Adiabatic chaos and particle diffusion

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(Submitted 16 March 1990; resubmitted 30 October 1990)
Zh. Eksp. Teor. Fiz. **99**, 763–776 (March 1991)

We study a class of systems in which chaos is connected with the non-conservation of the adiabatic invariant when one passes through a separatrix, using as example the problem of the motion of a charged particle moving at an angle to the magnetic field in the field of a wave packet. We give a method for the approximate description of the dynamics in such systems. A distinguishing feature of the dynamics is that in the adiabatic limit the measure of the region of chaotic motion remains finite. Transport processes in large volumes of phase space therefore turn out to be connected with adiabatic chaos. We show in this paper that in a well defined range of parameters the transport has the nature of Levi flights and is anomalous.

1. INTRODUCTION

Recently there has been considerable interest in the study of transport processes arising as the result of dynamic chaos.[1–4] This interest is connected with both fundamental problems of statistical physics and with the manifold applications in plasma physics (stochastic particle acceleration and heating of charged particles) and hydrodynamics (stochastic advection).

Relatively recently the kind of systems in which the production of chaos is connected with multiple changes in the nature of the motion when the parameters of the system are changed slowly has been the object of investigations. These are Hamiltonian systems with one degree of freedom and a slow time-dependence or (the case which is considered below) systems with two degrees of freedom where the variables corresponding to one of the degrees of freedom change rapidly and those of the other degree change slowly.

In the phase plane of the fast variables there are separatrices for fixed values of the slow variables. The drift of the slow variables induces the phase points to cross these separatrices. The motion far from the separatrices has an adiabatic invariant—an approximate integral the value of which along the trajectory undergoes only small, of order ε, oscillations where the small parameter ε characterizes the ratio of the rates of change of the slow and the fast variables. When one passes through a separatrix the adiabatic invariant undergoes a small change of order $\varepsilon \ln \varepsilon$ or, in some problems, of order ε.[5–8] It is important that this change depends strongly on the initial conditions and is therefore random in the sense of the theory of dynamic chaos. The summation of such random changes in the adiabatic invariant when one passes many times through a separatrix leads to a diffusion of the adiabatic invariant and the occurrence of a unique form of chaos which is called slow or adiabatic chaos.[9]

An important feature of adiabatic chaos is that in the adiabatic limit, as $\varepsilon \to 0$, the measure of the region of chaotic motion nevertheless stays finite (~ 1) and not small, as, for instance, in problems with overlapping resonances.[2] Indeed, as $\varepsilon \to 0$, the chaos region is approximately the same on each energy level surface of the system as the region (of measure ~ 1) of phase space in which the motion with conservation of the adiabatic invariant leads to a transition through a separatrix. It therefore turns out that there are transport processes connected with adiabatic chaos in large phase space volumes. Because of the fractal nature of the chaos region, the statistical properties of the motion are very complicated and it was established in many numerical experiments that they are, first of all, characterized by long-time correlations.[10,11] In multi-dimensional Hamiltonian systems the presence of such correlations can lead to anomalous particle transport.

We study in the present paper adiabatic chaos and the transport connected with it in a model problem about the motion of a charged particle in a magnetic field and the field of a wavepacket which propagates at an angle to the magnetic field.[12] Under cyclotron resonance conditions and in a well defined range of parameters this problem is described by a Hamiltonian system with two degrees of freedom, one of which corresponds to a fast motion and the other one to a slow motion. The dynamics of the slow variables is in the adiabatic approximation described by the average of the initial system over the fast motion. We show that these dynamics lead to multiple crossings of the separatrix of the fast motion and, hence, to the occurrence of adiabatic chaos. We obtain estimates for the diffusion rate of the adiabatic invariant and study the reconstruction of the diffusion region when the particle energy changes. We consider the anomalously fast transport process which appears due to the effect of the chaos-order boundary.

The acceleration of the diffusion-rate is caused by the existence of Levi flights in the phase plane of the fast variables. Random walks accompanying Levi flights were observed in Ref. 12 for the problem considered in what follows, of the motion of a charged particle in a magnetic field and in the field of a wavepacket. It was shown in Ref. 13 that such anomalous processes, connected with the multifractal nature of the dynamics of a nonintegrable system, are to a larger or lesser degree present in the majority of dynamic systems with chaos. In what follows we describe the mechanism of the onset of Levi flights: flights in the phase plane of the fast motion are caused by the "sticking" of a particle in a well defined region in the phase plane of the slow variable with a small measure. In that region there is no crossing of the separatrix, but the adiabatic tori are disrupted. For some values of the energy we determined numerically the way the probability P for the flight of a particle depends on its length l, and it turns out to be a power-law dependence: $P(l) \propto l^{-1-\alpha}$ with an index $\alpha < 1$. The anomalous transport obeys a free flight law, i.e., the mean square particle displacement increases proportional to the square rather than to the first power of the time, as occurs for normal diffusion.

