneous media which does not require separation of modes. It leads to the prediction of a superexponential attenuation for singular correlation functions with $\gamma \geqslant 2$. It is shown that for finite systems this effect yields a dependence of the extinction coefficient on the longitudinal size of the sample. We analyze effects caused in the angular distribution of the intensity of the scattered waves by multiple scattering by anomalously large-scale inhomogeneities. We show that the scattering function has different regimes, depending on the distance (z) traversed by the wave in the medium. As z increases, this function changes in shape from a function corresponding to the scattering function in the Born approximation to a Gaussian angular distribution.

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1. GENERAL THEORY OF WAVE PROPAGATION IN A RANDOMLY INHOMOGENEOUS MEDIUM

We consider a scalar harmonic wave field $u(\mathbf{r},t) = u(\mathbf{r})\exp(-i\omega t)$. The amplitude $u(\mathbf{r})$ is described by the Helmholtz equation

$$[\Delta+k_0^2(1+\varphi(\mathbf{r}))]u(\mathbf{r})=0, \qquad (1.1)$$

where $k_0=\omega\varepsilon_0^{1/2}/c$ is the wave number, c is the wave propagation velocity, $\varphi(\mathbf{r})=\delta\varepsilon(\mathbf{r})/\varepsilon_0$, $\varepsilon(\mathbf{r})=\varepsilon_0+\delta\varepsilon(\mathbf{r})$ is the dielectric constant of the medium, $\delta\varepsilon(\mathbf{r})$ is its fluctuating part, and ω is the frequency. The random field $\varphi(\mathbf{r})$ is assumed to be Gaussian with a zero mean and with a correlation function $\psi(\mathbf{r}'-\mathbf{r}'')=\langle\varphi(\mathbf{r}')\varphi(\mathbf{r}'')\rangle$, where the angle brackets mean a statistical average.

In this section of the paper we are interested in calculating the mean Green's function of Eq. (1.1), i.e., $\langle G(\mathbf{r})\rangle$. The function $\langle G(\mathbf{r}'-\mathbf{r}'')\rangle$ satisfies the Dyson equation[1]

$$\langle G(\mathbf{r})\rangle=G_0(\mathbf{r})+\int G_0(\mathbf{r}-\mathbf{r}')\Sigma(\mathbf{r}'-\mathbf{r}'')\langle G(\mathbf{r}'')\rangle\,d\mathbf{r}'\,d\mathbf{r}'', \qquad (1.2)$$

where $G_0(\mathbf{r})=(4\pi r)^{-1}\exp(ik_0r)$ is the Green's function of Eq. (1.1) in the case $\varphi=0$, and $\Sigma(\mathbf{r}'-\mathbf{r}'')$ is the kernel of the mass operator. In diagram form, Σ is the sum of all the strongly coupled diagrams:

(1.3)

where a solid line corresponds to the Green's function G_0, and a dashed line to the correlation function ψ. A factor of k_0^2 is to be understood at the vertices in (1.3).

If the terms in series (1.3) fall off sufficiently rapidly, the Dyson equation can usually be solved in the Bourret approximation. In this case, the first term of the series (1.3) is used as Σ (Ref. 2):

$$\Sigma_{Bt}(\mathbf{r}'-\mathbf{r}'')=k_0^4G_0(\mathbf{r}'-\mathbf{r}'')\psi(\mathbf{r}'-\mathbf{r}''). \qquad (1.4)$$

On occasion, the Kraichnan approximation,[6]

$$\Sigma_{Kn}(\mathbf{r}'-\mathbf{r}'')=k_0^4\langle G(\mathbf{r}'-\mathbf{r}'')\rangle\psi(\mathbf{r}'-\mathbf{r}''), \qquad (1.5)$$

is also used. The latter approximation corresponds to a partial summation of the series (1.3). The procedure for determining $\langle G\rangle$ in this case reduces to solving a nonlinear integral equation.

For a known kernel Σ, using the Fourier-transformation formulas

$$f(\mathbf{q})=\int d\mathbf{r}\,f(\mathbf{r})e^{-i\mathbf{q}\mathbf{r}},\quad f(\mathbf{r})=\int\frac{q\mathbf{q}}{(2\pi)^3}f(\mathbf{q})e^{i\mathbf{q}\mathbf{r}},$$

we can easily solve Eq. (1.2):

$$\langle G(\mathbf{q})\rangle^{-1}=G_0^{-1}(\mathbf{q})-\Sigma(\mathbf{q}), \qquad (1.6)$$

where $G_0^{-1}(\mathbf{q})=q^2-k_0^2-i0$. Going back to the $\mathbf{r}$ representation, we find

$$\langle G(\mathbf{r})\rangle=\frac{1}{4\pi^2ir}\int_{-\infty}^{\infty}\frac{qe^{iqr}\,dq}{q^2-k_0^2-\Sigma(\mathbf{q})}. \qquad (1.7)$$

If we adopt the customary assumption of $\max|\langle G(\mathbf{q})\rangle|\to0$ as $|q|\to\infty$ in the complex plane, then by closing the integration contour by means of an infinite semicircle in the upper half-plane we find that the value of the integral in (1.7) is determined exclusively by the singularities of the function $\langle G(\mathbf{q})\rangle$ with $\operatorname{Im} q>0$. Let us consider some simple situations.

If $\langle G(\mathbf{q})\rangle$ has a first-order pole at the point $q=k_{\rm eff}=k'+ik''$, then

$$\langle G(\mathbf{r})\rangle=Z_P^{(1)}\frac{e^{ik'r}}{4\pi r}e^{-k''r}, \qquad (1.8)$$

where $Z_P^{(1)}=(1-\partial\Sigma/\partial k_{\rm eff}^2)^{-1}$. For the case of an nth-order pole, the asymptotic behavior of $\langle G(\mathbf{r})\rangle$ at large distances is

$$\langle G(\mathbf{r})\rangle\sim Z_P^{(n)}\frac{e^{ik'r}}{4\pi r}r^{n-1}e^{-k''r}, \qquad (1.9)$$

where $Z_{\rm Br}^{(n)}=2nk_{\rm eff}i^{n-1}/(2\delta_{2n}-\partial^n\Sigma/\partial k_{\rm eff}^n)$, $n\geqslant2$, and δ_{mn} is the Kronecker delta.

If $\langle G(\mathbf{q})\rangle$ has a branch point of the type $(q-k_{\rm eff})^{-1+\nu}\ln^\mu(q/k_{\rm eff}-1)f_0(q)$, where $f_0(q)$ is a smooth function near $q=k_{\rm eff}$, then we can write

$$\langle G(\mathbf{r})\rangle\sim Z_{\rm Br}\frac{e^{ik'r}}{4\pi r}\frac{\ln^\mu(k_{\rm eff}r)}{r^\nu}e^{-k''r}, \qquad (1.10)$$

where $Z_{\rm Br}=2k_{\rm eff}f_0(k_{\rm eff})\exp[-i\pi(\mu+1/2\nu)]/\Gamma(1-\nu)$, and $\Gamma(x)$ is the gamma function.

If the function $\langle G(\mathbf{q})\rangle$ has several finite poles and branch points $q=k'+ik''$, the behavior of $\langle G(\mathbf{q})\rangle$ at large distances is dominated by the singularity with the smallest imaginary part k''.

To find the positions of the poles (and also the positions of the branch points in the case $\nu<1$ or $\nu=1$ and $\mu>0$), $q=k_{\rm eff}$, we need to solve the equation

$$q^2-k_0^2-\Sigma(\mathbf{q})=0. \qquad (1.11)$$

If $\Sigma\ll k_0^2$, we find $k_{\rm eff}=k_0+\Sigma(k_0)/(2k_0)$ in first order in Σ. If we use the Bourret approximation, (1.4), for Σ, go over to the $\mathbf{q}$ representation, and note that we have

$$(p^2-k_0^2-i0)^{-1}=i\pi\delta(p^2-k_0^2)+\mathscr{P}(p^2-k_0^2)^{-1}$$

according to the Sokhotskiĭ formula, where $\mathscr{P}$ means the principal value of the integral, we find

$$k'=k_0+\frac{k_0^3}{16\pi^3}\int_0^\infty dp\frac{p^2}{p^2-k_0^2}\int d\Omega_{\mathbf{p}}\,\psi(\mathbf{k}_0-\mathbf{p}), \qquad (1.12)$$

$$k''=\frac{k_0^3\pi}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\psi(\mathbf{p})\delta[(\mathbf{k}_0-\mathbf{p})^2-k_0^2]. \qquad (1.13)$$

The quantity $2k''$ in this case is the same as the total scattering cross section calculated in the Born approximation, σ_B.

2. MODEL OF AN INSTANTANEOUSLY HOMOGENEOUS MEDIUM

Let us consider wave propagation in a random medium with a correlation function

$$\psi(\mathbf{q})=(2\pi)^3a^2\delta(\mathbf{q}),\quad \psi(\mathbf{r})=a^2,$$

in which case all points of the medium are correlated identically strongly, regardless of the distances involved (an "instantaneously homogeneous medium"). In this model it is possible to construct, in closed form, an exact solution for $\langle G(\mathbf{r})\rangle$ as well as solutions in the Bourret approximation, $G_{Bt}(\mathbf{r})$, and the Kraichnan approximation, $G_{Kn}(\mathbf{r})$.

