Theory of charged-particle acceleration by a collection of shock-waves in a turbulent medium

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We develop a kinetic theory of charged-particle acceleration and transport in a plasma with strong MHD supersonic and super-Alfvénic turbulence. A feature of such turbulence is, besides the smooth eddies that vary in scale, the presence of abrupt discrete shock fronts, i.e., a clearly pronounced intermittent structure. Particle acceleration near the fronts by a first-order Fermi mechanism imparts an intermittent structure to the accelerated particle distribution function. The equation for the particle distribution function is averaged in two stages. In the first, account is taken of the nonperturbative contribution of the strong shock waves, and the connection is obtained between the intermittent part of the distribution function and the average distribution function. In the second stage, a closed integrodifferential equation is obtained for the particle distribution function after averaging over the smooth fluctuations between the fronts. It is shown that to describe the particle kinetics in the system it is not enough to specify one second-rank velocity-correlation tensor, but more detailed statistical information is needed on the correlations of the fronts and on their forms. The distribution functions of the accelerated particles are found and the influence of the parameters of the collection of shock fronts on the exponent of the particle energy spectrum is investigated.

1. THE PROBLEM

Analysis of a number of aspects of the evolution of scalar and vector fields in stochastic media has recently provided an insight into the important role of intermittency effects encountered in the statistical description of these fields. A great variety of physical examples, ranging from reaction kinetics to cosmological models, is given in Refs. 1–3. We consider here intermittent charged-particle distributions engendered by particle acceleration and transport in a medium with large-scale supersonic and super-Alfven fluctuations. The acceleration of particles by fluctuations of the electric field induced by plasma motion in a magnetic field (the Fermi mechanism)[4] is regarded as one of the basic mechanisms that form the spectra of superthermal particles. Particularly cosmic rays.

Particular interest attaches to particle acceleration by shock waves in a turbulent medium,[5-12] since such phenomena are directly observed near the front of a leading geomagnetic shock wave and in interplanetary space. There is no doubt that particle acceleration processes take place in large-scale phenomena involving shock waves, such as supernova flares and strong stellar winds from various classes of stars in the galaxy. Under these conditions, in view of the presence of multiple sources of strong perturbations, and also of strong inhomogeneities in the interstellar media, one can expect the formation of random collections of strong shock fronts superposed on large-scale compression and rarefaction waves and of various other smooth perturbations having quite broad spatial and temporal spectra. Since the principal energy carriers in the interstellar medium are shock waves, the presence of such waves of varying strength is a distinguishing feature of interstellar turbulence, therefore, naturally called supersonic.

Since the rate of star formation in the galaxy and the frequency of supernova flares vary little in the time over which the particles are accelerated, supersonic turbulence can be regarded in our problem as stationary. We constructed a model of homogeneous and stationary interstellar turbulence generated by supernova flares in the galactic disk earlier.[13,14] The distribution function of shock fronts was also calculated in this model, in which a uniform distribution of the supernovae over the galactic disk is assumed.

In a more accurate picture of interstellar-turbulence distribution it must be recognized that an appreciable fraction of type-II supernovae are gathered into relatively compact systems, OB-associations that evolve within a finite time. The lifetime of turbulence within a single association is of the same order. But this time for an individual association (on the order of 10^7 years) is larger by 2–3 orders of magnitude than the particle-acceleration time which will be estimated below. The notion of stationary turbulence is therefore applicable for the treatment of a strongly turbulent region within the confines of one OB association, as is the theory developed below for particle acceleration by a collection of shock waves and of large-scale turbulent motions. The accelerated particles are regarded here as passive impurities and their reaction on the shock fronts is disregarded. The influence of accelerated particles on the shock-wave fronts, which determines in a number of cases the structure of the front, is analyzed in Refs. 5, 8, 9, and 11.

The aim of the present paper is an investigation of the kinetics of charged particles interacting with a specified collection of shock waves.

Since particles are accelerated near an MHD shock front, their distribution can acquire variations whose spatial scale l is of the order of $\varkappa/u \approx v\Lambda/3u$, where u is the front velocity and $\varkappa$ is the local coefficient of diffusion across the front. This diffusion can be caused by small-scale fluctuations of macroscopic turbulent fields, and also by Coulomb collisions if the medium is dense enough. The kinetics of formation of the charged-particle spectrum of accelerated particles by an ensemble of fronts depends substantially not only on the strengths of the fronts, but also on the relation

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between the scale l and the average distance L between the fronts (which we shall identify with the maximum dimension of the turbulent cells, i.e., with the principal turbulence scale).

If $\beta = L/l = uL/\varkappa \ll 1$, the particle interacts within a characteristic time $\varkappa/u^2$ with several fronts, and the variation of the particle distribution is determined by the smaller of the two scales, i.e., by the value of L. The distribution function must be averaged over regions with dimensions of order L. Perturbation theory can be used under these conditions. The corresponding problem of calculating the kinetic coefficients and finding the form of the spectrum of the accelerated particles was solved by us earlier.[10] More complicated and more interesting is the inverse case $\beta \gg 1$. Under this condition a strong inhomogeneity has time to form in the distribution of the accelerated particles, and its spatial scale is small compared with the main scale of the turbulence.

