Anomalous resistance of randomly inhomogeneous Hall media

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The problem of the average resistivity $\hat{\rho}_e$ of a randomly inhomogeneous medium in a strong magnetic field is analyzed for the case in which fluctuations of the Hall (antisymmetric) components of the microscopic resistivity tensor $\hat{\rho}$ are much larger than the diagonal elements of this tensor. A possible anisotropy of the inhomogeneity stemming from a difference between the length scales λ_z and $\lambda_\perp$ (respectively along and across the magnetic field) is taken into account. An anomalous bulk resistivity ($\rho_{e\perp} \gg \rho_0$) arises at $\lambda_z/\lambda_\perp \gg (\beta\Delta^2)^{-1}$, where ρ_0 is the microscopic resistivity, $\beta \gg 1$ is the Hall parameter, and Δ is the relative fluctuation in the quantity $\rho_0\beta$ ($\beta\Delta \gg 1$). The reason for the anomalous resistivity is the pronounced twisting of the current lines, whose distribution may be both quasiuniform and fractal [under the condition $\lambda_z/\lambda_\perp \gg (\beta/\Delta)^{1/2}$]. The effective resistivity reaches a maximum $\rho_{e\perp} \approx \rho_0\beta\Delta$ at the transition between these two regimes. A variational principle is formulated for the Hall current flow. Size effects (a dependence of $\rho_{e\perp}$ on the longitudinal dimension of the sample, b_z) are studied in the various regimes. The use of these results to describe plasma current opening switches is discussed.

1. INTRODUCTION

The classical problem of the average characteristics of a conducting medium with random fluctuations in its microscopic conductivity tensor is still far from final solution. Aside from the trivial case of layered (one-dimensional) inhomogeneities, most of the progress on this problem has come from the work by Dykhne.[1-3] The problem of the average conductivity of a two-dimensional, two-phase system was taken up in Ref. 1. For the case in which the arrangements of these phases are statistically equivalent, an exact result was found for the effective conductivity: $\sigma_e = (\sigma_1\sigma_2)^{1/2}$, where σ_1 and σ_2 are the conductivities of the two phases. This result was generalized in Ref. 2 to the case of a two-dimensional, two-phase Hall medium. A "Hall medium" is to be understood here as an anisotropic conductor in a (uniform) external magnetic field which has a local resistivity tensor

$$\hat{\rho}(\mathbf{r}) = \rho_0 \begin{pmatrix} 1 & \beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\beta \gg 1$ is the Hall parameter. In the case of a plasma described in electron MHD,[4,5] for example, we would have $\rho_0 = m/(ne^2\tau_e)$ and $\beta = \omega_{Be}\tau_e$ (here m, e, and n are the mass, charge, and density of the electrons; τ_e is the collision rate; and ω_{Be} is the electron Larmor frequency).

A next step was taken in Ref. 3: A Hall medium of a general type, inhomogeneous in three dimensions, was analyzed. The primary result of Ref. 3 was the development of a methodology based on a transformation from an analysis of Ohm's law,

$$\mathbf{j} = \hat{\sigma}\mathbf{E}, \quad \operatorname{div}\mathbf{j} = 0, \quad \operatorname{rot}\mathbf{E} = 0, \tag{2}$$

to the problem of the diffusion of a passive scalar φ,

$$\frac{\partial\varphi}{\partial t} + \mathbf{v}\nabla\varphi = \operatorname{div}(\hat{D}\nabla\varphi), \tag{3}$$

which can be treated by a variety of qualitative analysis methods. This ansatz is important, so we will repeat the corresponding calculations. Breaking up the conductivity tensor σ_{ik} into a symmetric part σ^s_{ik} and an antisymmetric part σ^a_{ik}, we write Ohm's law (2) with $\mathbf{E} = -\nabla\varphi$ in the form $\partial(\sigma_{ik}\partial\varphi/\partial x_k)/\partial x_i = 0$. This equation is equivalent to (3) in the case $\partial\varphi/\partial t = 0$, with

$$v_i = \partial\sigma_{ik}{}^a/\partial x_k, \quad D_{ik} = \sigma_{ik}{}^s. \tag{4}$$

The "flow" which arises here is incompressible: $\operatorname{div}\mathbf{v} = 0$. By finding the asymptotic (effective) diffusion $\hat{D}_e$ in convection-diffusion problem (3), (4), we can thus find the effective conductivity in our original problem:

$$\hat{\sigma}_e = \hat{D}_e + \langle\hat{\sigma}^a\rangle. \tag{5}$$

This conductivity relates the average values of the current density and the electric field:

$$\langle\mathbf{j}\rangle = \hat{\sigma}_e\langle\mathbf{E}\rangle, \tag{6}$$

where the angle brackets mean a spatial average.

A similar problem was attacked from a completely different direction in Refs. 4 and 5: Conservation laws, the most important of which are associated with the approximate freezing of the magnetic field **B** in the electrons,

$$\frac{\partial B}{\partial t} = \operatorname{rot}[\mathbf{v}_e\mathbf{B}] - \operatorname{rot}(D_m \operatorname{rot}\mathbf{B}), \tag{7}$$

were used to analyze the current flow through an inhomogeneous plasma. Here $\mathbf{v}_e = -c \operatorname{curl}\mathbf{B}/(4\pi en)$ is the electron current velocity (the ions are assumed to be immobile), $D_m = c^2/(4\pi\sigma_0)$ is the magnetic viscosity, $\sigma_0 \equiv 1/\rho_0$, and c is the velocity of light. In the terminology of electron MHD (EMHD), the meaning here is that the energy and the momentum, which are concentrated for the most part in the magnetic field component ($B^2/8\pi \gg nmv_e^2/2$ and $|ne\mathbf{A}/c| \gg nmv_e$, respectively; $\mathbf{B} = \operatorname{curl}\mathbf{A}$), are entrained by the electron flow and are scattered by various obstacles. The effect is an increased dissipation, for which the term "EMHD resistance" was introduced in Ref. 4. In particular, if all the magnetic energy[1)] $B^2/8\pi$ is dissipated at an electrode or at a plasma-vacuum interface, a plasma diode ac-

