Fermions in an antiferromagnet and phase transition

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By using the phenomenological Hamiltonian for the fermions in an antiferromagnet (AFM), constructed by accounting for the symmetric properties of the AFM ordered state, we study the ground state of the system when a single fermion is introduced. It is shown that, depending on the parameters of the Hamiltonian, a self-localization or delocalization regime exists. The formation of a large-radius polaron is possible in an intermediate range. A phase transition is possible for a finite concentration of particles in the system. Twisting of the AFM order parameter is advantageous in the new phase. It is shown that if one considers this phase in the $2D$ case, when the order parameter vector **n** lies in the plane, vortex excitations must be taken into account.

1. INTRODUCTION

The question of the interaction of fermions with magnetic degrees of freedom has been widely discussed recently in connection with applications to the theory of high-temperature superconductivity.[1–4] Besides the microscopic approach, mainly on the basis of the Hubbard model, a symmetry approach[5,6] to studying fermions in an antiferromagnet (AFM) is possible. In this case, an essential element in describing the fermions is a representation by which a single particle excitation is transformed in an Heisenberg AFM. We will assume that the ground state of the AFM is described by a nonlinear σ-model for the AFM order parameter—the unit vector **n** giving the magnetization direction of the sublattices. The Hamiltonian of the single-particle AFM excitations[5] has been constructed for the invariance group of the AFM order parameter. A brief symmetry description of the AFM and derivation of the Hamiltonian are given in the Appendix. The Hamiltonian for a point Γ of the Brillouin zone (see Fig. 1) has the form

$$H=\hat\psi^+(\hat p^2/2m)\hat\psi-g_4|\hat\psi^+\tau^1\hat\psi|^2+J(\partial_\mu\varphi)^2/2 +g_2(\partial_\mu\varphi)^2|\hat\psi|^2+g_1\hat\psi^+\tau^3\hat p_\mu\hat\psi\partial_\mu\varphi, \qquad (1.1)$$

where $\hat\psi=\binom{\psi}{\chi}$, $\mathbf{n}=(\cos\varphi,\sin\varphi,0)=(n_x,n_y,n_z)$, φ is the tilt angle of the vector **n**, lying, as is assumed, in the xy-plane.

In this paper, the possible ground states of the system are investigated, depending on the parameters of the Hamiltonian. We show in Sec. 2 that localized states of two types are possible in the single particle problem. Depending on the parameters of the Hamiltonian, either a self-localized regime is possible at atomic scales where our phenomenological approach is inapplicable, or delocalization of the particles occurs. The formation of a large-radius polaron is possible in a certain interval of the parameters. It will also be shown that the motion of a localized particle leads to the creation of an excitation around the polaron, having an amplitude proportional to the velocity of this particle.

In Sec. 3 we consider the ground state of the system in a delocalized regime for a finite concentration of particles. It will be shown that due to the interaction of the magnetic degrees of freedom with the fermionic ones, spin-wave softening occurs, where the magnitude of the renormalization of J is proportional to the fermion concentration. For a sufficiently large interaction constant g_i, the renormalized stiffness vanishes at a certain concentration. If one always assumes smootheness of the configuration $\mathbf{n}(x,y)$ in the space, then a transition to a spiral phase is possible in the system.[4] We, will show, however, that for decreased effective stiffness, one must also consider vortex excitations. A Kosterlitz-Thouless transition creating vortices is possible at a finite temperature. At the critical point, where the stiffness vanishes, the order parameter **n** has a finite correlation radius and at this point our model is approximately described by a theory with the local symmetry group $U(1)$, leading to a logarithmic attraction of the two types of fermions in the AFM.

For the rest of the paper, we limit ourselves to considering a two-dimensional Heisenberg AFM on a square lattice. Moreover, it is always assumed that the vector **n** lies in the plane. This case is realized in an AFM with an anisotropy of easy plane type, for example in lanthanum AFM's.

2. SINGLE-PARTICLE STATES

Consider the state of the system obtained by introducing a single fermion into an AFM. In this case, we restrict ourselves to the self-consistent field approximation. The energy levels are determined from the Schrödinger equation

$$[\hat p^2/2m-g_4\hat\psi^+\tau^1\hat\psi\tau^1+g_2(\partial_\mu\varphi)^2+g_1\tau^3\partial_\mu\varphi\hat p_\mu]\hat\psi=E\hat\psi. \qquad (2.1)$$

Since we want to illustrate the main possibilities here, we limit ourselves below to considering different limiting cases.