In the Bourret approximation, (1.4), the kernel of the mass operator is

$$\Sigma_{Bt}(\mathbf{q})=a^2k_0^4G_0(\mathbf{q}). \qquad (2.1)$$

According to (1.6), the Green's function $\langle G(\mathbf{q})\rangle$ with this kernel has four first-order poles $k_{\text{eff}} = \pm k_0(1\pm a)^{1/2}$, of which two contribute to $\langle G(\mathbf{r})\rangle$ by virtue of the circumvention rule. These two are $k_{1,2P} = k_0(1\pm a)^{1/2}$. As a result, we find[11] from (1.8)

$$G_{Bt}(\mathbf{r})=(8\pi r)^{-1}\sum_{j=1,2}\exp(ik_{jP}r). \qquad (2.2)$$

The quantity $\operatorname{Im}\Sigma_{Bt}(\mathbf{k}_0)$, which corresponds to the total scattering cross section in the Born approximation, (1.13), is infinite in the case of kernel (2.1). On the other hand, $G_{Bt}(\mathbf{r})$ in (2.2) has no anomalies. The reason is that in deriving (2.2) we calculated not $\Sigma_{Bt}(\mathbf{k}_0)$ but the quantity $\Sigma_{Bt}(\mathbf{k}_{\text{eff}})$, without ignoring the distinction between k_{eff} and k_0. Incorporating this distinction thus eliminates the infrared divergence in the extinction coefficient in the Bourret approximation.

In the Kraichnan approximation, we have

$$\Sigma_{Kn}(\mathbf{q})=a^2k_0^4\langle G(\mathbf{q})\rangle \qquad (2.3)$$

in our model, and a quadratic equation arises in the determination of $G_{Kn}(\mathbf{q})$:

$$G_{Kn}^{-1}(\mathbf{q})=q^2-k_0^2-a^2k_0^4G_{Kn}(\mathbf{q}). \qquad (2.4)$$

The quantity $\Sigma_{Kn}(\mathbf{k}_0)$ is finite and equal to iak_0^2 here (when the circumvention rule is taken into account). In other words, in this case the Kraichnan approximation makes it possible to eliminate the divergence in the extinction coefficient (cf. Ref. 5). On the other hand, the quantity $\Sigma_{Kn}(\mathbf{k}_{\text{eff}})$ becomes infinite for the value of k_{eff} given by Eq. (1.11) (cf. Ref. 7). It is, however, possible to find the exact asymptotic behavior of $G_{Kn}(\mathbf{r})$ by analogy with (2.2).

Solving Eq. (2.4), we find that $G_{Kn}(\mathbf{q})$ has four branch points in the form of roots $k_{\text{eff}} = \pm k_0(1\pm 2a)^{1/2}$. Replacing ε_0 by $\varepsilon_0 + i0$, we find that we are left with two of these branch points in the upper half-plane: $k_{1,2\text{Br}} = k_0(1\pm 2a)^{1/2}$. If these points are far enough apart, and the condition $|k_{1B} - k_{2B}|r \gg 1$ holds, then by using (1.10) with $\nu = \frac{3}{2}$ and $\mu = 0$ we find (cf. Ref. 12)

$$G_{Kn}(\mathbf{r})\sim -(2\pi a)^{-1/2}k_0^{-9/4}r^{-3/2}\times\sum_{j=1,2}k_{\text{Br}}^{3/4}\exp[i(k_{\text{Br}}r+(-1)^j\pi/4)]. \qquad (2.5)$$

In this model we thus do not have the customary exponential-attenuation factor $\exp(-k''r)$ in the Bourret and Kraichnan approximations. For $a\ll 1$, each of the models exhibits beats, and an "algebraic" attenuation $r^{-3/2}$ is seen in the Kraichnan approximation, (2.5). We are assuming here that we have singled out a factor of r^{-1} in (2.5); for the Green's function in three-dimensional space, this factor is purely geometric and is unrelated to the attenuation. The derivation of an exact solution in this model is based on the circumstance that all the integration operations are carried out explicitly in the diagram series for $\langle G\rangle$ with the correlation function $\psi(\mathbf{q}) \propto \delta(\mathbf{q})$. The result is

$$\langle G(\mathbf{q})\rangle = G_0(\mathbf{q})\sum_{n=0}^{\infty}(2n-1)!![G_0^2(\mathbf{q})k_0^4a^2]^n. \qquad (2.6)$$

In deriving this result, we noted that in our case all the $(2n-1)!!$ diagrams of nth order contribute identically to $(G_0^2(\mathbf{q})k_0^4a^2)^nG_0(\mathbf{q})$. Series (2.6), a power series in $G_0^2k_0^4a^2$, has a zero convergence radius. If we instead look at it as an asymptotic series and use the identity

$$(2\pi)^{1/2}(2n-1)!!=\int_{-\infty}^{\infty}dx\exp(-x^2/2)x^{2n},$$

we can sum this series (cf. Refs. 13–15):

$$\langle G(\mathbf{q})\rangle=\frac{1}{(2\pi)^{1/2}a}\int_{-\infty}^{\infty}\frac{\exp(-x^2/2a^2)}{q^2-k_0^2-xk_0^2-i0}dx. \qquad (2.7)$$

Then taking inverse Fourier transforms, we find (cf. Refs. 11 and 13)

$$\langle G(\mathbf{r})\rangle=\frac{1}{(2\pi)^{1/2}a}\int_{-\infty}^{\infty}dx\frac{1}{4\pi r}\exp[ik_0r(1+x)^{1/2}]\exp\left(\frac{-x^2}{2a^2}\right). \qquad (2.8)$$

Expression (2.8) can also be derived on the basis of some almost obvious considerations. The idea is that a model of a medium with $\psi(\mathbf{q}) = (2\pi)^3a^2\delta(\mathbf{q})$ corresponds to a situation in which the probability for any inhomogeneous fluctuation is zero, and only homogeneous fluctuations are possible: $\varphi = \varphi_{\mathbf{q}=0}$. In the Gaussian case, the state distribution function, $x = \varphi_{\mathbf{q}=0}$, is

$$\rho(x)=[(2\pi)^{1/2}a]^{-1}\exp(-x^2/2a^2). \qquad (2.9)$$

Now noting that in this case the expression

$$G(\mathbf{r})=(4\pi r)^{-1}\exp[ik_0r(1+x)^{1/2}] \qquad (2.10)$$

determines the Green's function of an arbitrary unaveraged state with $\varepsilon = \varepsilon_0(1+\varphi)$, and taking an average of (2.10) with the distribution function (2.9), we find (2.8).

To find the asymptotic behavior of the integral in (2.8) at $k_0r\gg 1$, we use the method of steepest descent. Making the substitution $(1+x)^{1/2}=t$, we find the cubic equation $t_*^3 + t_* + k_0ra^2/2 = 0$ for the saddle points. It follows from the form of our integration contour in the t plane that the asymptotic behavior of (2.8) is determined by the root t_* for which the relations $3\pi/2 < \arg t_* < 5\pi/3$ hold. A representation uniform with respect to the parameter a^2 is

$$\langle G(\mathbf{r})\rangle\sim Z(r)(4\pi r)^{-1}\exp[i\phi(r)-h(r)], \qquad (2.11)$$

where

$$Z(r)=[(1+x_0^2)(1+3x_0^2)^{-1/4}(1+{}^3/_4x_0^2)^{-1/2}]^{1/2},$$

$$\phi(r)=k_0r\frac{(1+{}^3/_4x_0^2)^{3/2}}{1+x_0^2}-\phi_0,\quad h(r)=\frac{k_0rx}{4}\frac{1+{}^3/_2x_0^2}{1+x_0^2},$$

$$\phi_0=\operatorname{arctg}\frac{{}^1/_2x_0(5+3x_0^2)}{(1+x_0^2)(1+3x_0^2)^{1/2}+(1-x_0^2)(1+{}^3/_4x_0^2)^{1/2}}$$

$$-\operatorname{arctg}\frac{x_0(1+{}^3/_4x_0^2)^{1/2}}{1+{}^1/_2x_0^2},\tag{2.12}$$

and x_0 is the real root of the equation $x_0^3+x_0-k_0ra^2/2=0$. In particular, if $k_0ra^2\ll 1$, by retaining the first terms of the expansions of the quantities $Z(r)$, $\phi(r)$, and $h(r)$ in this small parameter, we find the approximate result

$$\langle G(\mathbf{r})\rangle\approx(4\pi r)^{-1}\exp[ik_0r-k_0^2r^2a^2/8].\tag{2.13}$$

Correspondingly, we find the following result for the mean field $\langle u(z)\rangle$ propagating along the z axis:

$$\langle u(z)\rangle\infty\exp[ik_0z-k_0^2z^2a^2/8].\tag{2.14}$$

In the opposite case, with $k_0ra^2\gg 1$ in (2.11), we find

$$i\phi(r)-h(r)\sim 3a^{4/3}(k_0r)^{4/3}(i\cdot 3^{1/2}-1)/8\cdot 2^{1/3},\quad h(r)\gg 1,$$

and the field disappears almost completely.