 0038-5646/91/030423-08$03.00

2. RESONANCE HAMILTONIAN OF A CHARGED PARTICLE IN THE FIELD OF A WAVEPACKET, PROPAGATING ALMOST TRANSVERSELY TO A MAGNETIC FIELD

The Hamiltonian of a charged particle of mass m and charge e in an electromagnetic field has the form

$$H=\frac{1}{2m}\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)^2+e\varphi, \qquad (1)$$

where $\mathbf{P}=\mathbf{p}+(e/c)\mathbf{A}$ is the generalized particle momentum and $\mathbf{A}$ and φ are, respectively, the vector and scalar potentials of the electromagnetic field. We choose the potentials in such a way that the constant magnetic field be directed along the Z axis and the nonstationary electric field lie in the XZ plane, i.e.,

$$\mathbf{A}=B_0X\mathbf{e}_y, \qquad \varphi=\varphi(X,Z,t), \qquad (2)$$

where B_0 is the magnetic field strength.

Because the Hamiltonian (1) is independent of the Y coordinate there is an additional integral of motion in the system—the generalized momentum component P_y is conserved. Thanks to this one can eliminate one degree of freedom and the Hamiltonian of the problem,

$$H=\frac{p_x^2}{2m}+\frac{p_z^2}{2m}+\frac{m}{2}\omega_0^2X^2+e\varphi(X,Z,t), \qquad (3)$$

where $\omega_0=eB_0/mc$ is the cyclotron frequency, corresponds to a dynamical system with $2\frac{1}{2}$ degrees of freedom (the half degree of freedom is added because the potential φ is nonstationary). We now specify Eq. (3) choosing the potential of the electric field in the form of a wavepacket:

$$\varphi(X,Z,t)=-\varphi_0\sum_{n=-\infty}^{\infty}\cos(k_xX+k_zZ-n\Delta\omega t)$$

$$=-\varphi_0T\cos(k_xX+k_zZ)\sum_{n=-\infty}^{\infty}\delta(t-nT), \qquad (4)$$

where $T=2\pi/\Delta\omega$ is a characteristic time interval between the pulses of the δ-function and k_x and k_z are components of the wavevector, while we have assumed uniformity and a sufficiently large spectral width for the wavepacket, as in Ref. 12. A detailed discussion of these assumptions was given in Ref. 14. If the characteristic frequency of the wavepacket is a multiple of the cyclotron frequency,

$$\Delta\omega=q\omega_0, \quad q=\pm1,\ \pm2,\ldots, \qquad (5)$$

after the time for one rotation of the particle in the magnetic field it undergoes exactly q "impacts" from the field of the waves. We can then in the Hamiltonian (3) split off the resonance terms under the condition that the cyclotron rotation is a high-frequency one and the motion of the particle along the magnetic field is close to being in resonance with the cyclotron frequency

$$\omega_0\gg\max\{|k_zv_z-\omega_0|,\ (e\varphi_0k_x^2/m)^{1/2}\}, \qquad (6)$$

where $v_z=p_z/m$ is the particle velocity along the magnetic field. In the general case when $q>2$ the resonance Hamiltonian (3) corresponds to a nonintegrable Hamiltonian system with dynamics which in many respects are determined by the angle at which the wavepacket propagates.

In the simplest case ($q=4$) the resonance Hamiltonian $\mathscr{H}$ has the following form:[12]

$$\mathscr{H}={}^1/_2\varepsilon^2P^2-(\sin v\sin z-\sin u\cos z), \qquad (7)$$

where P, z, and u, v are dimensionless canonically conjugate momenta and coordinates:

$$u=\frac{k_xp_x}{m\omega_0}, \quad v=-k_xX, \quad z=k_zZ-\omega_0t,$$

$$P=\frac{1}{k_z}\frac{k_x^2}{m\omega_0}\left(p_z-\frac{m\omega_0}{k_z}\right)$$

with the dimensionless time $\tau=\Omega^2t/2\omega_0$ [$(\Omega=(e\varphi_0k_x^2/m)^{1/2}$ is the frequency of the small oscillations in the wavepacket field], $\varepsilon=2^{1/2}k_z\omega_0/k_x\Omega$ is a parameter which depends on the angle of the wavepacket propagation. In what follows we shall be interested in the case of almost perpendicular propagation of the wavepacket with respect to the magnetic field, i.e., we shall assume the parameter ε to be small. When $\varepsilon=0$ the Hamiltonian (7) describes the dynamics of a particle with one degree of freedom. Including the longitudinal component of the field ($\varepsilon\neq0$) adds a second degree of freedom and makes the problem nonintegrable. When $\varepsilon\ll1$ the variables u, v are fast and ρ, z slow ($\rho=\varepsilon P$).

We call the system in the variables u, v with the Hamiltonian (7) with $\rho, z=$ const the fast one. The phase portraits of the fast system on the uv plane are shown in Fig. 1 for $z\in[0,\pi/4)$, $z=\pi/4$, and $z\in(\pi/4,\pi/2]$. When z increases further, the portraits repeat. The separatrices occurring in the portraits separate the phase plane into regions of oscillations and rotations, i.e., trapped and free particles, respectively. For $z=\pi/4+\pi n/2$ ($n=0,\pm1,\pm2,\ldots$; in what follows the values of z will be defined up to terms $\pi n/2$) the separatrices form a square lattice and for $z=\pi/2$ they degenerate into straight lines.