Consider the case $g_1=0$. By writing the matrix equa-

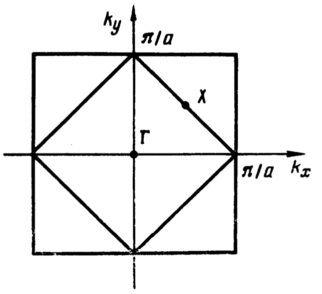

FIG. 1. Brillouin zone in AFM; Γ and X are characteristic points.

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tion (2.1) for the components ψ, χ, we obtain the coupling $\psi = \pm\chi$. The equations for the components coincide and have the form

$$-\Delta\psi/2m-2g_4|\psi|^2\psi+g_2(\partial_\mu\varphi)^2\psi=E\psi. \qquad (2.2)$$

It is evident that for $g_2 > 0$, the energy minimum (1.1) is attained for φ = const, since in the opposite case, the energy of the system is increased by

$$\int d^2r[J(\partial_\mu\varphi)^2/2+g_2|\psi|^2(\partial_\mu\varphi)^2]>0.$$

Equation (2.2) with φ = const reduces to the nonlinear Schrödinger equation with attraction. It is known that the eigenstates of

$$-\Delta\psi/2m-|\psi|^2\psi=\varepsilon\psi \qquad (2.3)$$

for $m < m_0 \approx 2\pi$ are delocalized with a continuous spectrum of eigenvalues $\varepsilon \geqslant 0$, and for $m > m_0$, a particle in the ground state is localized at microscopic scales. In the case under consideration, $m^* = 2mg_4$ plays the role of m. Therefore, for $m^* > m_0$ the fermion is localized at atomic scales, where the continuum theory under consideration is inapplicable, and for $m^* < m_0$, the fermion is delocalized. In both cases, we have φ = const in the ground state.

If $g_2 < 0$ holds, a solution is possible for a ground state with φ = const Eq. (2.2) differs from (2.3) by the additional attractive potential $g_2(\partial\mu\varphi)^2$. Therefore, for $m^* > m_0$, the fermion, as before, is localized. The distribution of the tilt angle φ of the magnetization vector $\mathbf{n}$ is determined from the equation obtained by varying the functional (1.1) with respect to φ:

$$J\Delta\varphi/2+g_2\partial_\mu(|\psi|^2\partial_\mu\varphi)=0. \qquad (2.4)$$

Since the fermion is localized at atomic scales, we have $|\psi|^2 \propto \delta(r - r_0)$, and Eq. (2.4) assumes the form

$$J\Delta\varphi/2=-g_2\partial_\mu\delta(r-r_0)\langle\partial_\mu\varphi\rangle-g_2\delta(r-r_0)\langle\Delta\varphi\rangle, \qquad (2.5)$$

where $\langle\cdots\rangle$ is the mean over the scale of the fermion localization length, which is not determinable in the continuum model. Solving (2.5), we obtain

$$\varphi=-\frac{g_2}{\pi J}\langle\partial_\mu\varphi\rangle\frac{x_\mu}{r^2}-\frac{g_2}{\pi J}\langle\Delta\varphi\rangle\ln r. \qquad (2.6)$$

A term growing as $r\to\infty$ is absent if $\langle\Delta\varphi\rangle = 0$.

In the case $g_2 < 0$, $m^* < m_0$, for φ = const the fermion is delocalized. The existence of a phase with $\varphi \neq$ const is favorable if

$$\int d^2r(J/2-|g_2|\,|\psi|^2)(\partial_\mu\varphi)^2<0.$$

If the fermion is delocalized ($|\psi| \propto L^{-1}$), this condition is not satisfied, and we have φ = const in the ground state.

In the case $m^* < m_0$, $g_2 < 0$ for sufficiently small values of J, localized polaron states with energy $W < 0$ can arise. Actually, in the field of attraction $g_2(\partial\mu\varphi)^2$, if φ is a localized function, there always exists in the two-dimensional case, a localized solution with an exponentially small negative energy level

$$E\propto-\exp\left\{-\left[mg_2\int d^2r(\partial\varphi)^2\right]^{-1}\right\}.$$

Therefore, the total energy of the system $W = E + J\int(\partial\varphi)^2$ can be negative for sufficiently small J. In the arguments here, we have not considered the nonlinear term $g_4|\psi|^2$, which lowers the value of W even more.