To take correctly into account the contribution made to the shock fronts, we can use the known solution[5,7] for an individual front of arbitrary force and then average, during the first stage, the distribution function over regions of spatial scale l near the front. This introduces into the transport equation an integral operator that describes the strong acceleration on an individual front. This is followed by averaging over regions on the order of the main scale L. In view of the inequality $\beta \gg 1$, perturbation theory is inapplicable here, too, and the kinetic equations must be renormalized. This was done by the method we developed in Ref. 12. Specification of the pair-correlated tensor of the turbulent velocity field does not suffice for the calculation of the kinetic coefficient, and information on the statistical properties of the fronts is needed, as well as on their correlations with the velocity field between the fronts. This means that higher-order correlators of the velocity field must be taken into account.

In the last section of the article we obtain the simplest solutions of the equation describing particle transport and acceleration by strong supersonic turbulence with shock fronts. We calculate the spectra of the accelerated particles and investigate the dependence of the power-law exponents on the turbulence parameters.

2. INTERACTION OF PARTICLES WITH STRONG FRONTS

Consider a random set of shock fronts with average distance between them L, and with Mach numbers such that $M - 1 \gg 1$. The set will henceforth be regarded as statically homogeneous and isotropic. An inhomogeneous cloud of accelerated particles with a power-law spectrum in a wide energy interval is produced near each front within the average time L/u between the collisions of the fronts (u is the front velocity and does not differ greatly from the characteristic velocity of the medium since $M \gg 1$ is unlikely).[5,7,8] The variations of this distribution have a scale of order $l \sim v\Lambda/u \gg \Lambda$, where v is the particle velocity, and Λ their transport path in the turbulent medium ahead of the front.[7] The distribution of the accelerated particles in space is thus strongly inhomogeneous: relatively narrow peaks of accelerated particles are observed near the fronts, but the front collisions diffuse them turbulently over the entire system. The arrival of a subsequent shock front of sufficient strength, however, produces again a strongly inhomogeneous particle distribution. This acceleration pattern is a natural consequence of the intermittance of supersonic turbulence, meaning the presence in it of strong discontinuities.

To formulate an accelerated-particle distribution function averaged over an ensemble of random fronts, correct account must be taken of these strong local inhomogeneities. Assuming the condition $l \ll L$ to be satisfied, we separate the scales Λ satisfying the inequality

$$l \ll \Delta \ll L, \tag{1}$$

and average the distribution functions over these scales. Since $\Lambda \ll L$, we can use in the regions between the fronts the transport equation:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial r_\alpha} \varkappa_{\alpha\beta} \frac{\partial N}{\partial r_\beta} - u_\alpha \frac{\partial N}{\partial r_\alpha} + \frac{p}{3} \frac{\partial N}{\partial p} \frac{\partial u_\alpha}{\partial r_\alpha}, \tag{2}$$

and must use on the front itself the condition that the distribution functions and the particle-flux densities that are differential in p be equal on both sides of the discontinuity (see Refs. 5, 7, and 11 for details). We denote the distribution function averaged over the scales Δ by $\overline{N}$. At distances from the fronts of order Δ and larger we have $\overline{N} \approx N$, since the inhomogeneity scale in these regions is large compared with Δ. Near the discontinuity, however, we can use the known solution[7] of Eq. (2) for a planar shock front and connect the local value of the distribution function with its value $\overline{N}$ far (at distances Δ) from the front:

$$N_i(z,p) = \theta(-z_i) \left\{ \overline{N}(p) + \left[(\gamma_i+2) p^{-(\gamma_i+2)} \int_0^p p'^{(\gamma_i+1)} \overline{N}(p')\, dp' - \overline{N}(p) \right] \exp\left(\frac{\Delta u_{ni} z_i}{\varkappa_i} \right) \right\} + \theta(z_i)(\gamma_i+2) p^{-(\gamma_i+2)} \int_0^p p'^{(\gamma_i+1)} \overline{N}(p')\, dp'. \tag{3}$$

Here z_i is the coordinate measured along the normal from the ith front ($z_i < 0$ ahead of the front, $z_i > 0$ behind the front), $\varkappa_i$ is the diffusion coefficient in the normal direction ahead of the front, Δu_{ni} is the discontinuity of the normal velocity component of the medium on the front,

$$\gamma = \frac{(\sigma+2)}{(\sigma-1)}$$

is the exponent of the "universal" spectrum on an individual front, and $\sigma = \rho_2/\rho_1$ is the degree of compression of the medium in the shock wave. The solution (3) can be easily expressed in terms of the Green's function derived in Refs. 5 and 7 for a planar shock front. $\overline{N}(p)$ in (3) is of course a function of the "large-scale" coordinates (determined accurate to Δ) and of the time. A stationary solution can be used in (3) in view of the rapid formation of a power-law spectrum on the shock-wave front: at constant $\varkappa$ the time Δt of its formation in the range from p_0 to p is estimated at[7]

$$\Delta t \approx \frac{3\varkappa}{u \Delta u_n} \ln \frac{p}{p_0} \approx \frac{l}{\Delta u_n} \ln \frac{p}{p_0}. \tag{4}$$

This time (for $\Delta u_n \sim u$) is much shorter than the average time, of order L/u, between the front collisions.