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quires a surface EMHD resistance

$$R \approx B/[4\pi enc \max(b_y, b_z)], \tag{8}$$

where b_x, b_y, b_z are the dimensions of the system along the coordinate axes; the external magnetic field **B** is assumed to be directed along the z axis; and the current I is flowing along the x axis. If there is no external magnetic field **B**, or if the magnetic field of the current is dominant, $\delta B \gtrsim B$, the result in (8) becomes dependent on I. The following formula was thus proposed in Ref. 4 for the EMHD resistance of a plasma current opening switch in the EMHD stage of the opening: $R = u/c^2 = 30u/c$ [Ω], where u is the average electron current velocity.

Although expression (8) can formally be written in terms of a resistivity,

$$\rho \approx \frac{B}{4\pi enc} \frac{\min(b_y, b_z)}{b_x}, \tag{9}$$

we should bear in mind that this is a surface resistivity, which is "connected in series" with the bulk EMHD resistivity due to the plasma inhomogeneity. The latter is totally unrelated to (9); in particular, it may be much higher (more on this below).

In this paper we systematically study the average resistivity of a randomly inhomogeneous Hall medium (the bulk EMHD resistivity in the case of a plasma). The inhomogeneities are assumed to be three-dimensional and, in general, anisotropic, with different length scales λ_z (along the magnetic field $\mathbf{B} = B\mathbf{e}_z$) and $\lambda_\perp$ (in the perpendicular direction). Our result agrees with that of Ref. 3 in the case of isotropic three-dimensional perturbations ($\xi \equiv \lambda_z/\lambda_\perp = 1$), but the results are different in the two-dimensional case ($\lambda_z = \infty$).

The paper is organized as follows. In Sec. 2 we evaluate the anomalous Hall resistivity of an unbounded current randomly inhomogeneous medium, making use of a convection-diffusion analogy, (3). If the anisotropy of the inhomogeneity is not too pronounced, $\xi \ll (\beta/\Delta)^{1/2}$, the current distribution is characterized by the same transverse (with respect to **B**) correlation length as characterizes the fluctuations of the medium ($\lambda_\perp$). The opposite case $\xi \gg (\beta/\Delta)^{1/2}$ corresponds to a fractal distribution of the current, characterized by anomalously large transverse correlation lengths for the current. Analysis of that regime requires invoking the methods of percolation theory.[6-8]

In Sec. 3 the geometric reasons for the occurrence of an anomalous resistivity, which are associated with the nature of the current paths, are discussed. In this connection, a variational principle is formulated for the current flow in an inhomogeneous Hall medium.

In Sec. 4 we examine the size effect for the bulk anomalous resistivity, i.e., the effect of the boundaries of a Hall conductor of finite size. In Sec. 5 we briefly summarize and discuss the results.

2. ANOMALOUS RESISTIVITY OF A BOUNDED, RANDOMLY INHOMOGENEOUS MEDIUM IN A STRONG MAGNETIC FIELD

In this section of the paper we make use of the convection-diffusion analogy,[3] which reduces Ohm's law (2) to the problem of the transport of a passive scalar, (3).

According to (1), the microscopic conductivity tensor $\hat{\sigma} = \hat{\rho}^{-1}$ is

$$\hat{\sigma} = \sigma_0 \begin{pmatrix} \frac{1}{1+\beta^2} & \frac{-\beta}{1+\beta^2} & 0 \\ \frac{\beta}{1+\beta^2} & \frac{1}{1+\beta^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

We are thus dealing with the diffusion of a passive impurity in an incompressible flow, specified by (4) (we recall that we have $\beta \gg 1$):

$$\mathbf{v}(x, y, z) = [\mathbf{e}_z, \nabla\psi(x, y, z)], \quad \psi = \sigma_0/\beta, \tag{11}$$

with a seed diffusion tensor

$$\hat{D} = \sigma_0 \begin{pmatrix} \beta^{-2} & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

We denote the length scales of the variation of the stream function ψ along and across the z axis by λ_z and $\lambda_\perp$, respectively, and we denote by Δ the characteristic depth of the relative fluctuations in $\psi(x, y, z)$. The convection velocity in (11) can then be estimated from

$$v \approx (\sigma_0/\beta)\Delta/\lambda_\perp. \tag{11'}$$

The convection effect in (3) outweighs the transverse seed diffusion if the Peclet number is large: $P \equiv \lambda_\perp v/D = \beta\Delta \gg 1$. We will make use of this inequality below as the major large parameter of the problem.

We begin with the case in which the anisotropy is not too large, $\xi \equiv \lambda_z/\lambda_\perp < (\beta/\Delta)^{1/2}$. This case corresponds to the situation in which the time scale of the longitudinal diffusion over a distance equal to the size of the inhomogeneity, $t_z = \lambda_z^2/\sigma_0$, is shorter than the transverse convection time $t_\perp = v/\lambda_\perp$; i.e., $t_z < t_\perp$. We thus have the important time interval $t_z < t < t_\perp$, in which the velocity **v** can be assumed independent of x and y; i.e., $\mathbf{v} = \mathbf{v}(z)$. To analyze the behavior of the passive impurity at these time scales we consider the auxiliary problem of the convection-diffusion transport, (3), in a shear flow $\mathbf{v} = \mathbf{v}(z) \perp \mathbf{e}_z$. We need to find an effective transverse diffusion. We introduce the new stream function

$$\boldsymbol{\Psi}(z) = \int^z \mathbf{v}(\zeta)\, d\zeta$$

and we initially assume that $\boldsymbol{\Psi}$ is bounded (without any loss of generality, we can assume that two averages over z vanish: $\langle \mathbf{v} \rangle_z = 0$, $\langle \boldsymbol{\Psi} \rangle_z = 0$). This problem was solved by Zel'dovich[9] in the particular case of a periodic plane-parallel flow, $\boldsymbol{\Psi} = \mathbf{e}_x \sin(kz)$. In general we would have, in place of (3),

$$\frac{\partial\varphi}{\partial t} + \frac{d\boldsymbol{\Psi}}{dz}\nabla\varphi = D_{\alpha\beta}\frac{\partial^2\varphi}{\partial r_\alpha \partial r_\beta} + D_{zz}\frac{\partial^2\varphi}{\partial z^2}; \quad \alpha, \beta = x, y. \tag{13}$$