Let us consider the effect of the term $g_1\hat\psi + \tau^3\partial\mu\varphi\hat p\mu\hat\psi$ on the AFM state. Let $m^* > m_0$. Then the fermion is localized and $|\psi|^2 \approx A\delta(r - r_0)$, $|\psi|^2 \approx B\delta(r - r_0)$. (For $g_1 \neq 0$ the condition $\psi = \pm\chi$ is not satisfied.) The self-consistency condition for φ assumes the form

$$J\Delta\varphi+g_2\partial_\mu(|\hat\psi|^2\partial_\mu\varphi)+g_1\partial_\mu j_\mu^z=0, \qquad (2.7)$$

where

$$j_\mu^z=\hat\psi^+\tau^z p_\mu\hat\psi+\text{c. c.}$$

In the ground state there is no current $j_\mu^z = 0$. However, if we consider a moving localized polaron, then in the quasi-classical approximation we have $j_\mu \propto v_\mu\delta(\mathbf{r} - \mathbf{v}t)$ and $\partial_\mu j_\mu \propto v_\mu\,\partial_\mu\,\delta(\mathbf{r} - \mathbf{v}t)$, where $\mathbf{v}$ is the velocity of motion. Substituting j_μ into (2.7) and solving, we obtain a contribution linear in the velocity,

$$\delta\varphi\propto\frac{g_1}{J}\frac{\mathbf{vR}}{|\mathbf{R}|^3}, \qquad (2.8)$$

where $\mathbf{R}$ is the vector taken from the polaron to the observation point.

In this manner, the moving polaron creates around itself an AFM excitation with an amplitude proportional to the velocity, and a power-law decay of the tilt angle φ of the magnetization vector.

3. MULTIPARTICLE EFFECTS

1. In this section, we consider the effects to which a finite concentration of additional particles leads. We will assume that the doped fermions in an AFM are described by Bloch states and are not self-localized. This case is realized if the term $g_4\hat\psi^4$ is absent or the mass m is sufficiently small, that is $mg_4 < m_0$. Ignoring the term with g_4, let us write the Hamiltonian (1.1) in the form

$$H=J_0A_\mu^2/2+\hat\psi^+\hat p^2\hat\psi/2m+g_1\hat\psi^+\tau^3\hat p_\mu\hat\psi A_\mu+g_2A_\mu^2|\hat\psi|^2, \qquad (3.1)$$

where $A_\mu = [\mathbf{n}(\partial_\mu\mathbf{n})]\,\hat{\mathbf{z}} = \partial_\mu\varphi$. As previously, it is assumed that the vector $\mathbf{n}$ lies in the xy-plane and the model is equivalent to an XY-model interacting with fermions.

The Hamiltonian (3.1) describes a field A_μ of bosons with mass $M^2 = J_0$ interacting with fermions. The term with g_1 describes the interaction of the fermionic spin current

$$j_\mu^z=\hat\psi^+\tau^3p_\mu\hat\psi+\text{c. c.}$$

with the field A_μ, and the term with g_2 is obvious. If by A_μ is understood a field of superfluid velocity: $(V_s)_\mu = A_\mu$, then the stiffness J_0 coincides with the superfluid density, and our Hamiltonian describes the interaction of the current with V_s. It is clear that for a sufficiently large current, superfluidity is destroyed and J_{eff} vanishes, since in superconductivity there is a critical current. Here, this (spin) current arises spontaneously for a sufficiently large constant g_1. Let us find the renormalization of the stiffness constant J_0, caused by the interaction of the magnetic system with the fermions:

$$J_{\text{eff}}(\mathbf{k},\omega)=-\delta^2H/\delta A_\mu^2(\mathbf{k},\omega)=J_0+2g_2n+\delta J(\mathbf{k},\omega), \qquad (3.2)$$