3. DRIFT OF THE SLOW VARIABLES IN THE ADIABATIC APPROXIMATION

Slow changes in z lead by virtue of the equations of motion to pulsations of the separatrices of the fast system with

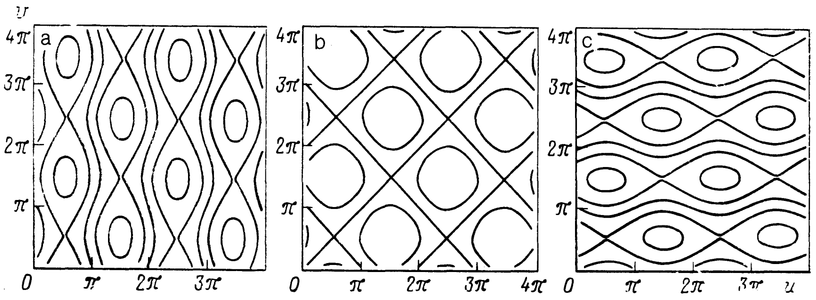

FIG. 1. Phase portraits in the fast motion uv plane; a: $0<z<\pi/4$; b: $z=\pi/4$; c: $\pi/4<z<\pi/2$.

time. When the particle moves it may therefore cross the separatrix net in both directions and as a result spend part of the time in rotational and part of the time in oscillatory regions.

To describe the drift of the slow variables z, ρ we use the adiabatic approximation. To do this we determine in each of the regions into which the separatrices of the fast system divide the phase space the function $I = I(z,\rho,\mathscr{H})$, the adiabatic invariant,[8] as follows: in the oscillatory regions of the fast system $I(z,\rho,\mathscr{H})$ is the same as half the action integral of the $\mathscr{H} = h$ trajectory of the fast system, i.e.,

$$I=\frac{1}{4\pi}\oint_{\mathscr{H}=h} u\,dv.$$

In the rotational regions I is equal to the action on the $\mathscr{H} = h$ trajectory of the fast system. For such a definition I is continuous on the separatrices. The equations for the evaluation of the adiabatic invariants I_{tr} of trapped and I_{fr} of free particles have the following form:

$$I_{tr}(z,\rho,h)=\frac{2}{\pi(\sin z\cos z)^{1/2}}\int_{|h-\rho^2/2|}^{\sin z+\cos z} dy\,K\left(\left[\frac{(\sin z+\cos z)^2-y^2}{4\sin z\cos z}\right]^{1/2}\right), \quad (8)$$

$$I_{fr}(z,\rho,h)=I_{tr}(z,\rho,h={}^1/_2\rho^2\pm(\sin z-\cos z))+\frac{2}{\pi}\int_{|h-\rho^2/2|}^{|\sin z-\cos z|} dy\,K\left(\left[\frac{4\sin z\cos z}{(\sin z+\cos z)^2-y^2}\right]^{1/2}\right)\times\frac{1}{[(\sin z+\cos z)^2-y^2)]^{1/4}}, \quad (9)$$

where $K(\varkappa)$ is the complete elliptic integral of the first kind, $z\in(0,\pi/2)$.

For motion far from the separatrices, I undergoes along an actual trajectory only small oscillations, of order ε.[5-7] In the adiabatic approximation the value of I is assumed to be constant along a trajectory and the change in z and ρ is described by the Hamiltonian system:

$$\dot z=\varepsilon\frac{\partial\mathscr{H}}{\partial\rho},\quad \dot\rho=-\varepsilon\frac{\partial\mathscr{H}}{\partial z}, \quad (10)$$

where the Hamiltonian $\mathscr{H}$ is expressed in terms of z, ρ and I (this system is obtained by averaging the differential equations for z, ρ over the fast motion). As $\mathscr{H}$ is an integral of the motion the $I(z,\rho,h) = \text{const}$ level lines determine the trajectories of the particle motion in the $z\rho$ plane. The set of such level lines is for a given value of $\mathscr{H} = h$ called the phase portrait of the slow motion. Two possible kinds of such portraits are shown in Fig. 2 It is clear from the figure that the values $z = \pi/4$ and $z = \pi/2$ correspond to equilibrium positions. One can study their stability analytically but the portraits shown suffice to describe qualitatively the dynamics of the system.

Each point in the $z\rho$ plane corresponds to a whole set of trajectories of the fast system, given by the equation $\mathscr{H} = h$. The set of points z, ρ corresponding to separatrices is called in Ref. 15 the indeterminacy curve. The indeterminacy curves are in Fig. 2 indicated by the heavy lines. These curves are periodic with a period $\pi/2$ and are for $0 < z < \pi/2$ given by the equation

$$^1/_2\rho^2\pm(\sin z-\cos z)=h, \quad (11)$$

which mean that along an indeterminacy curve the value of the Hamiltonian in one of the saddle points coincides with h. The indeterminacy curve separates in the $z\rho$ plane points which correspond to different kinds of motion of the fast system: rotational and oscillatory motions.

We now consider in detail the connection between the phase portraits in Fig. 2 and the motion in the uv phase plane. The motions of free particles in the phase plane of the fast motion correspond in Fig. 2a to four amygdaloidal sectors, bounded by the indeterminacy curves. For the sectors with centers at $z = 0$ and $z = \pi$ the free motion is infinite in the v coordinate (as in Fig. 1a) and for the sectors with centers at $z = \pi/2$ and $z = 3\pi/2$ the motion is infinite in the u direction (as in Fig. 1b). The points positioned outside these sectors correspond to the finite motion of trapped particles in the uv phase plane.