Proceeding in the spirit of the quasilinear theory, we separate the impurity density φ into a smooth component and an oscillatory component: $\varphi(x, z, t) = \langle\varphi\rangle_z + \tilde{\varphi}$. After a sufficiently long time $t \gg t_z$, the oscillations become small, $\tilde{\varphi} \ll \langle\varphi\rangle_z$, and (13) yields

$$\frac{\partial\langle\varphi\rangle_z}{\partial t} + \left\langle \frac{d\boldsymbol{\Psi}}{\partial z}\nabla\tilde{\varphi} \right\rangle_z = D_{\alpha\beta}\frac{\partial^2\langle\varphi\rangle_z}{\partial r_\alpha \partial r_\beta} + D_{zz}\frac{\partial^2\langle\varphi\rangle_z}{\partial z^2}, \tag{14}$$

$$\frac{\partial\tilde{\varphi}}{\partial t}+\frac{d\boldsymbol{\Psi}}{dz}\nabla\langle\varphi\rangle_z=D_{\alpha\beta}\frac{\partial^2\tilde{\varphi}}{\partial r_\alpha\,\partial r_\beta}+D_{zz}\frac{\partial^2\tilde{\varphi}}{\partial z^2}. \tag{15}$$

Under the condition $t\gg t_z$, we can use a quasisteady approximation in (15), in which we can discard the first term on the left side. Also asymptotically small is the first term on the right side, which describes the seed transverse diffusion. Omitting these terms, and expressing $\tilde{\varphi}$ in (15) in terms of $\langle\varphi\rangle_z$, we substitute $\tilde{\varphi}$ into (14). As a result we find a diffusion equation for $\langle\varphi\rangle_z$ in the (x, y) plane with an effective diffusion tensor

$$\hat{D}_e=\hat{D}_\perp+\frac{\langle(\Psi_\alpha(z)\Psi_\beta(z)\rangle_z}{D_{zz}}. \tag{16}$$

The result in (16) is also valid for a shear flow in a layer $0\leqslant z\leqslant b_z$ whose boundaries are impenetrable to the impurity φ, under our earlier condition that there is no average flow, $\boldsymbol{\Psi}(0)=\boldsymbol{\Psi}(b_z)$. In this case the average in (16) is over the interval $[0, b_z]$. We will make use of this comment in Sec. 4.

Going back to our average-resistivity problem, we recall that $\boldsymbol{\Psi}(z)$ is proportional to the integral of the randomly oscillating bounded function $\Psi(z)$. Consequently, $\boldsymbol{\Psi}$ increases without bound, in the manner of a coordinate of a Brownian particle: $\langle\Psi^2(z)\rangle\approx\langle v^2\rangle\lambda_z z$. According to (16), the transverse displacement of the impurity particle is given by

$$r_\perp{}^2(t)\approx\frac{\langle\Psi^2(z(t))\rangle}{D_{zz}}t\approx\frac{\sigma_0\Delta}{\beta\lambda_\perp}t^{3/2}\sigma_0{}^{1/2}. \tag{17}$$

[Here we have used (11′) and $z(t)\approx(D_{zz}t)^{1/2}$, $D_{zz}=\sigma_0$.] We wish to stress that the motion of the particles in this regime is a "superdiffusion" motion.[3]

Equation (17) describes the motion of the impurity particles as long as we can ignore the dependence of the velocity on the transverse coordinates, i.e., at $r_\perp(t)<\lambda_\perp$. Beyond this point, a decorrelation sets in. The correlation time t_c can thus be found from the equation $r_\perp(t_c)=\lambda_\perp$:

$$t_c\approx(\sigma_0/\lambda_z{}^2)^{1/3}(\lambda_\perp/v)^{4/3}.$$

At $t>t_c$, the impurity transport becomes a diffusion with an effective diffusion coefficient (or an effective conductivity)

$$D_{e\perp}\approx\sigma_{e\perp}\approx r_\perp{}^2(t_c)/t_c\approx\sigma_0\xi^{2/3}(\Delta/\beta)^{4/3},\quad(\beta\Delta^2)^{-1}<\xi<(\beta/\Delta)^{1/2}. \tag{18}$$

The first of these inequalities corresponds to the requirement that the effective diffusion exceed the seed diffusion. The second of the inequalities was discussed above. In the isotropic case, with $\xi=1$, result (18) corresponds to that derived in Ref. 3.

The case of highly anisotropic fluctuations, $\xi>(\beta/\Delta)^{1/2}$, is less trivial, since over the longitudinal diffusion time t_z a particle manages to traverse a transverse distance which is greater than the transverse correlation length of the medium, $r_\perp(t_z)>\lambda_\perp$, and the fractal structure of the current lines, i.e., the contour lines $\psi(x, y, z)=\text{const}$ at a given z, becomes important. We know[6,7] that among the contour lines of a random function there are both short-circuited lines (with a diameter on the order of $\lambda_\perp$) and percolation lines (with a diameter much greater than $\lambda_\perp$). The longer the contour lines, the smaller the fraction of the area they occupy. Nevertheless, in convection in the presence of a weak seed diffusion it is the percolation current lines which dominate the transport, despite the relatively small number of these lines.