where

$$\delta J(\mathbf{k},\omega)=-g_1^2\left\langle\sum_{\mathbf{q},\omega'} j_\mu^z(\mathbf{q},\omega')j_\mu^z(\mathbf{k}-\mathbf{q},\omega-\omega')\right\rangle$$
$$=-g_1^2\sum_{\mathbf{q},\omega'}G^0\left(\mathbf{q}+\frac{\mathbf{k}}{2},\omega'+\frac{\omega}{2}\right)$$
$$\times G^0\left(\mathbf{q}-\frac{\mathbf{k}}{2},\omega'-\frac{\omega}{2}\right)q^2=-g_1^2mn,$$

and n is the concentration of fermions. The evaluation of the correlator in (3.2) is elementary. However, one can also obtain the result (3.2), without integrating, by using Green's functions. For this, let us take advantage of an analogy with calculations in superconductivity theory. Observe that $\delta J(\mathbf{k},\omega)$ is the kernel in the expression for the average current

$$\langle j_\mu(\mathbf{k},\omega)\rangle=-\delta J_{\mu\nu}(\mathbf{k},\omega)A_\nu(\mathbf{k},\omega). \tag{3.3}$$

We add to the current operator

$$j_\mu^z=e(\psi^+\tau^3p_\mu\psi+\text{c. c.})/2m$$

a term $\hat\psi^+\hat\psi A_\mu e^2/m(e=g_1m)$, so that the total operator is invariant with respect to the local group $U(1)$: $A_\mu\to A_\mu+\partial_\mu\lambda$, $\hat\psi\to\exp(i\tau^3\lambda)\hat\psi$. In this case, Eq. (3.3) for the total gauge-invariant current must depend only on the transverse part of the vector potential. In the case considered by us, the field $A_\mu=\partial_\mu\varphi$ is purely longitudinal, therefore $j_\mu+g_2m\hat\psi^+\hat\psi A_\mu=0$, whence in the coordinate representation we must have $\delta J(r,t)=g_1^2\ m\langle\hat\psi^+\hat\psi\rangle$.

$$J_{\text{eff}}(\mathbf{k}\to0,\ \omega\to0)=J_0+f^2mn, \tag{3.4}$$

$$f^2=g_1^2-2g_2/m. \tag{3.5}$$

2. In this manner, the interaction of the AFM with the fermions leads to spin-wave softening and if the condition $f^2mn+J_0=0=J_{\text{eff}}$ is satisfied, it is natural to expect the unstable magnetic subsystem to form a spontaneous spin current. Such an instability was discovered in Ref. 4 for the point X. As the concentration $\mathbf{n}$ is increased further, the effective stiffness becomes negative and then the spiral phase with $\partial_\mu\varphi=A_\mu=$ const is more favorable. The self-consistency equations for A_μ are obtained from (3.2):

$$2g_2\sum(n_+-n_-)A_\mu-J_0A_\mu=f\sum(n^+-n^-)k_\mu,\qquad n_\pm=[\exp\beta(\varepsilon_0\pm fk_\mu A_\mu-\varepsilon_F)+1]^{-1}, \tag{3.6}$$

where n^+,n^- are the occupation numbers of the fermion states with different polarization, $n^+-n^-=\hat\psi+\tau^3\hat\psi$, $\beta=1/T$. Differentiating (3.6) with respect to A_μ, we find the condition for a critical concentration, for which a new minimum appears, which coincides with that obtained from (3.5). For $J_{\text{eff}}<0$ the energy functional (3.1) is a negative definite quadratic form. Therefore, for the existence of a minimum, it is necessary to add the term γA_μ^4. For $T=0$ the self-consistency conditions (3.6), taking into account the additional term, are exactly solved. Summing over the wave vectors k_μ, we obtain

$$A_\mu^2=-J_{\text{eff}}/2\gamma,\quad \varphi=\mathbf{Ar}.$$

3. Let us consider the critical point $J_0=f^2mn$ in more detail. Since the stiffness vanishes, the terms of the expansion of the AFM energy of higher orders become essential. In the previous section, we proposed that the decomposition proceed in powers of A_μ. One can point to yet another interesting possibility. Let the next terms of the decomposition be$F_{\mu\nu}^2$, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Generally speaking, for a purely longitudinal field $A_\mu=\partial_\mu\varphi$, it is necessary that $F_{\mu\nu}=0$. However, in the case of a nonsmooth configuration φ (vortex phase) we have $F_{\mu\nu}\neq0$, and therefore $F_{\mu\nu}^2$ makes sense. Let us note that the case under consideration occurs if there are no terms of the same order ($|A|^4$, $(\partial_\mu A_\mu)^2$ and others) in the energy decomposition, violating $U(1)$-symmetry. At the critical point, for a simple quadratic dispersion law of femions, we have

$$H=\frac{1}{2m}\hat\psi^+(p-e^*\tau^3A_\mu)^2\hat\psi+\Lambda F_{\mu\nu}^2,\qquad e^*=g_1m. \tag{3.7}$$