We see that the asymptotic form of the exact solution is fundamentally different from the asymptotic behavior in the Bourret and Kraichnan approximations.[1)] There are no beats in the exact solution, while there is a factor which causes a superexponential attenuation. It follows from (1.8)–(1.10) that an attenuation law of the form (2.11)–(2.14) cannot be derived through an analysis of any simple singularities of $\langle G(\mathbf{q})\rangle$ which lie in the region of finite q; i.e., $\langle G(\mathbf{q})\rangle$ has a singularity only at $q=\infty$ in this problem. On the other hand, since $\langle G(\mathbf{q})\rangle$ must tend toward zero as q tends toward infinity along the real axis, by virtue of the convergence of the integral (1.7), we conclude that $q=\infty$ is an essential singularity for $\langle G(\mathbf{q})\rangle$ [cf. the function $w(z)$ in Ref. 16]. This conclusion demonstrates, in particular, that this model is not valid as an initial approximation for finding finite singularities of the spectrum of the Green's function in the Kraichnan approximation for more realistic correlation functions (Refs. 17 and 18, for example).

Let us examine the physical mechanism for the onset of the unusual superexponential attenuation in (2.11)–(2.14). The customary mechanism for the attenuation of the mean field, which involves the escape of scattered radiation off to the sides of the original wave propagation direction (along the z axis), leads to the Bouguer law[1] $\langle u\rangle\propto\exp(-\tau z)$. In this model, each instantaneous state of the medium is homogeneous, and there is no loss of radiation off to the sides; all the scattered waves propagate strictly forward. The resultant field is a superposition of waves with randomly shifted phases. This superposition gives rise to a non-Bouguer attenuation in our case. The attenuation of the mean field results from a transition of the coherent component of the field into a random component, propagating in the same direction.

3. GENERAL CASE OF A SUPEREXPONENTIAL ATTENUATION OF THE MEAN FIELD

In a medium with a correlation function different from $\delta(\mathbf{q})$, two damping mechanisms operate: the escape of radiation off to the sides and random phase shifts of the forward-scattered waves. If large-scale inhomogeneities constitute a sufficiently large fraction of the spectrum $\psi(\mathbf{q})$, so that the total scattering cross section becomes infinite in the Born approximation, the resultant attenuation law is something between $\exp(-\tau r)$ and $\exp[-(\tau r)^2]$. One would naturally assume that in this case $\langle G(\mathbf{q})\rangle$ would again—as in the case $\psi(\mathbf{q})\propto\delta(\mathbf{q})$—have a singularity only at $q=\infty$. The use of a resummation of the Dyson type (based on a classification of diagrams on the basis of their degree of coupling) for the perturbation-theory series is ineffective in this situation. That this is true can be seen simply from the circumstance that all the $(2n-1)!!$ diagrams of order n in (2.6) (both strongly and weakly coupled) turn out to be identical.

It follows from (2.8) and (2.10) that in order to find a superexponential attenuation of $\langle G(\mathbf{r})\rangle$ we need to deal correctly with the phase relations between the forward-scattered waves. It is not difficult to see that to find the asymptotic behavior of (2.13) and (2.14) it is sufficient to expand $(1+x)^{1/2}$ in a series and retain up to the linear term in the argument of the exponential function in (2.10). For the field $u(\mathbf{r})$ this approach corresponds to the approximation

$$u(\mathbf{r})\propto\exp(ik_0r+ik_0rx/2).\tag{3.1}$$

For a medium with arbitrary large-scale inhomogeneities, the corresponding correction to the phase of the plane wave, $u_0(\mathbf{r})=U_0\exp(ik_0z)$, can be found in the eikonal approximation:

$$u(\mathbf{r})\approx U_0\exp\left[ik_0z+\frac{ik_0}{2}\int_0^z\varphi(\rho,z')\,dz'\right].\tag{3.2}$$

To use expression (3.2) is to assume that the fluctuations are relatively small, $\varphi^2\ll 1$ (weak coupling). We accordingly restrict the analysis below to this case alone. The mean value of the field in (3.2) for a Gaussian random field φ is

$$\langle u(\mathbf{r})\rangle=U_0\exp\left[ik_0z-\frac{k_0^2}{2}\int\frac{d\mathbf{q}}{(2\pi)^3}q_\parallel^{-2}\sin^2\frac{q_\parallel z}{2}\psi(\mathbf{q})\right].\tag{3.3}$$

[We are using the notation z, $\boldsymbol{\rho}$ and $q_\parallel$, $\mathbf{q}_\perp$ for the components of the vectors $\mathbf{r}$ and $\mathbf{q}$ respectively along and across z; we are also using $\psi(\mathbf{r})=\psi(\boldsymbol{\rho},z)$, $\psi(\mathbf{q})=\psi(\mathbf{q}_\perp,q_\parallel)$.] A necessary condition for the applicability of the eikonal approximation is the inequality $l_\varphi\gg\lambda$, where l_φ is a length scale of the inhomogeneities, and $\lambda=2\pi/k_0$ is the wavelength. If there is a length scale r_m in the system which characterizes the spectral region $q\lesssim r_m^{-1}$ in which $\psi(\mathbf{q})$ is significantly nonzero, the corresponding applicability condition becomes $r_m\gg\lambda$.

Below we discuss singular correlation functions of the type

$$\psi(\mathbf{q})=a^2r_0^3(qr_0)^{-\gamma}f(qr_m),\tag{3.4}$$

where r_0 is a dimensional parameter, and the function $f(x)$ falls off rapidly at $x\gg 1$ (unless the opposite is stipulated) and has a value $f(0)\neq 0$. Here $0\leqslant\gamma<3$.

If the condition $z \ll r_m$ holds, we can use the substitution

$$q_\parallel^{-2} \sin^2(q_\parallel z/2) \approx z^2/4 \tag{3.5}$$

over the entire region in (3.3) which is important to the integration. We then find

$$\langle u(\mathbf{r})\rangle \approx U_0 \exp\left[ik_0 z - \frac{k_0^2 z^2}{8} \int \frac{d\mathbf{q}}{(2\pi)^3} \psi(\mathbf{q}) \right]. \tag{3.6}$$

If the condition $z \gg r_m$ holds instead, the asymptotic behavior of the mean field is

$$\langle u(\mathbf{r})\rangle \approx U_0 \exp\left[ik_0 z - H(z)\right], \tag{3.7}$$

where the function $H(z)$ is given in the limit $z \to \infty$ to within terms $\sim$const by

$$H(z) = \begin{cases} {}^1/_2\sigma_{\rm BE} z, & 0 \leqslant \gamma < 1 \\ {}^1/_2\sigma_{\rm BE} z - s_1 \ln(z/r_m), & \gamma = 1, \\ {}^1/_2\sigma_{\rm BE} z + s_2 (z/r_0)^{\gamma-1}, & 1 < \gamma < 2, \\ s_3 (z/r_0) \ln(z/r_m) + \sigma_1 z, & \gamma = 2, \\ s_2 (z/r_0)^{\gamma-1} + \sigma_2 z, & 2 < \gamma < 3. \end{cases} \tag{3.8}$$

Here

$$\sigma_{\rm BE} = \frac{k_0^2}{4} \int \frac{d\mathbf{q}_\perp}{(2\pi)^2} \psi(\mathbf{q}_\perp, 0),$$

$$\sigma_1 = \frac{\pi A_0}{r_0} \left[(C-1) f(0) - \int_0^\infty \ln x \frac{\partial f}{\partial x} dx \right],$$

$$\sigma_2 = \frac{k_0^2}{8} \int \frac{d\mathbf{q}_\perp}{(2\pi)^2} \left[\psi(\mathbf{q}_\perp, 0) - a^2 r_0^3 (q_\perp r_0)^{-\gamma} f(0) \right],$$

$$s_1 = 2A_0 f(0), \quad s_2 = 2A_0 f(0) \frac{\Gamma(1-\gamma)}{2-\gamma} \sin\frac{\pi\gamma}{2}, \quad s_3 = \pi A_0 f(0), \tag{3.9}$$

$A_0 = (k_0 r_0 a/4\pi)^2$, and $C = 0.5772...$ is Euler's constant. It follows from (3.8) that in the case $\gamma < 2$ the asymptotic behavior of the mean field is exponential [the quantity $\sigma_{\rm BE}$ is the total cross section for single scattering in the eikonal approximation; in the limit $r_m \gg \lambda$, it is the same as the Born total cross section, (1.13); Ref. 1]. Actually, in calculating the leading term of $H(z)$ in the case $\gamma < 2$ it is sufficient to use the replacement

$$q_\parallel^{-2} \sin^2(q_\parallel z/2) \approx (\pi z/2)\, \delta(q_\parallel) \tag{3.10}$$

in (3.3). For values $1 \leqslant \gamma < 2$, the singularity of the correlation function is manifested in correction terms proportional to s_1 and s_2. If, on the other hand, we have $2 \leqslant \gamma < 3$, then the asymptotic behavior of $\langle u(z)\rangle$ is superexponential: The singularity of the correlation function is manifested in the leading terms, proportional to s_3 and s_2. As $\gamma \to 3$ in (3.8), an attenuation $\langle u\rangle \propto \exp(-\text{const}\cdot z^2)$ arises in (3.8), as in the exactly solvable model, (2.14). The reason is that in the limit $\varepsilon \to 0$ the correlation function $\psi(\mathbf{q}) \propto \varepsilon/\mathbf{q}^{3-\varepsilon}$ in a sense acquires a behavior $\propto \delta(\mathbf{q})$.