We first consider the two-dimensional case, $\lambda_z=\infty$. The corresponding problem of the effective diffusion in a random two-dimensional incompressible flow $\mathbf{v}(x, y)$ was solved in Ref. 6 under the assumption that there is a unique length scale of the flow, $\lambda_\perp$. The following asymptotic expression was found for the effective diffusion in the limit of a large Peclet number:

$$D_{e\perp}\approx\lambda_\perp v(D/\lambda_\perp v)^{3/13}, \tag{19}$$

where $D=\sigma_0/\beta^2$ is the seed diffusion, and the exponent 3/13 is expressed in terms of the critical exponents of the two-dimensional percolation problem.

The case of finite λ_z can also be dealt with by means of (19), with a suitable renormalization of D. We note in this connection that a diffusive motion of a particle along the coordinate z corresponds to an additional diffusive displacement of the particle with respect to the separatrices of the flow:[2)]

$$\delta\approx z(t)\lambda_\perp/\lambda_z=(\sigma_0 t)^{1/2}/\xi=(Dt)^{1/2},\ D=\sigma_0/\xi^2.$$

We thus find (19) with $D\approx\sigma_0/\beta^2+\sigma_0/\xi^2$:

$$D_{e\perp}=\sigma_{e\perp}\approx(\Delta/\beta)^{10/13}(\beta^{-2}+\xi^{-2})^{3/13},\ \xi>\sqrt{\beta/\Delta}. \tag{20}$$

Switching from the effective conductivity tensor to its inverse, the effective resistivity tensor, we write the results in (18) and (20) in the following form:

$$\hat{\rho}_e=\begin{pmatrix}\rho_{e\perp} & \rho_{eh} & 0\\ -\rho_{eh} & \rho_{e\perp} & 0\\ 0 & 0 & \rho_{e\parallel}\end{pmatrix}, \tag{21}$$

where

$$\rho_{e\parallel}\approx\rho_0,\quad\rho_{eh}\approx\langle 1/(\beta\rho_0)\rangle^{-1}, \tag{22a}$$

$$\rho_{e\perp}=(\rho_{eh})^2\sigma_{e\perp}\approx\rho_0\begin{cases}(\xi\beta\Delta^2)^{2/3}, & 1/\beta\Delta^2<\xi<(\beta/\Delta)^{1/2}, \quad (22b)\\ (\xi\beta/\Delta)^{3/13}\beta\Delta, & (\beta/\Delta)^{1/2}<\xi<\beta,\\ (\beta\Delta)^{10/13}, & \beta<\xi. \quad (22c)\end{cases}$$

Figure 1 shows the effective transverse resistivity as a function of the degree of anisotropy of the fluctuations, ξ. Note the unexpected maximum in the resistivity, $\rho_{e\perp}(\xi)\approx\rho_0\beta\Delta$, at $\xi\approx(\beta/\Delta)^{1/2}$, which is equal in order of magnitude to the fluctuations of the Hall components of tensor (1). We turn now to a discussion of the reasons for this behavior.

3. VARIATIONAL PRINCIPLE; QUASIUNIFORM AND FRACTAL DISTRIBUTIONS OF THE CURRENT

The average resistivity tensor (21), (22) thus by no means reduces to a simple spatial averaging of microscopic tensor (1). The transverse resistivity $\rho_{e\perp}$ may be many times ρ_0. Since the Joule dissipation power

$$W=\int(\mathbf{j}\hat{\rho}\mathbf{j})\,d^3\mathbf{r}=V\langle\mathbf{j}\rangle\hat{\rho}_e\langle\mathbf{j}\rangle \tag{23}$$

(V is the volume of the medium) is determined exclusively by the symmetric part of the microscopic resistivity,

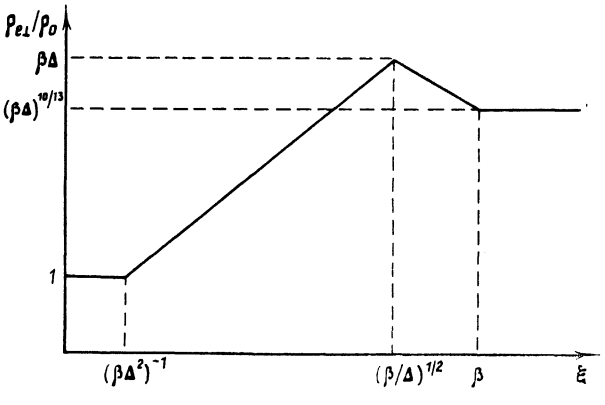

FIG. 1. Effective transverse resistivity as a function of the degree of anisotropy of the fluctuations.

$\rho^s_{ik} = \rho_0 \delta_{ik}$, the anomalous average resistivity in (22) can be explained in terms of an anomalous microscopic current distribution $\mathbf{j}(\mathbf{r})$.

It is well known that in the case of a symmetric resistivity tensor $\hat{\rho} = \hat{\rho}^s$, for the actual current distribution, the dissipation integral in (23) has a minimum under the condition that incompressibility ($\operatorname{div} \mathbf{j} = 0$) is retained. If there is an antisymmetric component, $\hat{\rho}^a \neq 0$, the situation changes. An elementary calculation shows that the second variation of (23) is again positive definite, while the first variation is given by the generally nonzero expression

$$\delta W = -2 \int (\mathbf{j}\hat{\rho}^a \delta\mathbf{j})\, d^3\mathbf{r}. \qquad (24)$$

Using the condition $\operatorname{div} \mathbf{j} = 0$, we reach the conclusion that (24) vanishes, i.e., that W has a minimum, for a certain class of current variations, which satisfies the auxiliary condition

$$\operatorname{rot} (\hat{\rho}^a \delta\mathbf{j}) \equiv \operatorname{rot} \rho_0\beta [\delta\mathbf{j}, \mathbf{e}_z] = 0. \qquad (25)$$

Condition (25) is none other than Ohm's law for a purely antisymmetric resistivity tensor $\hat{\rho} = \hat{\rho}^a$. In the limit of interest here, $\beta\Delta \gg 1$, this part is dominant, and the substitution $\hat{\rho} = \hat{\rho}^a$ represents the zeroth approximation for Ohm's law. Since the solution of this problem is multivalued in this approximation, the variational principle which we have formulated may be thought of as a principle for selecting the "correct" zeroth approximation in the current percolation problem. This correct approximation should be regarded as that "zeroth" distribution of $\mathbf{j}$ which minimizes the ohmic dissipation. Throughout the rest of this section of the paper, we are implicitly using this requirement.