There are three different kinds of trajectories in Fig. 2a. Trajectories which do not intersect the indeterminacy curves belong to the first kind. They correspond to particles which do not change the nature of their motion, i.e., they are all the time trapped or all the time free.

The second type are trajectories intersecting (in four points) only a single indeterminacy curve. On this kind of trajectory a particle which initially is free, for instance, becomes trapped after its trajectory crosses the indeterminacy curve and it either retains its initial direction or changes it to the opposite one after the next crossing of this curve (see Fig. 3a). These two variants of its motion are equally probable due to the symmetry of the problem (the probability arises here because the initial conditions corresponding to a conservation or a change in the direction of the motion are interchangeable when $\varepsilon\to 0$; a probability approach to such problems was proposed in Ref. 16). The motion in the plane of the fast uv variables has the form of random walks along a straight line parallel either to the u axis or to the v axis. The step of the walk (its magnitude is $\sim\varepsilon$) is to a first approximation constant and equal to the distance over which the particle is displaced in the interval between two captures.

To the third type belong trajectories which cross all indeterminacy curves. Such trajectories, after the particle is trapped in a cell in the uv plane and subsequently has left it, change the direction of their motion with equal probabilities by $\pm\pi/2$. Since by virtue of the conservation of the adiabatic invariant the step of the walk is constant to a first approximation and is the same for different directions, the motion in the plane of the fast variables has the form of random walks over a square lattice (Fig. 3b).

It will be shown in what follows that multiple crossings of the separatrix lead to a diffusion of the adiabatic invariant and therefore the time during which the particle is in a regime of the second or the third kind of motion is proportional to the area corresponding to each kind in the phase portrait of the slow motion. The first kind of trajectories, which do not cross the indeterminacy curves, always conserve the adiabatic invariant[17] and are in the whole phase space for most initial conditions wrapped on the invariant tori.

4. DIFFUSION OF THE ADIABATIC INVARIANT

In actual fact, the adiabatic invariant is conserved only for motion far from the separatrix. On the separatrices the

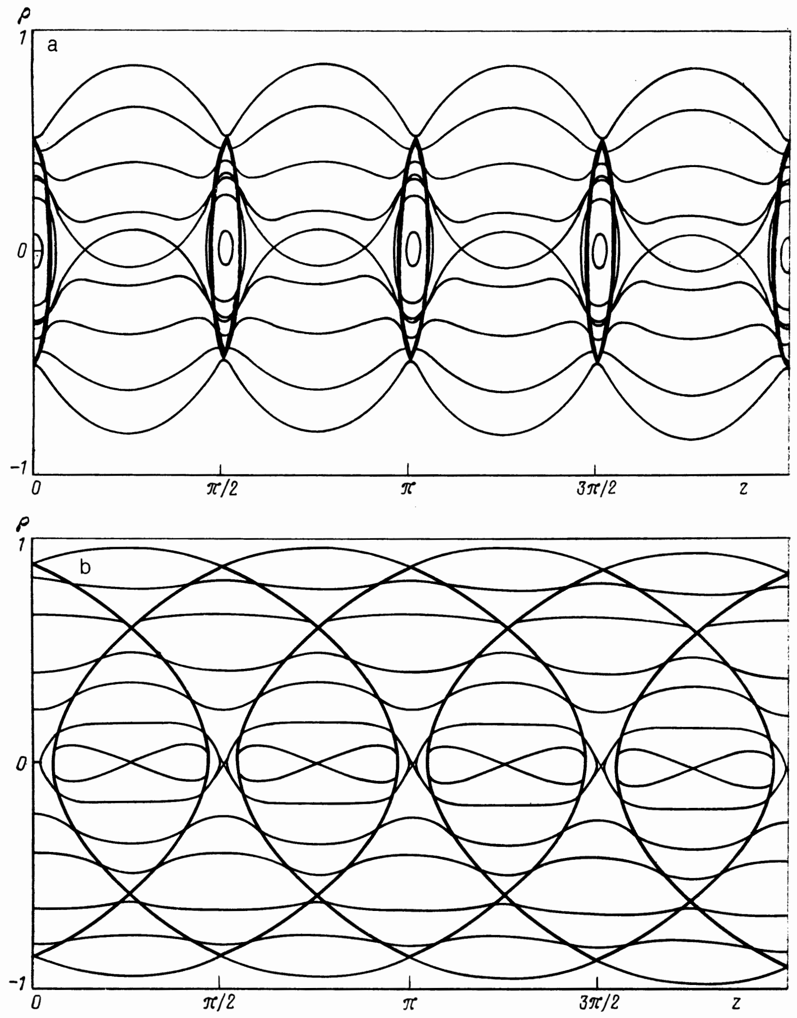

FIG. 2. Phase portraits of the slow motion in the $z\rho$ plane; a: $h = -0.83$; b: $h = 0.88$.