There are, on the other hand, systems with long-range correlation functions of the form (3.4), for which the concept of a length scale r_m would be difficult to introduce, since the function $f(x)$ does not fall off sufficiently rapidly as $x \to \infty$ [e.g., $f(x) = \text{const}$]. When one applies the eikonal method to such systems one runs into the difficulty that a substantial fraction of the spectrum $\psi(\mathbf{q})$ consists of the small-scale part [the weak decay of $\psi(\mathbf{q})$ as $q \to \infty$]. By virtue of the condition for the applicability of the eikonal approximation, $l_\varphi \gg \lambda$, the integration in (3.3) is justified only in the region $q \ll \lambda^{-1}$. It thus becomes necessary to introduce a truncating upper limit $k_m \lesssim \lambda^{-1}$ in the integral. [A corresponding situation arises in the general case of correlation (3.4) if $r_m < \lambda$.] In this case it is natural to require, as the condition for the applicability of the eikonal approximation, that the leading terms of the asymptotic behavior of $\langle u(z)\rangle$ be independent of the parameter k_m.

It follows from (3.8) and (3.9) that with $\gamma < 2$ for systems of this sort the coefficient of the leading term in $H(z)$—the attenuation rate of $\sigma_{\rm BE}/2$—depends on k_m; i.e., the eikonal method is not valid. Actually, in this case it is sufficient to use the Bourret approximation, since σ_B is finite. In the case $\gamma \geqslant 2$, the coefficients s_1 and s_2 of the leading terms of the asymptotic behavior of $H(z)$ do not depend on k_m, and we can say that the eikonal approximation is valid. The correction coefficients σ_1 and σ_2, on the other hand, are functions of k_m. If the contribution of the small-scale inhomogeneities is also taken into account correctly—to do so requires going beyond the eikonal approximation—then one can calculate the contribution to k_m from the constants σ_1 and σ_2. One known method of this type is the method of separating the modes of a fluctuating field.[9,19,20] In this method, the field φ is broken up into two independent parts: a "soft" part ($q \ll k_*$) and a "hard" part ($q > k_*$). In other words, one writes $\varphi = \varphi_< + \varphi_>$. For the unaveraged field $u(\mathbf{r},\varphi)$, one uses the approximation[19]

$$u(\mathbf{r}, \varphi) \approx u(\mathbf{r}, \varphi_>) \exp\left[i\frac{k_0}{2} \int_0^z \varphi_<(\rho, z')\, dz' \right]. \tag{3.11}$$

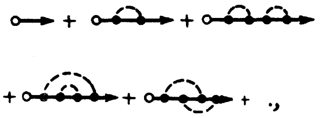
In an evaluation of the mean field $\langle u(\mathbf{r},\varphi)\rangle$, the factors in (3.11) are averaged independently. The Bourret approximation is used for $\langle u(\mathbf{r},\varphi_>)\rangle$, while an exact average is taken of the exponential eikonal term. A definite shortcoming of this method is the need to introduce an additional parameter to separate the degrees of freedom, k_*, and to verify that this parameter disappears in the leading orders of the resulting equations.

We wish to propose a calculation method which makes it possible to derive an expression for $\langle u(z)\rangle$ which does not contain parameters of the form k_* and k_m and which is valid in both the far zone and the near zone.

We wish to call attention to the circumstance that in all the weak-coupling cases which we have discussed [the Bourret approximation, (1.12), (1.13); the eikonal method (3.3); the exactly solvable model (2.13), (2.14); and the mode separation method] the asymptotic behavior of the mean field is

$$\langle u(z)\rangle \sim u_0(z) \exp\left[a^2 g(z)\right]. \tag{3.12}$$

From the standpoint of perturbation theory, the meaning of the exponential representation of solution (3.12) is as follows: In contrast with a direct expansion of the mean field in powers of a^2,

$\langle u \rangle =$  (3.13)

in which the first term corresponds to the incident field $u_0(z)$, the corrections in the argument of the exponential function in (3.12) do not contain secular terms. In other words, the corrections of higher order in a do not increase more rapidly than $g(z)$ [at least as long as the corresponding expansion parameter is small; see (3.16)]. Adzhemyan *et al.*[21] give a rigorous proof that there are no secular terms in the perturbation theory restructured in the necessary way.

Because of this comment, we can seek a solution for $\langle u \rangle$ in the form

$$\langle u(z) \rangle = u_0(z) \exp[\Phi_u(z)], \tag{3.14}$$

and we can expand $\Phi_u(z)$ in powers of the coupling constant a^2:

$$\Phi_u(z) = a^2 g_1(z) + a^4 g_2(z) + \dots \,. \tag{3.15}$$

Terms with fractional powers of a^2 may also appear in expansion (3.15) (Ref. 9), but the first term is always of order a^2. [More precisely, the expansion parameter in (3.13), (3.15) is the quantity

$$\xi_\gamma \sim \frac{a^2 f(0)}{16\pi} (k_0 r_0)^{3-\gamma} \frac{1}{3-\gamma} \frac{(k_0 z)^{\gamma-2}-2}{\gamma-2}, \quad 0 \leqslant \gamma < 3; \tag{3.16}$$

cf. the expression for $\gamma = 2$ in Ref. 9 and expression (2.11) of the present paper for $\gamma \to 3$.] The first term in (3.15) is easily reconstructed through a comparison with the direct expansion in (3.13):

$$u_0(z) + u_0(z) a^2 g_1(z) \tag{3.17}$$

$=$ 

Retaining only the first term, $a^2 g_1(z)$, in (3.15) (the expansion parameter ξ_γ is $\sim 10^{-4} - 10^{-5}$, even for such highly opalescent systems as liquid crystals), we find

$$\Phi_u(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi)^3} K_\mathbf{r}(\mathbf{p}) \psi(\mathbf{p}), \tag{3.18}$$

where

$$K_\mathbf{r}(\mathbf{p}) = k_0^4 \exp(-ik_0 z) \int_V \int_V d\mathbf{r}_1 \, d\mathbf{r}_2 \, G_0(\mathbf{r}-\mathbf{r}_1)$$
$$\times G_0(\mathbf{r}_1-\mathbf{r}_2) \exp[i\mathbf{p}(\mathbf{r}_1-\mathbf{r}_2) + ik_0 z_2]. \tag{3.19}$$

If the volume V is a plane slab of a medium of thickness z_0, i.e., if $0 \leqslant z \leqslant z_0$ [to avoid having to deal with refraction at the boundary, we assume, here and below, that the medium outside the sample has a dielectric constant ε_0, while that inside has $\varepsilon(\mathbf{r}) = \varepsilon_0 + \delta\varepsilon(\mathbf{r})$], we find the following expression for $K_\mathbf{r}(\mathbf{p}) = K_z(\mathbf{p}, z_0)$:

$$K_z(\mathbf{p}, z_0) = -\frac{k_0^3}{2\alpha} \left\{ \frac{\sin^2[(\alpha-\beta) z_0/2]}{(\alpha-\beta)^2} + \frac{\sin^2[(\alpha+\beta) z_0/2]}{(\alpha+\beta)^2} + \frac{i}{2} \left[\frac{z_0}{\alpha-\beta} - \frac{\sin[(\alpha-\beta) z_0]}{(\alpha-\beta)^2} + \frac{z_0}{\alpha+\beta} - \frac{\sin[(\alpha+\beta) z_0]}{(\alpha+\beta)^2} \right] \right\}, \tag{3.20}$$

where $\alpha = (k_0^2 - p_\perp{}^2)^{1/2}$, $\beta = k_0 - p_\parallel$, $\mathrm{Im}\, \alpha \geqslant 0$, $z \geqslant z_0$.

If the volume V is the half-space $z \geqslant 0$, for $z \in V$ we find

$$K_z(\mathbf{p}) = -\frac{k_0^3}{4\alpha} \exp(-ik_0 z) \int_0^\infty \int_0^\infty dz' \, dz''$$
$$\times \exp[ik_0 |z-z'| + i\alpha |z'-z''|$$
$$+ ip_\parallel (z'-z'') + ik_0 z'']. \tag{3.21}$$

We find a similar behavior of $K_z(\mathbf{p})$ in the problem of the propagation of a wave through an infinite medium, from a source in the $z = 0$ plane, which creates a plane wave $\exp(ik_0|z|)$.

Working in precisely the same manner, writing the average Green's function in the form $\langle G(\mathbf{r}) \rangle = G_0(\mathbf{r}) \exp[\Phi_G(\mathbf{r})]$, we can calculate the first term of an expansion of $\Phi_G(\mathbf{r})$ in powers of a^2. In an infinite medium, the kernel $K_r(\mathbf{p})$ for this case is

$$K_r(\mathbf{p}) = \frac{i}{8\pi k_0} \int d\mathbf{r}' \frac{\exp(ik_0 r')}{4\pi r'}$$
$$\times \exp(i\mathbf{p}\mathbf{r}') \exp[ik_0(|\mathbf{r}-\mathbf{r}'| - r)]. \tag{3.22}$$

The results which we have derived here, (3.14) and (3.18)–(3.22), are generalizations of earlier results. Specifically, if we have $p^2 \psi(\mathbf{p}) \to 0$ in the limit $\mathbf{p} \to 0$, then we can take the limit $k_0 z_0 \to \infty$ in (3.20) [and in (3.21) and (3.22) we can take the limits $k_0 z \to \infty$, $k_0 r \to \infty$, respectively]. For kernels (3.20)–(3.22) we find

$$K_r(\mathbf{p}) \approx {}^1/_2 k_0^3 z_0 [-\pi\delta(\alpha^2-\beta^2) + i\mathscr{P}(\alpha^2-\beta^2)^{-1}], \tag{3.23}$$

where $\alpha^2 - \beta^2 = k_0^2 - (\mathbf{k}_0 - \mathbf{p})^2$, $\mathbf{k}_0 = k_0 \mathbf{z}/z$, in agreement with the result of the Bourret approximation, (1.12), (1.13). [To derive (3.23), we could use (3.10) in (3.20), (3.21) as $z, z_0 \to \infty$, while in the limit $r \to \infty$ it is sufficient to set $|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{r}'\mathbf{r}/r$, $\mathbf{k}_0 = k_0 \mathbf{r}/r$ in (3.22).]