The Hamiltonian (3.7) describes the spin electrodynamics of two fermionic fields ψ_1, ψ_2 having opposite charges with respect to the "gauge" field A_μ. In the two-dimensional case, the Coulomb potential of two point charges is logarithmic. Therefore, the particles ψ_1 and ψ_2 attract each other logarithmically, which can lead to a superconductive condensate $\langle\psi_1,\psi_2\rangle\neq0$ for a sufficiently large concentration $n_c=J_0/f^2m$, destroying the AFM ordering. The higher terms of the original fermionic Hamiltonian and the nonlinearity weakly violate $U(1)$-symmetry. Therefore, the logarithmic potential of the two particles has a finite radius, which can be quite large.

Let us observe that there have been many attempts to obtain the attraction between fermions in an AFM with the help of magnetic subsystem excitations. However, they are primarily based on the BCS-mechanism.[1] In this case, the issue quickly becomes one of the "confinement" of the excitations. At the point X, where to first order, there is no interaction of the fermions with the vector $\mathbf{m}$, this $U(1)$-symmetry, if it occurs, can not be violated at the critical point. Furthermore, at the critical point, we have strong fluctuations of the vector $\mathbf{n}$.

$$\langle n(R)n(0)\rangle\propto\exp(-R/M).$$

In this sense, the AFM is, at the critical point, in a spin liquid phase.[7] On the whole, however, the possibility that a local symmetry exists, is a consequence of the quadratic spectrum of the fermions and the interaction j_mA_m.

4. VORTEX PHASE

It follows from what has been said above that the interation of a magnetic subsystem with fermions leads to a softening of the spin-wave stiffness constant. Let us consider the possibility of Kosterlitz-Thouless vortices[7] arising. From this analogy with superconductivity, it is natural to expect a mechanism of superconductivity destruction, using the creation of vortices, which in the 2D case are Kosterlitz-Thouless vortices. For this, let us find how the stiffness constant of the vortices is renormalized by interaction with the fermions, namely, let us show that it is renormalized similarly to J. Since we assume that the vecjtor $\mathbf{n}$ lies in the xy-plane, everything said below relates in equal degree as well to the XY-model.

Consider in A_μ the vortex contribution

$$A_\mu=\partial_\mu\varphi+\sum_i \frac{e_i(\hat{\varphi})_\mu}{|\mathbf{r}-\mathbf{r}_i|}, \quad \sum e_i=0, \tag{3.8}$$

where e_i is the charge of a vortex, $2\pi e_i = \oint A_\mu dr_\mu$, $\mathbf{r}_i$ is the center of the vortex, and $\hat{\varphi}_\mu$ is the unit vector corresponding to the angle in the cylindrical coordinate system. In the Hamiltonian, it is necessary to consider additional terms corresponding to the interaction of the vortices between themselves and with the fermions:

$$H=-K_0\sum_{i,j} e_i e_j \ln\frac{|\mathbf{r}_i-\mathbf{r}_j|}{a}+g_1\sum_i \hat{\psi}^+\tau^3 i\partial_\varphi\hat{\psi}\frac{e_i}{|\mathbf{r}-\mathbf{r}_i|}$$
$$-4\pi g_2 n\sum e_i e_j \ln\frac{|\mathbf{r}_i-\mathbf{r}_j|}{a}. \tag{3.9}$$