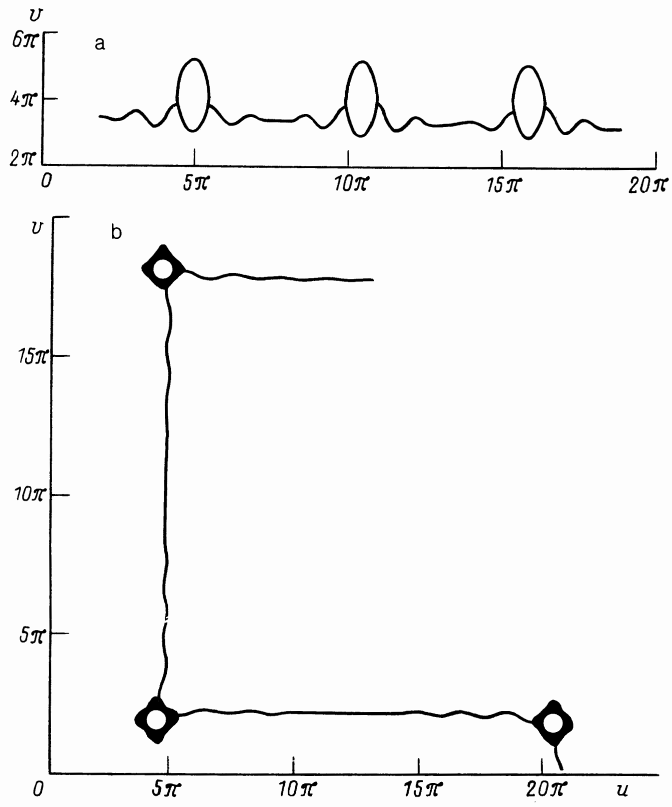

FIG. 3. Projections of two trajectories corresponding to different kinds of motion in the fast variables uv plane.

frequency of the fast motion becomes zero and the adiabatic invariant I has a singularity as function of the energy. The vicinity of the separatrices is a zone of nonadiabaticity and when one passes through this vicinity the value of I changes.

The problem of the magnitude of the change in the adiabatic invariant when one passes through a separatrix has been considered in a number of papers.[5-8] This change turned out to depend strongly on the initial conditions, i.e., to be random in the sense of dynamic chaos theory. This small change, of order ε (for the present problem), must be separated from the oscillations of the same order in ε which the adiabatic invariant executes far from the separatrix. To do this we introduce an improved adiabatic invariant

$$J = I + \varepsilon G(u, v, z, \rho),$$

which for motion far from the separatrix undergoes only oscillations of order ε^2. Here G is a smooth function of its arguments and its average value on a trajectory of the fast motion is equal to zero. A formula for G is given in Ref. 8. When one passes through the vicinity of the separatrices J undergoes a change ΔJ of order ε. The general formula from Ref. 8 gives to first approximation

$$\Delta J = -\varepsilon \frac{\rho a}{4\pi} \frac{\partial S}{\partial z} \ln(2 \sin \pi \xi). \tag{12}$$

Here $a = (-\nu)^{-1/2}$, where ν is the determinant of the matrix of the second derivatives of the Hamiltonian $\mathscr{H}$ of the fast system in the saddle point; $S = S(z)$ is the area of the

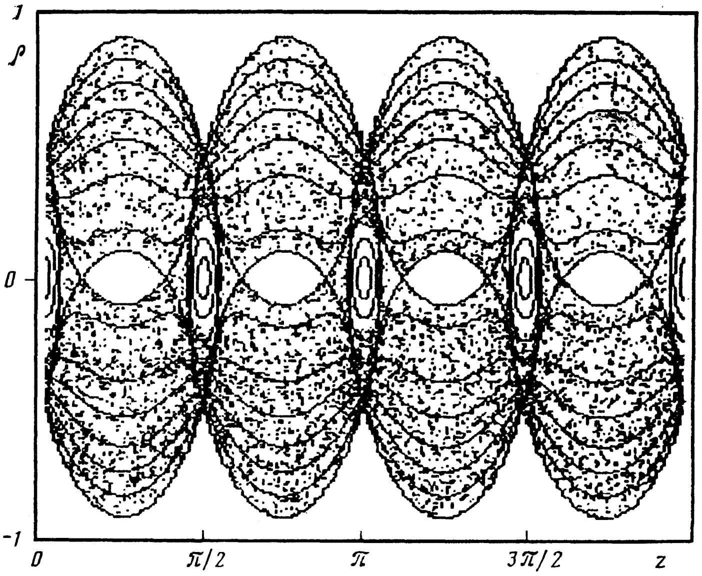

FIG. 4. In the $z\rho$ plane the positions of the particle trajectories over various time intervals are drawn for $h = -0.83$ in the phase portrait of the slow motion.

oscillatory region in the phase portrait of the fast system; ξ is a random quantity uniformly distributed in the interval $(0,1)$; the values of ρ, a, and $\partial S/\partial z$ are evaluated at the moment of passing through the separatrix (one assumes that $z \neq \pi/4$). The calculations give

$$a = \frac{2}{(2|\sin 2z|)^{1/2}},$$

$$\left|\frac{\partial S}{\partial z}\right| = \frac{8}{|\sin 2z|} \ln \left| \frac{|\cos z|^{1/2} + |\sin z|^{1/2}}{|\cos z|^{1/2} - |\sin z|^{1/2}} \right|. \tag{13}$$

The summation of the quantities ΔJ for multiple passages through the separatrix leads to a diffusion of J and, hence, to a diffusion of I (since far from the separatrices I and J differ only by an amount of order ε) and a violation of the adiabatic approximation. The projection of the phase point of the total system on the slow variables plane far from the indeterminacy curve oscillates close to the $I =$ const level line (see Fig. 2) and when one crosses the vicinity of the indeterminacy curve it is displaced along that curve by a small random distance in agreement with Eq. (12) and starts to oscillate close to another $I =$ const level line. After $1/\varepsilon^2$ passages through the indeterminacy curve, i.e., after a time $\sim 1/\varepsilon^3$, the value of I along the trajectory is changed by an amount of order 1 and a pattern of chaos emerges which is called slow or adiabatic chaos. In particular, after a time $\sim 1/\varepsilon^3$ the two regimes of random walk in the fast variables plane become mixed up with one another: along a straight line or along a square lattice (see Fig. 3).