We turn now to the geometric reasons for the anomalously large average resistivity. We introduce $\mathbf{u}(x, y, z) = \rho_0\beta\mathbf{j}$ (in the plasma case, $\mathbf{u}$ would be proportional to the electron current velocity). We write the exact microscopic Ohm's law in the form

$$\operatorname{rot} ([\mathbf{u}, \mathbf{e}_z] + \mathbf{u}/\beta) = 0. \qquad (26)$$

In the zeroth approximation in β^{-1} we have

$$\partial\mathbf{u}_\perp/\partial z = 0, \quad \operatorname{div} \mathbf{u}_\perp = 0, \qquad (27)$$

and thus, by virtue of $\operatorname{div} \mathbf{j} = 0$,

$$\psi u_z = -\int^z \mathbf{u}_\perp \nabla_\perp \psi(x, y, z')\, dz', \qquad \psi \equiv 1/\rho_0\beta. \qquad (28)$$

It follows from the second equation in (27) that in the two-dimensional case the current flows along contour lines of $\psi(x, y)$, whose pronounced twisting may substantially increase the resistivity. In the three-dimensional case, an additional degree of freedom arises, and the current may also flow around obstacles along the z axis. For random perturbations, however, the current must undergo a random walk along the z direction over a very large distance (more on this below). The current channels contract, since their volume (the product of the cross-sectional area of the channel and its length) is fixed by virtue of the incompressibility of the current. The ratio of $|\mathbf{u}_z|$ to $|\mathbf{u}_\perp|$ gives the factor by which the length of the channels increases (and the cross-section area of the channels decreases) in comparison with the corresponding value for planar current flow. According to the discussion above, the average resistivity increases by a factor of $(u_z/u_\perp)^2$ in comparison with that in the planar-flow case. In the zeroth approximation, this factor is infinite: For random perturbations of ψ, the integral in (28) diverges in diffusive fashion,

$$u_z = \left| (\mathbf{u}_\perp/\psi) \int_0^z \nabla_\perp \psi(x, y, z')\, dz' \right| \approx u_\perp \Delta (\lambda_z z)^{1/2}/\lambda_\perp, \qquad (28')$$

so the next approximation must be considered. We introduce the longitudinal current correlation length a_z, which determines the dependence of $\mathbf{u}_\perp$ on z, which is not present in the zeroth approximation: $|\partial\mathbf{u}_\perp/\partial z| \approx |\mathbf{u}_\perp|/a_z$. The integration in (28′) should then be limited to this dimension:

$$u_z/u_\perp \approx \Delta (\lambda_z a_z)^{1/2}/\lambda_\perp. \qquad (29)$$

A more rapid increase in $u_z/u_\perp$ would contradict the variational principle.

We thus find

$$\rho_{e\perp} \approx \rho_0 (u_z/u_\perp)^2 \approx \rho_0 \frac{\Delta^2 \lambda_z a_z}{\lambda_\perp^2}, \qquad \lambda_z < a_z. \qquad (30)$$

The limitation written out here corresponds to the assumption that the integral in (28) has a "Brownian behavior." This inequality also expresses the relatively slight sensitivity of the transverse current component $\mathbf{j}_\perp$ to the fluctuations of ψ, telling us, according to the variational principle, that there is no substantial tangling of $\mathbf{j}_\perp$. In accordance with this picture, we call the current percolation regime at $a_z > \lambda_z$ "quasiuniform."

To determine a_z we replace the zeroth approximation in (27) by a first-approximation equation which follows from (26):

$$\partial\mathbf{u}_\perp/\partial z = [\nabla (u_z/\beta), \mathbf{e}_z]. \qquad (31)$$

Setting $\partial/\partial z \approx 1/a_z$ in (31), and using (29), we find

$$a_z \approx \lambda_z [\lambda_\perp/\lambda_z (\beta/\Delta)^{1/2}]^{4/3}, \quad \xi < (\beta/\Delta)^{1/2}. \qquad (32)$$

Substituting (32) into (30), we see that a result derived above, (22a), is valid.

At $\xi \equiv \lambda_z/\lambda_\perp \approx (\beta/\Delta)^{1/2}$, the dimensions λ_z and a_z are comparable. As ξ increases further, the longitudinal correlation length of the current, a_z, becomes smaller than λ_z (more on this below), and the currents are forced to adjust in each cross section $z = z_0$ to accommodate the perturbations of the medium, running approximately along the contour lines of $\psi(x, y, z_0)$. Although this situation could be analyzed by an approach like that taken for the quasiuniform limit, it is simpler to invoke a convection-diffusion analogy here. According to the results derived above and in Refs. 6 and 7, it can be concluded that most of the current is concentrated in a "hot region" with a characteristic width

$$\delta\approx\lambda_\perp h=\lambda_\perp\left(\frac{D}{\lambda_\perp v}\right)^{3/13}\ll\lambda_\perp,\quad D\approx\sigma_0/\xi^2+\sigma_0/\beta^2. \tag{33}$$

The intersection of this region with the (x,y) plane is a web-shaped fractal whose dimension in our approximation of a single transverse length scale for the fluctuations, $\lambda_\perp$, is $d_h=7/4$ (Refs. 6 and 7). In the more general case of a power-law spectrum of inhomogeneities we would have[8] $1\leqslant d_h\leqslant 7/4$. An upper limit on the inertial interval of self-similarity (the transverse correlation length of the currents) is given by

$$a_\perp\approx\lambda_\perp h^{-1/3}=\lambda_\perp\left(\frac{D}{\lambda_\perp v}\right)^{-1/13}\gg\lambda_\perp. \tag{34}$$

In contrast with the quasiuniform case, we call this regime of an anomalous resistivity $[\xi>(\beta/\Delta)^{1/2}]$ a "fractal" regime.