From (3.9), similarly to the renormalization of $J_{\rm eff}$, using (3.3), (3.4), we obtain

$$\delta K_0=-f\sum_i \langle j_\varphi\rangle\frac{e_i}{|\mathbf{r}-\mathbf{r}_i|}\approx -f^2mn\sum_{i,j} e_i e_j\int d^2r\frac{1}{|\mathbf{r}-\mathbf{r}_i|\,|\mathbf{r}-\mathbf{r}_j|}$$
$$=-f^2mn\sum_{i,j} 2\pi e_i e_j \ln\frac{a}{|\mathbf{r}_i-\mathbf{r}_j|},$$

whence follows the renormalization

$$K_{\rm eff}=K_0(1-2\pi f^2mn/K_0). \tag{3.10}$$

Since the logarithmic divergence accumulates on large scales, we let $\mathbf{k}\to 0$ in $\delta J(\mathbf{k},\omega)$ for the derivation of (3.10). In the long-wave limit with one stiffness constant J for the spin waves, we have $K_0 = 2\pi J_0$ and (3.10) coincides with the renormalization of J. Therefore, in the critical domain, where $J_{\rm eff} = 0$, it is necessary along with the smooth excitations of the magnetic system to also consider vortex excitations. And what is more, the finite temperature causes vortex melting with a finite concentration of vortices to occur in the system before the instability in the spin-wave spectrum develops. Actually, the Kosterlitz-Thouless (KT) phase transition occurs at a temperature $T=\pi J_{\rm eff}$, that is when the entropic contribution to the free energy renders the creation of vortices favorable, although the magnetic system is still stable. If $n = n_c' = n_c - \delta n$, $\delta n = \pi T/mf^2$, a vortex phase arises. The phase diagram, therefore, has the form shown in Fig. 2. If vortices are present in the system, the spiral order parameter has a finite correlation radius R_c:

$$\langle A_\mu(x)A_\mu(x')\rangle \propto \exp(-|x-x'|/R_c), \quad R_c\propto(n-n_c')^{1/2}.$$

A more accurate study would require taking into account the correlation length of the pure AFM.[8] We assume that the finite correlation radius of the AFM order parameter does not change the physical picture considered here.

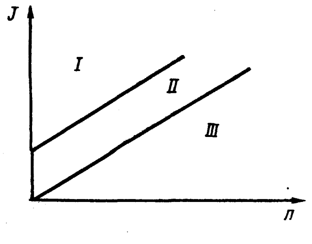

FIG. 2. Phase diagram of a $2D$-AFM taking into account the interaction of the magnetic subsystem with the fermions: I—AFM, II—KT, III—KT + spiral phase.

CONCLUSION

In the paper, we have investigated the ground state of a two-dimensional Heisenberg AFM doped by fermions (electrons or holes). The study is based on a phenomenological Hamiltonian, obtained by starting from the symmetry properties of the system.[5]

Moreover, we constructed the Hamiltonian of the fermions for the point Γ of the Brillouin zone. This assumption is justified for superconductors of electron type (for example, $Nd_{2-x}Ce_xCuO_4$), where the minimum of the electron spectrum is found to be in the center of the Brillouin zone.[9]

We showed that a single particle in the AFM, depending on the choice of the Hamiltonian parameters, is either localized on atomic scales, or delocalized. In the localized regime in the ground state, the tilt angle of the magnetization vector is either constant or decays in a power fashion with separation from the fermion. In a certain range of the parameters, polaron states with a more complex configuration are possible.

Owing to the interaction of the fermions with the antiferromagnetic system in the delocalized regime, a softening of the spin-wave stiffness J occurs, for a finite concentration of fermions, where the magnitude of the renormalization of J is proportional to the fermion concentration (3.4). As a result, the effective stiffness can vanish or become negative, which indicates the instability of the AFM ground state under consideration. It is shown that for $T=0$ a phase transition to a spiral phase is possible in the system if one does not consider vortices.

The possibility of creating vortex excitations in an AFM has been considered. It has been shown that the effective interaction constant of the vortices is renormalized the same as the spin-wave stiffness. Therefore, for a finite temperature $T=K_0$, a Kosterlitz-Thouless transition creating vortices is possible in the system.

For a specific choice of Hamiltonian parameters at the critical point, where the AFM stiffness vanishes, the Hamiltonian becomes invariant with respect to the local symmetry group $U(1)$, that leads to a logarithmic attraction of the two types of fermions in the AFM.