Figure 4 demonstrates the process described above. Here we have drawn, in the phase portrait of Fig. 2a, the positions of the particle at equal time intervals and the time of the recording was chosen sufficiently large (everywhere in the examples $\varepsilon^2 = 0.00025$). During that time only those regions occupied by the $I =$ const lines which do not cross the indeterminacy curves remain not covered by trajectories. As one expected, the representative point covered the space rather uniformly. This enables us to state that multiple passages through a separatrix lead to a loss of memory about the initial conditions. In the regions unoccupied by trajectories the quantity I is conserved and the motion is regular (for most initial conditions) according to Arnol'd's theorem about the continuing conservation of the adiabatic invariant.[17] The dimensions of the stability region are determined by the relative position of the $I =$ const lines and the indeterminacy curves.

It is interesting to note that we here encounter a new kind of bifurcations, connected with the relative arrange-

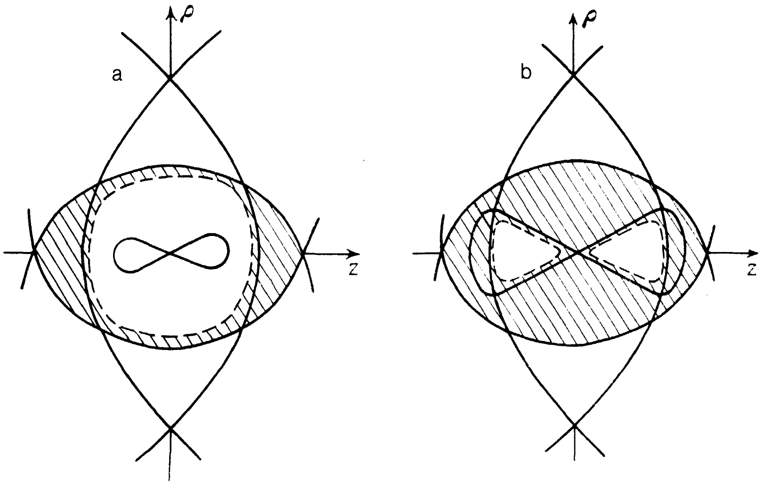

FIG. 5. Bifurcations arising due to a change in the relative positions of the $I =$ const lines and the indeterminacy curves.

ment of the separatrices of the slow motion and the indeterminacy curves (see, e.g., Fig. 5). On the fragment *a* the separatrix does not intersect the indeterminacy curve and there exists then on the portrait only one regular motion region. It is not shaded in the figure. On the fragment *b* the separatrix intersects the indeterminacy curve and there are now two different regular motion regions in the system.

5. PARTICLE DIFFUSION IN THE FAST-MOTION PLANE

We discuss in more detail the random walk of particles in the uv fast-motion plane. It was shown above that a typical picture of the motion with unstable trajectories consists in bounded oscillations with an amplitude of the order of the size of the cell and flights of various durations along the directions $u = \text{const}$ and $v = \text{const}$ due to crossing a separatrix. The duration of the rotational (free flight) stage depends on the magnitude of the adiabatic invariant I: it is (in first approximation) equal to the time of motion in the $z\rho$ plane (in the region corresponding to free trajectories) along an $I = \text{const}$ level line between two consecutive intersections of this line with indeterminacy curves; this time can be calculated from the solution of the set (10). Since I changes diffusively, the free flight length (the step of the random walk) also changes with time. The average step of the random walks ($\sim\varepsilon^{-1}$) is for $\varepsilon \ll 1$ much larger than the size of the cell (~ 1) which leads to a fast chaotic particle transport.

However, the picture described here is not valid for all values h of the energy. For instance, there is no diffusion at all for $-2^{1/2} < h < -1$. For $-1 < h < 0$ there are closed trajectories (see Fig. 2a) which do not intersect the indeterminacy curve in the region bounded by this curve. These trajectories correspond to regular rotation in the uv plane. The size of the region of regular motion is, generally speaking, determined by the position of the indeterminacy curve.

Actually the boundary curve separating the regions of regular and chaotic motions is not exactly the same as the $I = \text{const}$ curve tangent to the indeterminacy curve, and lies somewhat deeper in the rotational region. The area included between these curves tends to zero as $\varepsilon^2\ln^2(1/\varepsilon)$ as $\varepsilon \to 0$; the probability for the particle to be in that region is therefore also small. However, notwithstanding the smallness of this probability, the region near the boundary of the regular and the chaotic motions can to a large degree determine the statistical properties of the dynamical system. Due to the diffusion of the adiabatic invariant the particle occasionally lies in that region and is in a rotational regime until the changes of the adiabatic invariant remove it from this region. It is important that for such a motion the trajectory does not intersect but only touches the indeterminacy curve and therefore the change in the adiabatic invariant is in the case of this touching no longer described by Eq. (12).