If we use kernel (3.20) for singular correlation function (3.4), then the real part of $\Phi_u(\mathbf{r})$ in (3.18) takes a form similar to (3.8) in the limit $z_0 \to \infty$. For $\gamma < 2$, the only change is a replacement of σ_{BE} by σ_B in (3.8), while the coefficients s_1 and s_2 remain. For $\gamma \geqslant 2$, the coefficients s_2 and s_3 remain, while σ_1 and σ_2 become $\sigma_1 = \sigma_{01} + \sigma_\Delta$, $\sigma_2 = \sigma_{02} + \sigma_\Delta$, where

$$\sigma_{01} = s_3 r_0^{-1} [\ln(2k_0 r_m) + C - 1], \quad \sigma_{02} = s_3 r_0^{-1} (2k_0 r_0)^{2-\gamma}/(2-\gamma),$$

$$\sigma_\Delta = \frac{\pi k_0^3}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} [\psi(\mathbf{p})$$
$$- a^2 r_0^3 (q r_m)^{-\gamma} f(0)] \delta[(\mathbf{k}_0 - \mathbf{p})^2 - k_0^2]. \tag{3.24}$$

On the other hand, the imaginary part of $\Phi_u(\mathbf{r})$ is linear in z_0 in the limit $z_0 \to \infty$ for all $\gamma < 3$, as can be seen from (3.20)–(3.23). The proportionality factor is determined from (1.12). In particular, for $f(x) \equiv$const we find from (1.12)

$$k' = k_0 + s_3 r_0^{-1} (2k_0 r_0)^{2-\gamma} \operatorname{tg}(^1/_2\pi\gamma)/(2-\gamma), \quad 2<\gamma<3,$$

$$k' = k_0 - s_3 r_0^{-1}\pi/2, \quad \gamma=2. \tag{3.25}$$

In the case $\gamma = 2, f(x) \equiv$const, Eqs. (3.24), (3.25), and (3.8) become the same as the results of Refs. 8 and 9.

We wish to point out that for bounded systems a superexponential attenuation may be thought of as a dependence of the extinction coefficient τ on the longitudinal dimension of the sample, z_0 (the transverse dimension $L_\perp$ is unimportant if $k_0 L_\perp \gg 1$). Since we restricted the calculation of the kernel K in (3.17) to the single-loop diagram, the dependence $\tau(z_0)$ for such systems can be found (to within corrections $\sim 1/k_0 z_0$) from the ordinary Bourret approximation, provided that we allow for the finite value of z_0. In other words, we use the following expression in (1.7):

$$\Sigma_{Bt}(\mathbf{q}, z_0) = \int_V d\mathbf{r}\, G_0(\mathbf{r}) \psi(\mathbf{r}) e^{i\mathbf{qr}}. \tag{3.26}$$

4. ANGULAR DISTRIBUTION OF THE SCATTERING INTENSITY

We assume that a scalar monochromatic wave $u(\mathbf{r}) = U_0 \exp(i\mathbf{k}_s \mathbf{r})$ is propagating along the $\mathbf{k}_s = k_0 \mathbf{z}/z$ direction and is incident on a sample of finite volume V. We assume that the sample is a plane slab of a medium of thickness z_0 $(0 \leqslant z \leqslant z_0)$ with sufficiently large transverse dimensions. In this case the field at a point $\mathbf{r}$ outside the sample can be found most conveniently from its value at the surface of the sample, as in diffraction theory.[22] The field at point $\mathbf{r}$ is related to the field at the boundary, $u(\mathbf{r}')$, by

$$u(\mathbf{r}) = -2\,\mathrm{sign}(z - z_b) \int_{S_0} d^2\rho'\, u(\mathbf{r}') \frac{\partial}{\partial z} \frac{\exp(ik_0 R)}{4\pi R}. \tag{4.1}$$

Here S_0 is the surface of the sample, which coincides with the $z = z_b$ plane ($z_b = z_0$ or $z_b = 0$, for scattering into the forward and rear hemispheres, respectively), and $R = |\mathbf{r} - \mathbf{r}'|$ is the distance from the observation point to the point $\mathbf{r}' = (\boldsymbol{\rho}', z_b)$, which lies on surface S_0. For the field in the far zone we find from (4.1) the expression

$$u(\mathbf{r}) \approx -\frac{ik_0}{2\pi} \frac{|z - z_b|}{r} \frac{\exp(ik_0 r)}{r}$$

$$\times \int_{S_0} d^2\rho' \exp(-i\mathbf{k}_s \mathbf{r}' u(\rho', (z_b), \tag{4.2}$$

where $\mathbf{k}_s = k_0 \mathbf{r}/r$ is the wave vector of the scattered wave.

We are now interested in the angular distribution of the average scattering intensity:

$$I(\theta) \sim \langle u(\mathbf{r}) u^*(\mathbf{r})\rangle - \langle u(\mathbf{r})\rangle\langle u^*(\mathbf{r})\rangle, \tag{4.3}$$

where θ is the angle between $\mathbf{k}_s$ and $\mathbf{k}_i$, i.e., the scattering angle, and u^* is the complex-conjugate field. Making use of the statistical homogeneity of the medium in planes transverse with respect to z, using the change of variables $\boldsymbol{\rho} = \boldsymbol{\rho}' - \boldsymbol{\rho}''$, $\boldsymbol{\rho}_+ = (\boldsymbol{\rho}' + \boldsymbol{\rho}'')/2$ in (4.2), and integrating over $\boldsymbol{\rho}_+$, we find

$$I(\theta) \sim \frac{k_0^2 S_\perp \cos^2\theta}{4\pi^2 r^2} \int_{S_0} d^2\boldsymbol{\rho} \exp(-i\mathbf{k}_s \boldsymbol{\rho})$$

$$\times [\Gamma(\boldsymbol{\rho}, z_b) - |\langle u(\boldsymbol{\rho}, z_b)\rangle|^2], \tag{4.4}$$

where $\Gamma(\boldsymbol{\rho}' - \boldsymbol{\rho}'', z_b) = \langle u(\boldsymbol{\rho}', z_b) u^*(\boldsymbol{\rho}'', z_b)\rangle$, and $S_\perp$ is the cross-sectional area of the sample.

For a medium with large-scale inhomogeneities, it is sufficient to consider the scattering into the forward hemisphere alone ($z_b = z_0$). To calculate the field $u(\boldsymbol{\rho}, z_0)$ at the plane $z = z_0$, we can use the eikonal approximation, (3.2). Substituting (3.2) into (4.4), and assuming that the fluctuations are Gaussian, we find

$$I(\theta) = B_0 \int_0^{L_\perp/2} d\rho\, \rho J_0(\varkappa\rho) [e^{F(\rho) - F(0)} - e^{-F(0)}], \tag{4.5}$$

where

$$F(\rho) = k_0^2 \int \frac{d\mathbf{q}}{(2\pi)^3} q_\parallel^{-2} \sin^2 \frac{q_\parallel z_0}{2} \cos(\mathbf{q}_\perp \boldsymbol{\rho}) \psi(\mathbf{q}), \tag{4.6}$$

$J_0(x)$ is the Bessel function of index zero, $\varkappa$ is the transverse component of the vector $\mathbf{k}_s$, $\varkappa = k_0\theta$, $L_\perp$ is a characteristic transverse dimension of the sample, $B_0 = I_0 S_\perp k_0^2/2\pi r^2$, and I_0 is the intensity of the incident field u_0. Here we have noted that by virtue of the applicability of the eikonal approximation we have $\theta \ll 1$, and we have replaced the factor of $\cos^2\theta$ by unity and the factor of $\sin\theta$ by θ.

Because of the oscillations in the function $J_0(\varkappa\rho)$, only the region $\rho \sim \varkappa^{-1}$ contributes noticeably to the integral in (4.5). In this case, the finite value of the upper limit of the integration over ρ is manifested only for $L_\perp \sim \varkappa^{-1}$, i.e., only for scattering angles θ on the order of the angle of the diffraction by the overall sample, $\theta_{\text{diff}} \sim \lambda/L_\perp$. So that we can subsequently ignore the particular features of the cross-sectional shape of the sample, we consider angles $\theta \gg \theta_{\text{diff}}$, replacing the upper limit of the integration in (4.5) by infinity.[2)] It is not difficult to see that, after an integration of ρ, the second term in square brackets in (4.5) makes a contribution to the intensity which is proportional to $\delta(\varkappa)$, and this term can be discarded in an analysis of the scattering through finite angles. Actually, the role of this term is simply one of canceling the corresponding δ-function from the first term in brackets in (4.5).