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APPENDIX

Let us briefly consider a symmetric approach to constructing the Hamiltonian of the fermions in an AFM. Let us restrict ourselves to the case of a $2D$ AFM on a square lattice. Then, the invariance group of the AFM order parameter of a unit vector $\mathbf{n}$ consists of the elements

$$G_{\rm AFM}:\ \{T^2;\ RT;\ D_4\}, \tag{A1}$$

where T_a is the group of translations by the unit vector of the crystal lattice, R is the time-inversion operator, and D_4 is the point symmetry group of the crystal. As shown in Ref. 5, the fermionic representations $G_{\rm AFM}$ in the case of a wave of spin density (the weak coupling model) and in the case of a strong coupling (Hubbard model) form the same irreducible representation. Since we are concerned with a two-valued representation, the fermion in the AFM is described by the two-component column $\hat{\psi} = (\psi,\chi)$, and its components transform under the action of $G_{\rm AFM}$ according to the following law:

$$RT\hat{\psi}=\exp(i\lambda)\, i\tau_y\hat{\psi}^*, \qquad (A2)$$

$$T^2\hat{\psi}=-\exp(2i\lambda)\hat{\psi}, \qquad (A3)$$

$$U(1):\ \hat{\psi}\to\exp(i\tau\mathbf{n}\phi/2)\hat{\psi}, \qquad (A4)$$

where τ^i are the Pauli matrices in the space (ψ,χ) and $U(1)$ denotes the global group corresponding to rotation around $\mathbf{n}$, since this symmetry remains unviolated in an AFM. The new Brillouin zone signifies periodicity in the k-space with period $\mathbf{Q} = (\pi,\pi)$ (see Fig. 1). For $\mathbf{k}\to\mathbf{k}+\mathbf{Q}$ we have

$$\hat{Q}:\ \hat{\psi}\to(\tau\mathbf{n})\hat{\psi}. \qquad (A5)$$

This symmetry is essential for the boundary of the Brillouin zone, for example, in considering the point X (Fig. 1).

We do not consider the representations of D_4 in an AFM, since we restrict ourselves to the case of those operators in the Hamiltonian which are invariant with respect to D_4. In other words, we are interested purely in exchange effects in disregarding spin-orbital interaction. To second order in the gradients and first order in the concentration, it is easy to obtain the phenomenological Hamiltonian of the fermions in an AFM:

$$H=\sum \hat{\psi}^+(p^2/2m)\hat{\psi}+a\hat{\psi}^+(\tau\mathbf{m})\hat{\psi}$$
$$+g_2(\partial_\mu\mathbf{n})^2\hat{\psi}^+\hat{\psi}+g_1\hat{\psi}^+\tau i\partial_\mu\hat{\psi}\mathbf{j}_\mu+\text{c. c.}, \qquad (A6)$$

where $j_\mu = [\mathbf{n}(\partial_\mu\mathbf{n})] = z\partial_\mu\varphi$ is the Noether current in the AFM. The magnetic subsystem is described by the nonlinear σ-model Hamiltonian

$$H=\{{}^1/_2J(\partial_\mu\mathbf{n})^2+\varkappa[\mathbf{m}^2-(\mathbf{mn})^2]\}d^2x, \qquad (A7)$$

where $\mathbf{m}$ are the rotation generators of the group $SU(3)$ acting on the variables in the spin space. We assume below that $\mathbf{n}$ lies in the plane: $\mathbf{n} = (n_x, n_y) = \cos\varphi, \sin\varphi)$. Furthermore, in writing the Hamiltonian (A6) we assumed that the energy minimum lies at the point Γ of the Brillouin zone (Fig. 1). After Gaussian integration with respect to $\mathbf{m}$, we obtain the Hamiltonian (1.1), with which we will work (in this case $g_4<0$). For the point X, the Hamiltonian can be similarly obtained, and in the momentum representation has the form

$$H=\sum \varepsilon_k\hat{\psi}_k^+\hat{\psi}_k+g_2(\partial_\mu\mathbf{n})^2\hat{\psi}^+\hat{\psi}+g_1\hat{\psi}_k^+\tau\hat{\psi}\sin k_\mu\mathbf{j}_\mu. \qquad (A8)$$

For the point X, a semiphenomenological derivation of the Hamiltonian based on the Hubbard model, was given in Ref. 4; the result of this derivation coincides with (A7).

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