We turn to the result of the numerical simulation. In Fig. 6, corresponding to an energy $h = -0.83$, we show the positions of a point in the fast motion phase plane during a long time interval. We depict 10 trajectories with initial conditions chosen randomly in the chaos region. For each of the trajectories the recording time is much longer than the time for the diffusion of the adiabatic invariant I. In this figure there are trajectories for which the diffusive process alternates with sections of long flights along the $u = \text{const}$ and $v = \text{const}$ directions. These flights correspond exactly to a lengthy sticking near the critical invariant curve in the $z\rho$ plane which separates the regular and the chaotic motions.

Extensive analytical and computational studies[1,11] have been devoted to such statistical anomalies of chaotic motion, connected with the presence of a boundary for the chaos; in those studies one observed that the correlation functions of the chaotic trajectories can increase much more slowly than exponentially, in particular, according to a pow-

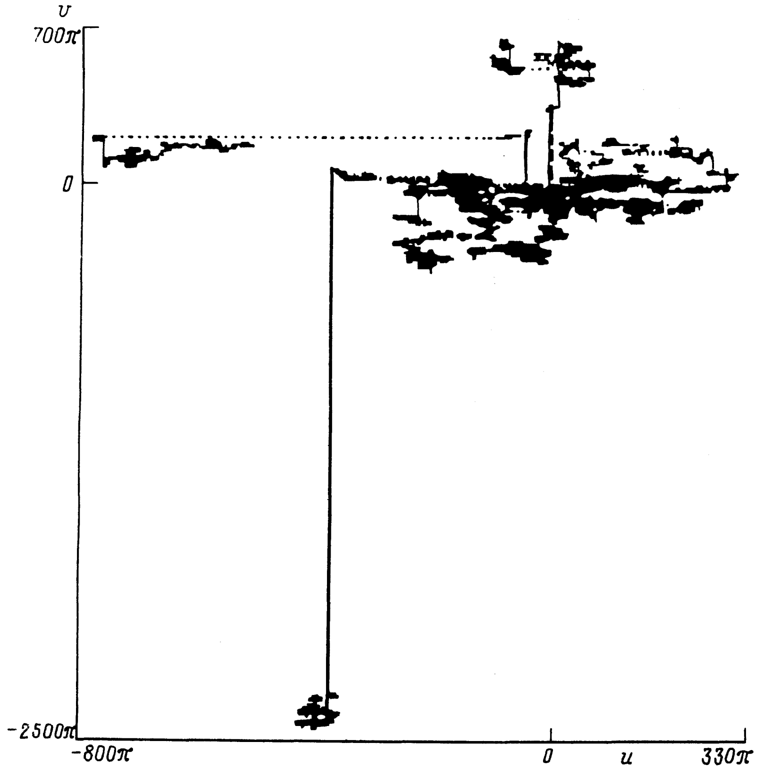

FIG. 6. Particle flights in the fast motion uv plane for $h = -0.83$.

er law. Another aspect of this effect is the possibility of anomalously fast diffusion[3,13] along some degrees of freedom (in our case in the fast variables uv plane) caused by the existence of lengthy, quasi-regular insertions in the particle trajectory which were observed in Refs. 12 and 13.

To find the transport coefficients, i.e., to determine the time-dependence of the mean square displacement of a particle in the uv plane one must determine the way the probability density $P(l)$ for the flight, which a particle carries out when it is "stuck" near the boundary curve, depends on the length l of the flight. To determine $P(l)$ we gave 600 initial conditions, uniformly distributed in z for $\rho = 0$ in some range from $z_{\min}$ to $z_{\max}$. The initial conditions were chosen close to $z = \pi/2$ and $z_{\min} > z_c$, where

$$z_c = \pi/2 - {}^1/_2 \arcsin(1-h^2)$$

is the coordinate of the intersection of the line $\rho = 0$ with the indeterminacy curve. The initial conditions in the uv plane were chosen for $u = 3\pi/2$, i.e., near the maximum range of the separatrix. For values $z > z_{\min}$ anomalously long flights were observed in the chaos region. For particles in free flight we constructed a Poincaré section in the $z\rho$ plane for $u = 3\pi/2 (\mathrm{mod}\ 2\pi)$. This section is shown in Fig. 7. Inside the "ring" there is a stability region surrounding the elliptical point $z = \pi/2$. Small stability regions, which form a "necklace" of the ring, correspond to a resonance of order $1/\varepsilon$ between motions along the fast and the slow degrees of freedom.

From statistical considerations it is preferable to calculate the integral distribution

$$\tilde{P}(l) = \sum_{l'>l} P(l').$$

The $\tilde{P}(l)$ distribution is shown in Fig. 8 in a doubly logarithmic scale for $h = -0.15$. The asymptotic nature of the required distribution $P(l)$ can be obtained from $\tilde{P}(l)$ through differentiation. The procedure described here was repeated later with twice the number of trajectories with initial conditions in the same range of z values. The results in the two cases differed slightly and for the h-dependence of the distribution function over the flight lengths $P(l)$ we found

$$P(l) \approx l^{-1-\alpha}, \quad \alpha \approx 0{,}9\ (h=-0{,}83), \alpha \approx 0{,}7\ (h=-0{,}15). \quad (14)$$

For large values of the flight length the distribution function $P(l)$ thus decreases as a power law. The maximum path lengths, fixed for those two energy values are $17000/2\pi$ and $28000/2\pi$, respectively.