We can now show that the angular distribution of the scattering intensity has different regimes, depending on the distance traversed by the wave in the medium. We assume $k_0 z_0 \gg 1$. For $\theta \neq 0$ we have

$$I(\theta) = B_0 \int_0^\infty d\rho\, \rho J_0(\varkappa\rho) e^{-\tilde{F}(\rho)}, \tag{4.7}$$

where

$$\bar{F}(\rho)=F(0)-F(\rho)$$

$$=\frac{k_0^2}{4\pi^2}\int_{-\infty}^{\infty}dq_\parallel\int_0^{\infty}dq_\perp q_\perp[1-J_0(q_\perp\rho)]q_\parallel^{-2}\sin^2\frac{q_\parallel z_0}{2}\psi(\mathbf{q}). \tag{4.8}$$

We introduce an effective argument for the exponential function in (4.7), $F_{\text{eff}}\equiv\tilde{F}(\varkappa^{-1})\sim F(\rho)$, and we consider the following limiting cases.

1. $F_{\text{eff}}\ll 1$. In this case the exponential function in (4.7) can be expanded in a series, in which we retain terms up to the term linear in $\tilde{F}(\rho)$. In this case the scattering intensity is

$$I(\theta)=B_0\int_0^{\infty}d\rho\,\rho J_0(\varkappa\rho)F(\rho). \tag{4.9}$$

Using the identity

$$p\int_0^{\infty}d\rho\,\rho J_0(p\rho)J_0(q\rho)=\delta(p-q),$$

we find

$$I(\theta)=\frac{B_0k_0^2}{2\pi}\int_{-\infty}^{\infty}\frac{dq_\parallel}{2\pi}q_\parallel^{-2}\sin^2\frac{q_\parallel z_0}{2}\psi(\varkappa,q_\parallel). \tag{4.10}$$

If the distance traversed by the wave in the medium is much greater than the correlation radius, $z_0\gg r_c$, we can use replacement (3.10) in (4.10). For the intensity we find

$$I(\theta)=\frac{B_0z_0k_0^2}{8\pi}\psi(\varkappa,0). \tag{4.11}$$

This result corresponds to the Born approximation.

If the condition $z_0\ll r_c$ holds instead, then we can use the replacement (3.5) in (4.10). The result in this case is

$$I(\theta)=\frac{B_0z_0^2k_0^2}{8\pi}\int_{-\infty}^{\infty}\frac{dq_\parallel}{2\pi}\psi(\varkappa,q_\parallel). \tag{4.12}$$

The difference between the factor $q_\parallel^{-2}\sin^2(q_\parallel z_0/2)$ and a δ-function in (4.10) corresponds to the incorporation of the finite dimension of the scattering system along z in the Born approximation. In this case, (4.12) corresponds to the limit of spatial homogeneity of the fluctuations along z. A feature which distinguishes (4.12) from (4.11) is the unusual volume dependence $I(\theta)\propto V^{4/3}$; another is the smoother scattering function.

2. $F_{\text{eff}}\gg 1$. In this case we can use the Laplace method for an asymptotic evaluation of the integral in (4.7). Expanding the argument of the exponential function in a power series in ρ near the point of the minimum, $\rho=0$, and retaining the first nonvanishing term ($\sim\rho^2$), we find

$$I(\theta)=2B_0M^{-1}(z_0)\exp[-\varkappa^2/M(z_0)], \tag{4.13}$$

or

$$M(z_0)=\frac{k_0^2}{4\pi^2}\int_0^{\infty}dq_\perp q_\perp^3\int_{-\infty}^{\infty}dq_\parallel q_\parallel^{-2}\sin^2\frac{q_\parallel z_0}{2}\psi(q_\perp,q_\parallel). \tag{4.14}$$

In particular, for $z_0\gg r_c$ we find (cf. Ref. 23)

$$M(z_0)=\frac{k_0^2z_0}{8\pi}\int_0^{\infty}dq_\perp q_\perp^3\psi(q_\perp,0), \tag{4.15}$$

and in the limit $z_0\ll r_c$ we find

$$M(z_0)=\frac{k_0^2z_0^2}{16\pi^2}\int_0^{\infty}dq_\perp q_\perp^3\int_{-\infty}^{\infty}dq_\parallel\psi(q_\perp,q_\parallel). \tag{4.16}$$

What features are caused in the angular distribution of the scattering intensity by the singularity of the correlation function $\psi(\mathbf{q})$? Since we are interested only in effects stemming from the long-wavelength part of the spectrum, we consider a correlation function

$$\psi(\mathbf{q})=a^2r_0^3(qr_0)^{-\gamma}\Theta(r_m^{-1}-q),$$

where $r_m\gg\lambda$, and $\Theta(x)$ is the unit step function. We rewrite expression (4.7) for $I(\theta)$ as follows:

$$I(\theta)=B_1\frac{1}{\theta^2}\int_0^{\infty}dt\,tJ_0(t)e^{-\tilde{F}(t)}, \tag{4.17}$$

where $B_1=B_0/k_0^2$, and where $\tilde{F}(t)$ is [we are using (3.10) and (3.9)]

$$\tilde{F}(t)=\begin{cases}d_1z_0\left(\dfrac{t}{\theta}\right)^{\gamma-2}\displaystyle\int_0^{N(t)}dx\,x^{1-\gamma}[1-J_0(x)], & \theta\gg(k_0z_0)^{-1};\\[2ex] d_2z_0^2\left(\dfrac{t}{\theta}\right)^{\gamma-3}\displaystyle\int_{-N(t)}^{N(t)}dy\int_0^{\mathcal{N}(t)}dx\,x(x^2+y^2)^{-\gamma/2}[1-J_0(x)], & \theta\ll(k_0z_0)^{-1}.\end{cases} \tag{4.18}$$

Here $d_1=a^2(k_0r_0)^{3-\gamma}k_0/8\pi$, $d_2=d_1k_0/2\pi$, $N(t)=\theta_1t/\theta$, $\mathcal{N}(t)=[N^2(t)-y^2]^{1/2}$, and $\theta_1=(k_0r_m)^{-1}$ is an angle on the order of magnitude of a characteristic single-scattering angle.

We first consider the case $\theta\gg(k_0z_0)^{-1}$. For $d_1z_0\ll 1$ we have

$$I(\theta)=B_1d_1z_0\cdot\begin{cases}\theta^{-\gamma}, & & \gamma<2,\\ \theta^{-2+d_1z_0}, & \theta\ll\exp(-1/d_1z_0), & \gamma=2,\\ \theta^{-2}, & \theta\gg\exp(-1/d_1z_0), & \gamma=2,\\ (d_1z_0)^{\gamma/(2-\gamma)}c_\gamma, & \theta\ll(d_1z_0)^{1/(\gamma-2)}c_\gamma^{-1/2}, & \gamma>2,\\ \theta^{-\gamma}, & \theta\gg(d_1z_0)^{1/(\gamma-2)}c_\gamma^{-1/2}, & \gamma>2,\end{cases}\tag{4.19}$$

where

$$c_\gamma=\left[-2^{1-\gamma}\frac{\Gamma(1-\gamma/2)}{\Gamma(\gamma/2)}\right]^{2/(\gamma-2)}.$$

For $\gamma<2$, the scattering function is identical to that in the Born approximation. As soon as we reach $\gamma=2$, however, the degree of singularity of the scattering function decreases at small angles (the $1/\theta^2$ law gives way to $1/\theta^{2-d_1z_0}$), and at $\gamma>2$ the scattering intensity at $\theta\to0$ becomes finite. The reason for this unusual behavior of $I(\theta)$ is that the condition for the applicability of the single-scattering treatment is violated for such singular correlation functions in the limit $\theta\to0$, and all scattering multiplicities must be taken into account.

In the opposite limit, $d_1z_0\gg1$, the higher scattering multiplicities become important for all values of γ. The singular nature of $I(\theta)$ persists only at $\gamma<2$, and in a very narrow angular region $\theta\lesssim\exp[-d_1z_0\theta_1{}^{2-\gamma}/(2-\gamma)\gamma]$ we can write

$$I(\theta)=B_1d_1z_0\theta^{-\gamma}\exp[-d_1z_0\theta_1{}^{2-\gamma}/(2-\gamma)].\tag{4.20}$$

In all other cases, there is a universal Gaussian law as in (4.13), (4.15):

$$I(\theta)=2B_1\frac{(4-\gamma)}{d_1z_0\theta_1^{2-\gamma}}\exp\left[\frac{-\theta^2(4-\gamma)}{d_1z_0\theta_1^{4-\gamma}}\right].\tag{4.21}$$

For $d_1z_0\sim1$, there is a smooth transition from (4.19) to (4.21). In particular, for $\gamma=2$, $\theta\ll\theta_1$, and $d_1z_0<2$ we have

$$I(\theta)=2B_1(e^C\theta_1)^{-d_1z_0}\frac{\Gamma(1-{}^1/_2d_1z_0)}{\Gamma({}^1/_2d_1z_0)}\theta^{-2+d_1z_0}.\tag{4.22}$$

With $d_1z_0=2$ we have $I(\theta)\propto\ln\theta^{-1}$, and for $d_1z_0>2$ the value of $I(0)$ becomes finite. The angular distribution in (4.22) was found for the scattering of neutrons near the critical point by the Glauber method[25] in Ref. 24. It was also found in Ref. 26 for the case of critical opalescence in the small-angle approximation for the radiation-transport equation.

For $\theta\ll(k_0z_0)^{-1}$ and $d_2z_0^2\ll1$, we find from (4.12) that $I(\theta)$ is singular in the limit $\theta\to0$ only for $1\leqslant\gamma<3$:

$$I(\theta)=\pi^{1/2}B_1d_2z_0{}^2\theta^{-\gamma+1}\Gamma\left(\frac{\gamma}{2}-\frac{1}{2}\right)\Big/\Gamma\left(\frac{\gamma}{2}\right)\tag{4.23}$$

[with $\gamma=1$ we have $I(\theta)\propto\ln\theta^{-1}$, while for $\gamma<1$ the quantity $I(0)$ is finite].