The fractal regime is subdivided into two distinct subregimes. The first, which is sensitive to the size of the longitudinal inhomogeneity, λ_z, corresponds to the case in which the first term in expression (33) for D is predominant, and it exists under the condition $(\beta/\Delta)^{1/2}<\xi<\beta$. A distinctive feature of this regime is a pronounced twisting of the current lines both in the (x,y) plane and along the z axis.[3)] Another feature of this regime is that there are two length scales for a longitudinal random walk of the current lines (Fig. 2). The smaller, $\tilde{a}_z$, is equal to the longitudinal displacement of the current line accompanying a corresponding transverse displacement $\lambda_\perp$. The large scale a_z, which sets the maximum amplitude for the longitudinal random walk of the current lines, corresponds to a displacement of these lines in the (x,y) plane over a transverse correlation length $a_\perp$. The size a_z can also be taken to be the minimum distance for which the pattern of $\psi(x, y, z_0)$ contour curves with a diameter on the order of (34) is indistinguishable from the $\psi(x, y, z_0+a_z)$ contour curves. According to Ref. 7, we have $a_z\approx\lambda_z h$. Using (33), we thus find

$$a_z\approx\lambda_z(\beta/\Delta\xi^2)^{3/13},\quad (\beta/\Delta)^{1/2}<\xi<\beta. \tag{35}$$

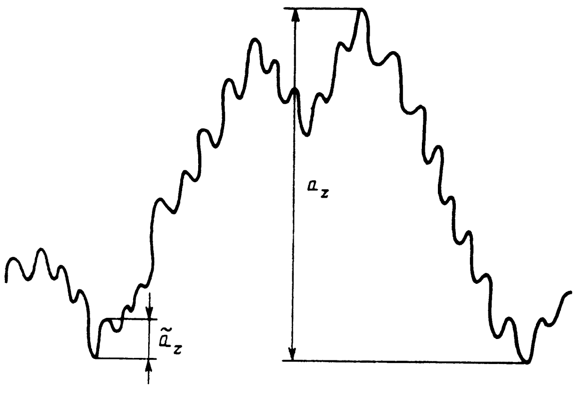

FIG. 2. Longitudinal random walk of the current lines in the case $(\beta/\Delta)^{1/2}<\xi<\beta$.

The dimension $\tilde{a}_z$ can be found as the distance over which a particle of the passive impurity is displaced along the z axis over the time corresponding to a transverse convection over a distance $\lambda_\perp$: $\tilde{a}_z\approx(D\lambda_\perp/v)^{1/2}$, i.e.,

$$\tilde{a}_z\approx\lambda_\perp(\beta/\Delta)^{1/2}=\lambda_z\xi^{-1}(\beta/\Delta)^{1/2},\quad (\beta/\Delta)^{1/2}<\xi<\beta. \tag{36}$$

The factor by which the average resistivity increases, which is related to the longitudinal random walk of the current, is determined by specifically this dimension. The ratio $u_z/u_\perp$ can be estimated from (31) and (28), in which we should set $|\nabla_\perp|\approx 1/(\lambda_\perp h)$, $\partial/\partial z\approx 1/a_z$:

$$u_z/u_\perp\approx\beta\lambda_\perp/\lambda_z,\quad (\beta/\Delta)^{1/2}<\xi<\beta. \tag{37}$$

Again, we wish to stress that in this λ_z-sensitive fractal regime an anomalous resistivity is caused by both the longitudinal random walk of the current and the twisting and contraction of the current in the (x,y) plane. According to (37), the longitudinal anomalous-resistivity factor disappears at the upper boundary of this ξ interval. In the second fractal subregime ($\xi>\beta$), there is thus no substantial random walk of the currents along the magnetic field ($u_z/u_\perp\leqslant 1$). This situation corresponds to the transition to the purely two-dimensional case. In this limit, the longitudinal correlation lengths a_z and $\tilde{a}_z$ no longer have a definite meaning.

We conclude this section of the paper by pointing out that the existence of a maximum of the effective resistivity $\rho_e(\xi)$ (Fig. 1) does not look particularly surprising in light of this discussion, particularly in view of the modification made in the variational principle for current flow in the Hall case.

4. SIZE EFFECT

Up to this point we have been discussing an unbounded Hall medium. In this section of the paper, we consider the average resistivity of a Hall medium which is bounded along the z direction.[4)]

We assume that our medium fills the region $0\leqslant z\leqslant b_z$ (nonplanar boundaries do not introduce any new effects[10]). The boundary condition

$$\mathbf{j}_z|_\Gamma=0 \tag{38}$$

can obviously have a significant effect on the average bulk characteristics only if the sample has a fairly small dimension: $a<a_z$ or $a<\tilde{a}_z$, where the dimensions a and $\tilde{a}_z$ are given by (32), (35), and (36). In the opposite case, the current could undergo a random walk along the z coordinate over most of the volume, in accordance with "its own needs," not sensing the boundary Γ. In addition, in the first fractal regime $[((\beta/\Delta)^{1/2}<\xi<\beta)]$ with $\tilde{a}_z<b_z<a_z$ there is no strong size effect. In other words, the expression for the average resistivity can differ from that in the case of an unbounded medium only by a numerical factor on the order of unity (more on this below).