For random walks in the plane with variable steps and a probability density distribution depending on the flight length l following Eq. (14) the mean square particle displacement depends, according to Ref. 18, on the time as follows:

$$\langle R^2 \rangle = \langle (u-u_0)^2 \rangle + \langle (v-v_0)^2 \rangle \approx \begin{cases} t^2, & 0<\alpha<1, \\ t^{3-\alpha}, & 1<\alpha<2. \end{cases} \quad (15)$$

For the cases, studied numerically, with energies $h = -0.83$ and $h = -0.15$ we have $0 < \alpha < 1$ and the anomalous transport behaves following a free flight law independently of the fact that the motion has a random walk character. We note that random walks with distribution functions such as (14) leading to such a hyperfast diffusion, are called Levi flights.[18]

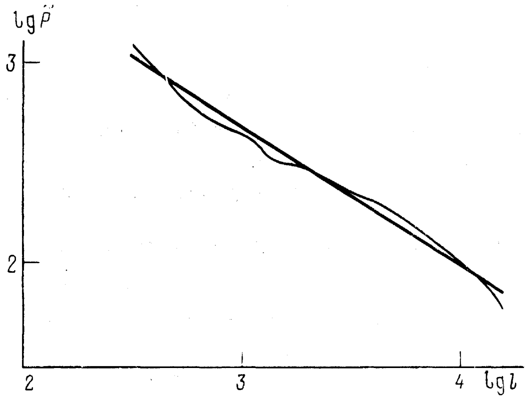

FIG. 8. Integral distribution $\tilde{P}(l)$ over flight lengths in a doubly logarithmic scale for $h = -0.15$ ($\alpha = 0.7$).

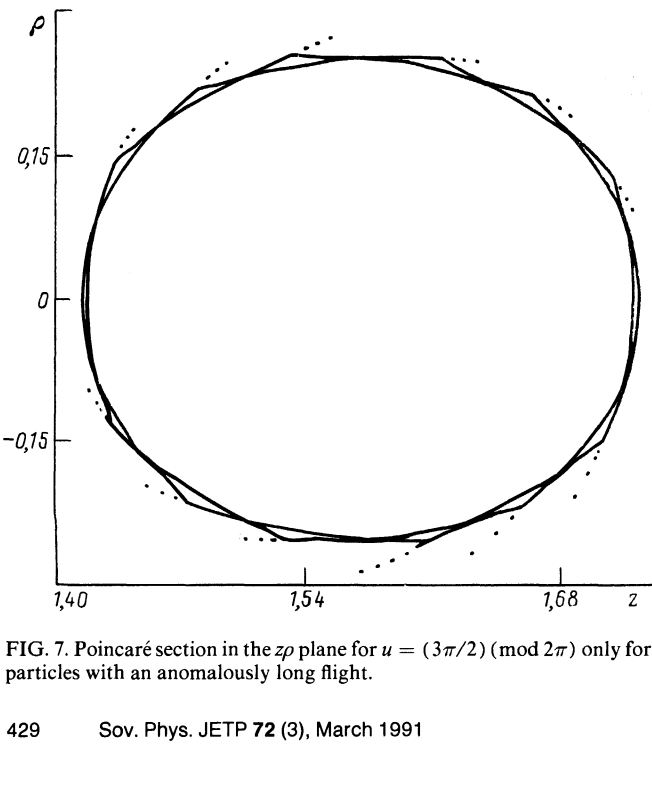

FIG. 7. Poincaré section in the $z\rho$ plane for $u = (3\pi/2) (\mathrm{mod}\ 2\pi)$ only for particles with an anomalously long flight.

6. CONCLUSION

In conclusion we note some general properties of systems in which chaos is adiabatic. The dynamics in such systems can be qualitatively described by constructing phase portraits of the motion in the slow plane averaged over the fast variables. The $I =$ const lines are in the adiabatic limit apart from small oscillations the same as the projections of the true trajectories of the motion in the slow variables plane everywhere where these lines do not intersect the indeterminacy curves. When there is an intersection the adiabatic invariant changes by an amount of order ε, where ε is the ratio of the frequencies of the slow and the fast motions. Multiple intersections of the indeterminacy curve lead to a diffusion of the adabatic invariant and to adiabatic chaos.

An important factor is the fact that as $\varepsilon \to 0$ the region of chaotic motion is not small. We have shown that in the problems considered small regions in phase space—the vicinities of the points where the $I =$ const lines touch the indeterminacy curves—may turn out to be the dominating influence on the diffusion. In these regions a particle is able for a long time to stick at the chaos-order boundary. Such a sticking of the particle leads to anomalously long flights in the fast variables plane. For the cases studied numerically the distribution of the lengths of these flights turned out to be a power-law one which leads to an anomalously fast diffusion.

The authors express their gratitude to G. M. Zaslavskiĭ for useful remarks and discussions of the results of this paper.

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Translated by D. ter Haar