Finally, for $\theta\ll(k_0z_0)^{-1}$ and $d_2z_0^2\gg1$ the angular distribution of the intensity is Gaussian, as in (4.13), (4.16), for all $\gamma(0<\gamma<3)$:

$$I(\theta)=B_1\frac{3(5-\gamma)}{2d_2z_0{}^2\theta_1^{5-\gamma}}\exp\left[-\frac{3(5-\gamma)\theta^2}{4d_2z_0{}^2\theta_1^{5-\gamma}}\right].\tag{4.24}$$

In contrast with (4.21), however, the half-width of the Gaussian function varies linearly with z_0, rather than as $z_0^{1/2}$.

To conclude this section of the paper we would like to point out that it is not legitimate to take a limit $r_m\to0$ in (4.20)–(4.22), (4.24). The reason is that Eqs. (4.5) and (4.7), which are based on the eikonal approximation, are intended to deal correctly with only the largest-scale inhomogeneities. To improve the treatment of intermediate- and small-scale inhomogeneities, we could use approximations for a boundary field $u(\rho,z_b)$ which are better than the eikonal approximation. In particular, if we use Rytov's smooth-perturbation method[1] to calculate $u(\rho,z_0)$, we find

$$u(\mathbf{r})\approx-ik_0U_0\cos\theta\frac{\exp(ik_0r)}{2\pi r}\int_{S_0}d\rho'\exp[-i(\mathbf{k}_s-\mathbf{k}_p)\mathbf{r}']$$

$$\times\exp\left[\int_0^{z_0}dz''\int d\rho''G^0_{\rm SPM}(\rho'-\rho'',z_0-z'')\varphi(\rho'',z'')\right],\tag{4.25}$$

where

$$G^0_{\rm SPM}(\rho,z)=\frac{k_0^2}{4\pi z}\exp\left(\frac{ik_0\rho^2}{2z}\right).$$

Interestingly, Eq. (4.25) may be thought of as a refinement of the well-known Glauber approximation[25] in the theory for the scattering of high-energy particles by a potential $\varphi(\mathbf{r})$ [the surface S_0 should be chosen in a region in which $\varphi(\mathbf{r})$ is vanishingly small in this case]. The Glauber approximation corresponds to the replacement of $\cos\theta$ by unity in (4.25), the replacement of $G^0_{\rm SPM}(\rho,z)$ by $(ik_0/2)\delta(\rho)$, and the discarding of the small longitudinal component of the scattering vector $\mathbf{k}_s-\mathbf{k}_i$.

The use of (4.25) in (4.4) eliminates the divergence as $r_m\to0$. In particular, if we calculate the mean field $\langle u(\rho,z)\rangle$ in this approximation, we find that the constants σ_1, σ_2, and $\sigma_{\rm BE}$ in (3.8) are finite. It is important to note, however, that the values found for these constants by this method are not correct, since the quantities σ_1, σ_2, and σ_B are determined by the entire region of the spectrum $\psi(\mathbf{q})$ $(0\leqslant q\leqslant2k_0)$, while the smooth-potential method takes only the region $q\ll k_0$ into account correctly. Of importance for our purposes here is that the errors in the description of the attenuation of the mean field $\langle u(\rho,z)\rangle$ lead to corresponding errors in the angular distribution of the intensity, (4.4). The errors afflict not only the second term in brackets in (4.4) [$|\langle u(\rho,z)\rangle|^2$; this point is important only in the limit $\theta\to0$] but also the first term [$\Gamma(\rho,z_0)$], because of distortion of the amplitude of the mean field during propagation between successive scattering events.

Effects stemming from the attenuation of the mean field between successive multiple-scattering events in (4.4) in the case $\theta \ll 1$ can be dealt with most easily by writing $\Gamma(\mathbf{\rho},z_0)$ in the form

$$\Gamma(\mathbf{\rho},\ z_0)=\tilde{\Gamma}(\mathbf{\rho},\ z_0)\exp\left[2\ \mathrm{Re}\ \Phi_u(z_0)\right], \tag{4.26}$$

with $\Phi_u(z)$ from (3.14). In $\tilde{\Gamma}(\mathbf{\rho},z_0)$, in contrast with $\Gamma(\mathbf{\rho},z_0)$, we can legitimately take the limit $r_m \to 0$. It turns out that an effective way to calculate $\tilde{\Gamma}(\mathbf{\rho},z_0)$ is to use the exponential representation

$$(\Delta_\perp + k_0^2)\tilde{\Gamma}(\mathbf{\rho},\ z_0) = k_0^2\,|U_0|^2 \exp\left[\Phi_{\tilde{\Gamma}}(\mathbf{\rho},\ z_0)\right], \tag{4.27}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\{x,y\} = \mathbf{\rho}$, and to then calculate $\Phi_{\tilde{\Gamma}}(\mathbf{\rho},z_0)$ by perturbation theory. In lowest order in ψ we find

$$\Phi_{\tilde{\Gamma}}(\mathbf{\rho},\ z_0) = k_0^2 \int \frac{d\mathbf{q}}{(2\pi)^3}\frac{\alpha^2}{|\alpha|^2}\left|\frac{\sin\left[(\alpha-\beta)\,z_0/2\right]}{\alpha-\beta}\right|^2$$

$$\times \exp\left(-z_0\ \mathrm{Im}\ \alpha\right)\cos\left(\mathbf{q}_\perp\mathbf{\rho}\right)\psi(\mathbf{q}), \tag{4.28}$$

where α and β are given in (3.20) (with $\mathbf{p}$ replaced by $\mathbf{q}$). The result for the intensity in this case is

$$I(\theta) = B_0 \int_0^\infty d\rho\rho J_0(k_0\rho\sin\theta)\{\exp\left[\Phi_{\tilde{\Gamma}}(\mathbf{\rho},\ z_0)\right.$$

$$\left.+\ 2\mathrm{Re}\ \Phi_u(z_0)\right] - \exp\left[2\mathrm{Re}\ \Phi_u(z_0)\right]\}. \tag{4.29}$$

In contrast with (3.14), (3.15), for the corrections to $\Phi_{\tilde{\Gamma}}(\mathbf{\rho},z_0)$ of higher order in ψ we cannot guarantee that there are no secular terms. An important point, however, is that only the small-scale part of the spectrum $\psi(\mathbf{q})$ contributes to the secular terms. Specifically, if we consider only large-scale inhomogeneities then we can use the approximation $\alpha - \beta \approx q_\parallel$, $\alpha + \beta \approx 2k_0$, for α and β in (4.28), (3.20). Expression (4.29) then becomes the eikonal expression, (4.5). Furthermore, if we retain the term quadratic in $\mathbf{q}_\perp$ in the expansion, i.e., if we assume $\alpha - \beta \approx q_\parallel + q_\perp{}^2/2k_0$, $\alpha + \beta \approx 2k_0$, then we find from (4.29) an expression corresponding to the use of the smooth-perturbation method, (4.25), in (4.4).

By calculating $I(\theta)$ from (4.29) we can take the limit $r_m \to 0$ correctly. In particular, it turns out that this limit corresponds to the replacement $\theta_1 \to 2$ in (4.20)–(4.22), (4.24).

5. TOTAL CROSS SECTION FOR MULTIPLE SCATTERING

The total cross section for scattering by a unit volume, σ_s, is given by[1]

$$\sigma_s = P_s/I_0V, \tag{5.1}$$

where P_s is the total scattered power. Far from the sample, at a distance $r \gg V^{1/3}$, we have

$$P_s = r^2 \int d\Omega_n I_s(\mathbf{n}),$$

where $I_s(\mathbf{n})$ is the intensity (the absolute value of the energy flux density) of the scattering in direction $\mathbf{n}$. If we write the field u in the form $u = \langle u\rangle + u_s$, where u_s is the scattered field, we find that the energy flux density $\mathbf{J} = C_0\ \mathrm{Im}(u^*\nabla u)$ breaks up into a sum of three terms:

$$\mathbf{J}=\mathbf{J}_m+\mathbf{J}_s+\mathbf{J}_{ms}, \tag{5.2}$$

where

$$\mathbf{J}_m=C_0\ \mathrm{Im}(\langle u\rangle^*\nabla\langle u\rangle),\quad \mathbf{J}_s=C_0\ \mathrm{Im}(u_s^*\nabla u_s),$$
$$\mathbf{J}_{ms}=C_0\ \mathrm{Im}(\langle u\rangle^*\nabla u_s+u_s^*\nabla\langle u\rangle),$$

and C_0 is a constant. The energy conservation law in integral form in this case is

$$\int_S \mathbf{J}(\mathbf{r})\,d\mathbf{S}=0 \tag{5.3}$$

for an arbitrary closed surface S.