This section of the paper is also convenient place for a discussion in terms of the diffusion of a passive scalar φ [see (3) and (4)]. It is easy to see that in these terms the boundary condition (38) means that there is no flux of φ across the boundary:

$$\partial\varphi/\partial z|_{z=0,\,b_z}=0.$$

We will take the average in transport equation (3) in two steps. We first take an average over z in the region $0<z<b_z$. Our arguments are identical to those used in the derivation of Eq. (16), except that now the velocity averaged over z is nonzero, and it introduces an additional convection term in the effective diffusion equation. $\langle \varphi \rangle_z$:

$$\frac{\partial\langle\varphi\rangle_z}{\partial t}+(\mathbf{v}_0(x,y)\nabla)\langle\varphi\rangle_z=D_e\Delta\langle\varphi\rangle_z,$$
$$\mathbf{v}_0(x,y)=\langle\mathbf{v}(x,y,z)\rangle_z=\frac{1}{b_z}\int_0^{b_z}\mathbf{v}(x,y,z)\,dz. \tag{39}$$

This procedure is valid as long as the oscillatory component of the velocity (oscillating along z) can be regarded as independent of x and y, i.e., under the condition $vb_z^2/\sigma_0<\lambda_\perp$, which holds in all cases in which there is a pronounced size effect.

The three-dimensional problem thus reduces to a corresponding two-dimensional problem in which the "seed" diffusion D_e is given by (16), in which we should set

$$\boldsymbol{\Psi}(z)=\int_0^z[\mathbf{v}(x,y,z')-\mathbf{v}_0(x,y)]\,dz', \qquad D_{zz}=\sigma_0, \qquad D_\perp=\sigma_0/\beta^2.$$

Introducing the dimensionless parameter $\eta\equiv b_z/\lambda_z$ for brevity, we have

$$v_0\approx\begin{cases} v\eta^{-1/2}, & \eta>1,\\ v, & \eta<1.\end{cases} \tag{40}$$

$$\Psi\approx\begin{cases} v\lambda_z\eta^{1/2}, & \eta>1,\\ v\lambda_z\eta^2, & \eta<1.\end{cases} \tag{41}$$

According to (16), we then have the modified seed diffusion

$$D_e\approx\sigma_0/\beta^2+\Psi^2/\sigma_0. \tag{42}$$

We have thus reduced the problem of the average resistivity of a bounded Hall medium with random three-dimensional inhomogeneities to the known problem of the effective diffusion in a two-dimensional random flow, (39), with seed diffusion (42). The solution of this problem is given by (19), in which we should set $\mathbf{v}\to\mathbf{v}_0$ and $D\to D_e$. Some caution must be exercised in making this replacement, since the intermediate effective diffusion D_e can arise over a finite time b_z^2/σ_0, which must be small in comparison with the particle revolution time in the velocity field $\mathbf{v}_0$: $\lambda_\perp/v_0$. This requirement is the same as a requirement which we used above in the derivation of an expression for D_e. In addition, it is necessary to satisfy the condition that the displacement of the particle due to the velocity component which oscillates along z over a time b_z^2/σ_0 be smaller than the width of the diffusion boundary layer, (33). This requirement of a spatial establishment of D_e is satisfied at $b_z<a_z$.

Figure 3 illustrates the limiting cases of this result. For a thin conducting layer, the current flow always occurs in a fractal regime, even if the corresponding bulk flow is quasiuniform at the same value of ξ.

5. CONCLUSION

We have thus examined, at the level of estimates, all regimes of the anomalous resistivity of a randomly inhomogeneous medium in a strong magnetic field. The strongest assumption which we have used here and which might restrict the applicability of the results is the assumption that the spatial distribution of the fluctuations of the medium can be characterized by simply two parameters: the inhomogeneity length scales λ_z and $\lambda_\perp$. In principle, there would be no particular difficulty in examining fluctuations characterized by power-law spectra of length scales (although such an analysis would be extremely laborious). The basic problem of a passive scalar was solved in Ref. 8 in that formulation. In addition, it was shown there that the single-scale approximation can be of fairly broad applicability if the Fourier spectrum of the fluctuations has a sufficiently sharp maximum.

Again in the example of the problem of the anomalous resistivity of an inhomogeneous Hall medium, we find two types of collective transport processes in random media:

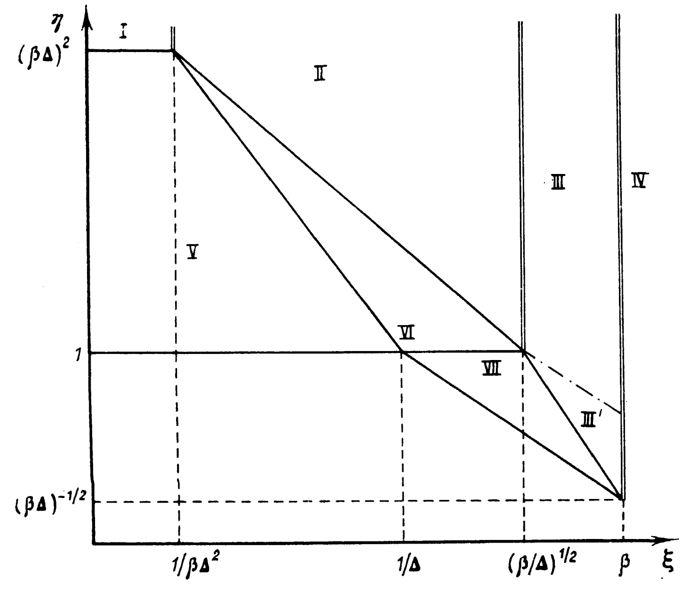

FIG. 3. Regimes of an anomalous resistance of a randomly inhomogeneous Hall medium which is bounded along z. The double lines separate bulk regimes I–IV, which correspond to the approximation of an unbounded medium. The solid lines separate different regimes of the strong size effect. When any solid line is crossed, the expressions for the effective resistivity change literally. The dot-dashed line is the line of the weak size effect. When it is crossed, the average resistivity changes by only a factor of a few units. Here are the expressions for $\gamma=\rho_{e1}/\rho_0$ in the various regimes: I—$\gamma=1$; II—(22a); III, III′—(22b); IV—(22c); V—$\gamma=(\beta\Delta\eta^{-2})^{10/13}$; VI—$\gamma=\beta\Delta(\beta^{-1}\Delta\xi^2\eta^{-2/3})^{3/13}$; VII—$\gamma=\beta\Delta(\beta^{-1}\Delta\xi^2\eta^4)^{3/13}$. Here are the equations for the boundaries between regimes: V/VI—$\eta=(\Delta\xi)^{-2}$; VI–II—$\eta=(\Delta\xi^2/\beta)^{-2/3}$; IV/VII—$\eta=(\Delta\xi)^{-1/2}$; III/VII—$\eta=(\Delta\xi^2/\beta)^{-1/2}$; III/III′—$\eta=(\Delta\xi^2/\beta)^{-3/13}$.

quasiuniform and fractal. A similar classification would be meaningful for several other problems, such as two-dimensional turbulent diffusion[7] (the high-frequency and low-frequency cases) and the anomalous transverse thermal conductivity of a plasma in a stochastically perturbed magnetic field[11] (the quasilinear and percolation limits).

In contrast with the first type of transport, the transport in the fractal regime is characterized by anomalously large correlation lengths, with the correlation length of the flow, a, being much larger than that of the medium, λ. On the one hand, this fact discourages attempts to apply standard perturbation-theory methods and averaging methods; on the other, it leads to the appearance of a broad inertial interval $[\lambda, a]$ in which the transport has scale-invariant properties, i.e., is of a fractal nature,[12] since there are no other length scales. If we introduce the concept of a hot region as the minimum volume which is responsible for half the total flow, we find that its fractal dimension for single-scale fluctuations is[5)] $d_{\rm hot} = 2.75$ (according to the discussion above, we would have $2 \leqslant d_{\rm hot} \leqslant 2.75$) in the multiscale case), while in the quasiuniform regime we would have $d_{\rm hot} = 3$. We wish to stress that the dimension $d_{\rm hot}$ does not change upon a variation of the parameters of the system; all that changes is the width of the inertial interval, which determines the range of applicability of the concept of a fractal dimension (in the quasiuniform regime we would have $a_\perp \approx \lambda_\perp$).

With regard to the quasiuniform regime of current flow in a Hall conductor in the interval $(\beta\Delta^2)^{-1} < \xi < (\beta/\Delta)^{1/2}$, the large longitudinal mixing length in (32), $a_z > \lambda_z$, does not make the hot region a fractal region ($d_{\rm hot} = 3$), but it does give rise to a scale-invariant longitudinal random walk of the current lines. In this approximation, the behavior of these current lines in the inertial interval $[\lambda_z, a_z]$ is identical to a plot of a Brownian function (Fig. 2). The fractal dimension of the current lines is therefore 1.5 (Ref. 12).

One field of application of this theory is the field of plasma current opening switches, which are used to create steep-front, high-power electrical pulses for inertial fusion and other applications.[13-15] According to the scenario described in Ref. 4, as a plasma suffers erosion (a decrease in density) because of current flowing through it, the plasma-filled gap at some point goes into an EMHD regime. This transition is accompanied by a sharp increase in the resistance of the gap. The magnetic field in the opening switch is usually the field of the current flowing through it, but there are also arrangements with an external magnetic field.[15] The maximum average resistivity of a randomly inhomogeneous plasma with $\Delta \approx 1$ is $\max(\rho_{e\perp}) \approx \rho_0 \beta \approx B/nec$, the same as the surface EMHD resistivity in (9) in the absence of small geometric parameters (i.e., in the case $b_x \approx b_y \approx b_z$). At large values of b_x, volume effects outweigh the surface effects. It should also be noted that for inhomogeneities of certain special types, e.g., for layered inhomogeneities of a plasma across the average current, the average resistivity may increase far more rapidly, to a level $\rho_{e\perp} \approx \rho_0 (\beta\Delta)^2$ (Ref. 4).

Another field of application of this theory might be the magnetoresistance effects in nonuniformly doped semiconductors, which are of the same plasma nature.[16]

A diagram of the size effect (Fig. 3) shows that suppression of the anomalous resistivity of an inhomogeneous Hall medium by no means requires that all the dimensions be greater than the length scale of the inhomogeneity of the medium; in particular, the anomalous bulk resistivity is observed even under the condition $b_z < \lambda_z$.

[1)] More precisely, the part of the magnetic energy due to the current, $\mathbf{B}\delta\mathbf{B}/4\pi$, where $\delta\mathbf{B}$ is the field of the current I flowing through the plasma.

[2)] Actually, it is not a matter of the particle undergoing a displacement with respect to the flow, but instead a change in the flow itself. This flow depends on z. Correspondingly, expression (20) could be derived in a different way, by examining how the separatrices of the time-varying velocity field $\mathbf{v}[x, y, z(t)]$ reconnect.[7] The result is the same.

[3)] This situation is quite natural, since in the quasiuniform regime there is a twisting of the current only in the z direction, while in the purely two-dimensional case ($\xi = \infty$) there is a twisting only in the (x, y) plane. In the intermediate regime which we are discussing here, these two types of random walks coexist.

[4)] The size effect in terms of b_x was analyzed in Ref. 6 for the case of two-dimensional inhomogeneities ($\xi = \infty$).

[5)] The fractal dimension of a nondegenerate planar cross section of a three-dimensional fractal is smaller by one than the dimension of the entire fractal.[12] We recall that the fractal dimension of two-dimensional percolation contour lines is $d_h = 1.75$.

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