As before, we consider a volume V bounded by two planes, $z = 0$ and $z = z_0$, on which a plane wave is incident along the z axis from the left. Substituting (5.2) into (5.3), integrating over the planes $z = 0$ and $z = z_0$, and taking a statistical average, using (3.14), we find

$$\sigma_s = \frac{1}{z_0}\Big\{\exp\left[2\ \mathrm{Re}\ \Phi_u(0)\right]\Big[1$$

$$+\frac{1}{k_0}\mathrm{Im}\ \Phi_u{}'(0)\Big]-\exp\left[2\ \mathrm{Re}\ \Phi_u(z_0)\right]$$

$$\times\Big[1+\frac{1}{k_0}\mathrm{Im}\ \Phi_u{}'(z_0)\Big]\Big\}. \tag{5.4}$$

There is a fundamental distinction between the first and second terms in braces (curly brackets) in (5.4). In the limit $z_0 \to \infty$ we have $\mathrm{Re}\ \Phi_u(z_0) \to \infty$. The quantity $\mathrm{Re}\ \Phi_u(0)$, in contrast, which is nonzero by virtue of the large-angle scattering, tends toward a finite limit with increasing thickness of the sample. In addition, the limit $z_0 \to \infty$ is allowed by the two quantities $\mathrm{Im}\ \Phi_u'(0)$ and $\mathrm{Im}\ \Phi_u'(z_0)$. [In first order in φ^2, these assertions can be verified easily through a direct calculation of the first two diagrams in (3.13).]

We can therefore replace $\exp[2\ \mathrm{Re}\ \Phi_u(0)]$ in (5.4) by one in the weak-coupling limit ($\varphi^2 \ll 1$), and we can ignore the quantities $k_0^{-1}\ \mathrm{Im}\ \Phi_u'$. As a result we find

$$\sigma_s = \frac{1}{z_0}\{1-\exp\left[2\ \mathrm{Re}\ \Phi_u(z_0)\right]\}, \tag{5.5}$$

where $\Phi_u(z_0)$ is given in (3.18), (3.20).

An expression similar to (5.5) has been derived previously[27] for the case of an exponential attenuation of the mean field, $2\ \mathrm{Re}\ \Phi_u(z_0) = -\tau z_0$. Expression (5.4) generalizes that earlier result to the case of superexponential attenuation of $\langle u\rangle$. Expression (5.5) determines the behavior of the total multiple-scattering cross section σ_s as a function of the sample thickness. At small values of z_0, the quantity σ_s is the same as the extinction coefficient (which is a function of z_0 in the case of superexponential attenuation of $\langle u\rangle$), while at large values of the thickness "saturation" sets in: The total cross section for the scattering of a unit volume tends toward $1/z_0$, while the total cross section for scattering by bulk inhomogeneities tends toward the geometric value $S_\perp = V/z_0$. We might add that in the case of exponential attenuation of $\langle u\rangle$, under the condition $\tau z_0 \ll 1$, the quantity

σ_s is the same as the Born value σ_B, but this is not true for superexponential attenuation. In particular, σ_B diverges in this case, while σ_s remains finite at all times.

CONCLUSION

We conclude with an examination of possible generalizations of the results derived here to the case of the scattering of electromagnetic waves, in particular, visible light, for which features of multiple scattering by anomalously large-scale inhomogeneities have been observed experimentally.[28] The primary distinction between that case and the case which we have treated is the vector nature of the electromagnetic field.

If we are interested in the attenuation of the mean field in a medium, we find no particular difficulty in making the corresponding generalization to the vector case. In particular, for each of the natural waves of the medium we can independently calculate the attenuation law by a method like that in Sec. 3. [For example, Eq. (3.26) remains in force; all that we have to do is take account of the tensor nature of the corresponding Green's function $\hat{G}_0$ and the corresponding correlation function $\hat{\psi}$ in this equation.]

With regard to the angular distribution of the scattering intensity, the situation is more complex. There are, however, two different cases in which the vector structure of the field does not lead to fundamental changes. The first is the case of an isotropic medium with fluctuations in a scalar order parameter. There is no depolarized scattering in this case, so the polarization is essentially conserved in the course of multiple rescattering through small angles. For this reason, the equation for the electromagnetic field in the problem of the scattering of light by large-scale isotropic inhomogeneities reduces (see Ref. 23, for example) to a Helmholtz equation (1.1). The second case is that of an anisotropic medium. In this case we can independently analyze multiple scattering by anomalously large-scale inhomogeneities of the individual natural wave by virtue of the difference between the values of the wave numbers for these waves. For the case (of practical interest) of the scattering of light by uniaxial nematic liquid crystals ($\gamma = 2$), for example, the pole as $\theta \to 0$ in the Born approximation, $I(\theta) \propto (\mathbf{k}_s - \mathbf{k}_i)^{-2}$ arises only if both k_i and k_s correspond to extraordinary waves.[7] If $\mathbf{k}_i$ and $\mathbf{k}_s$ correspond to waves of different types—one ordinary and one extraordinary—then we have $\mathbf{k}_s - \mathbf{k}_i \neq 0$ at $\theta = 0$ (if $\mathbf{k}_s$ and $\mathbf{k}_i$ correspond to ordinary waves, the equality $\mathbf{k}_s - \mathbf{k}_i = 0$ is possible at $\theta = 0$, but the intensity vanishes for geometric reasons).

If we instead consider scattering of light by fluctuations of a tensor order parameter in an isotropic medium (or in an anisotropic medium, for the same degenerate directions, in which the wave numbers of the natural waves are equal), then the interaction between modes of different polarizations arises because of the presence of a depolarized component in the scattering. This problem requires independent analysis. In particular, a direct generalization of the eikonal approximation would not be valid in this case.

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[1)] That $G_{Bt}(\mathbf{r})$ and $G_{Kn}(\mathbf{r})$ do not correspond to the exact solution $\langle G(\mathbf{r})\rangle$ was pointed out by Sekistov,[11] who carried out a numerical analysis of expressions (2.2), (2.5), and (2.8).

[2)] If we use the replacement (3.10) in (4.6) in this case, we find an expression for $I(\theta)$ which corresponds to a description of the scattering by means of the radiation-transport equation in the small-angle approximation.[1]

[1] S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarskii, *Introduction to Statistical Radiophysics*, Part II, Nauka, Moscow, 1978.

[2] R. C. Bourret, Nuovo Cim. **26**, 1 (1962).

[3] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Nauka, Moscow, 1982 (Pergamon, Oxford, 1984).

[4] A. Z. Patashinskii and V. L. Pokrovskii, *Fluctuation Theory of Phase Transitions*, Nauka, Moscow, 1982 (Pergamon, Oxford, 1979).

[5] N. A. Nemov, Opt. Spektrosk. **45**, 617 (1978) [Opt. Spectrosc. (USSR) **45**, 349 (1978)].

[6] R. H. Kraichnan, J. Math. Phys. **2**, 124 (1961).

[7] A. Yu. Val'kov and V. P. Romanov, Zh. Eksp. Teor. Fiz. **90**, 1264 (1986) [Sov. Phys. JETP **63**, 737 (1986)].

[8] L. Ts. Adzemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **68**, 323 (1986).

[9] L. Ts. Adzhemyan, A. N. Vasil'ev, M. M. Perekalin, and Kh. Yu. Reittu, Zh. Eksp. Teor. Fiz. **96**, 1278 (1989) [Sov. Phys. JETP **68**, 726 (1989)].

[10] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part I, Nauka, Moscow, 1976 (Pergamon, New York, 1980).

[11] V. N. Sekistov, Radiotekh. Elektron. **28**, 1262 (1983).

[12] N. A. Armand and V. N. Sekistov, Radiotekh. Elektron. **24**, 615 (1979).

[13] V. I. Tatarskii, Zh. Eksp. Teor. Fiz. **46**, 1399 (1963) [Sov. Phys. JETP **19**, 946 (1964)].

[14] A. L. Efros, Zh. Eksp. Teor. Fiz. **59**, 880 (1970) [Sov. Phys. JETP **32**, 479 (1971)].

[15] L. I. Volkovysskii, G. L. Dunts, and I. G. Aramanovich, *Collection of Problems on the Theory of Functions of a Complex Variable*, Nauka, Moscow, 1982, Problem 782.

[16] M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions*, Dover, New York, 1964 (Russ. Transl. Nauka, Moscow, 1979, Chap. 7.1.

[17] N. A. Armand and V. N. Sekistov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **23**, 555 (1980).

[18] N. N. Zernov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **25**, 520 (1982).

[19] V. N. Popov, *Path Integrals in Quantum Field Theory and Statistical Physics*, Atomizdat, Moscow, 1976.

[20] A. G. Vinogradov and Yu. A. Kravtsov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **16**, 1055 (1973).

[21] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'ma, Teor. Mat. Fiz. **74**, 360 (1988).

[22] M. Born and E. Wolf, *Principles of Optics*, Pergamon, New York, 1959 (Russ. Transl. Nauka, Moscow, 1970).

[23] M. I. Ryazanov, *Electrodynamics of Condensed Matter*, Nauka, Moscow, 1984.

[24] H. A. Gersch and W. A. Holm, Phys. Rev. B **14**, 1307 (1976).

[25] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, Nauka, Moscow, 1989, §§152, 131 (previous editions of this book have been published in English translation by Pergamon, New York).

[26] Yu. N. Barabanenkov and E. G. Stainova, in *Abstracts, All-Union Conference on Current Problems in Statistical Physics* (Lvov, 1987) Part II, ITF Akad. Nauk UkrSSR, Kiev, 1987, p. 110.

[27] Yu. N. Barabanenkov and E. G. Stainova, Zh. Eksp. Teor. Fiz. **88**, 1967 (1985) [Sov. Phys. JETP **61**, 1166 (1985)].

[28] A. Yu. Val'kov, L. A. Zubkov, A. P. Kovshik, and V. P. Romanov, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 281 (1984) [JETP Lett. **40**, 1064 (1984)].

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