After calculating the local variation of the distribution function near the fronts (3), we average Eqs. (2) directly over the scales Δ. To this end we identify the singular terms of the quantity $\partial u_\alpha/\partial r_\alpha$:

$$\frac{\partial u_\alpha}{\partial r_\alpha}=\left(\frac{\partial u_\alpha}{\partial r_\alpha}\right)_0-\sum_i \Delta u_{ni}\delta(z_i). \tag{5}$$

Here $(\partial u_\alpha/\partial r_\alpha)_0$ is the continuous part of the velocity divergence and is of the order of u/L; the δ-function terms are due to velocity jumps at the discontinuities. We write

$$\overline{\frac{p}{3}\frac{\partial N}{\partial p}\frac{\partial u_\alpha}{\partial r_\alpha}}=\frac{p}{3}\frac{\partial \bar N}{\partial p}\frac{\partial \bar u_\alpha}{\partial r_\alpha}+\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar N)\frac{\partial u_\alpha}{\partial r_\alpha}} \tag{6}$$

and calculate the last term. Obviously, its order of magnitude far from the front is

$$\frac{\Delta}{L}\frac{\partial \bar N}{\partial p}\frac{\partial \bar u_\alpha}{\partial r_\alpha}$$

and can be discarded in view of the small $\Delta/L\ll 1$. Near the ith front (in an layer of thickness Δ) we have with allowance for (3)

$$(N-\bar N)_i=-p^{-(\gamma_i+2)}\int_0^p \frac{\partial \bar N_i}{\partial p'}(p')p'^{(\gamma_i+2)}dp'\Big[\theta(z_i)+\theta(-z_i)\exp\left(\frac{\Delta u_{ni}z_i}{\varkappa_i}\right)\Big]. \tag{7}$$

With the aid of (7) and (5) (only the singular terms are taken into account in the latter) we get

$$\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar N)\frac{\partial u_\alpha}{\partial r_\alpha}}=\sum_i\frac{\Delta u_{ni}}{3\Delta}\left[p\frac{\partial \bar N_i}{\partial p}-\frac{(\gamma_i+2)}{3}p^{-(\gamma_i+2)}\int_0^p\frac{\partial \bar N_i}{\partial p'}p'^{(\gamma_i+2)}dp'\right]. \tag{8}$$

Although the right-hand side has the form of a sum over all fronts, the ith term differs from zero only near the ith front.

Next, we average the term

$$\overline{u_\alpha\frac{\partial N}{\partial r_\alpha}}=\bar u_\alpha\frac{\partial \bar N}{\partial r_\alpha}+\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar N)}. \tag{9}$$

We have

$$\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar N)}=\frac{1}{\Delta}\sum_i\int_{-\Delta/2}^{\Delta/2}u_{i\alpha}\frac{\partial}{\partial r_\alpha}(N-\bar N)_i dz_i$$

$$=\sum_i\frac{u_{i\alpha}}{\Delta}(N-\bar N)\Big|_{-\Delta/2}^{\Delta/2}-\sum_i\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2}(N-\bar N)_i\frac{\partial u_{i\alpha}}{\partial z_i}dz_i.$$

At a distance on the order of $\Delta/2$ from the front we have $N-\bar N\approx 0$, so that the integral term vanishes and we ultimately obtain with the aid of (5)

$$\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar N)}=-\sum_i\frac{\Delta u_{ni}}{\Delta}p^{-(\gamma_i+2)}\int_0^p p'^{(\gamma_i+2)}\frac{\partial \bar N_i}{\partial p'}dp'. \tag{10}$$

Combining the contributions (8) and (10) we have

$$\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar N)\frac{\partial u_\alpha}{\partial r_\alpha}}-\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar N)}=\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\frac{\Delta u_{ni}}{3\Delta}\times p^{(1-\gamma_i)}\int_0^p p'^{(\gamma_i+2)}\frac{\partial \bar N_i}{\partial p'}dp'. \tag{11}$$

Terms of the form

$$\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial}{\partial r_\beta}(N-\bar N)$$

contribute significantly on averaging neither near the fronts nor far from them. Therefore, averaging Eq. (2) with the aid of the results (6) and (9) above, we get

$$\frac{\partial \bar N}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar\varkappa_{\alpha\beta}\frac{\partial \bar N}{\partial r_\beta}-\bar u_\alpha\frac{\partial \bar N}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar N}{\partial p}\frac{\partial \bar u_\alpha}{\partial r_\alpha}+\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\frac{\Delta u_{ni}}{3\Delta}p^{1-\gamma_i}\int_0^p p'^{(\gamma_i+2)}\frac{\partial \bar N_i}{\partial p'}dp'. \tag{12}$$

Equation (12) can be simplified. We consider a collection of fronts of equal strength ($\gamma_i=\gamma$) and introduce the function

$$\overline{\sum_i\Delta u_{ni}\delta(z_i)},$$

which is equal to $\Delta u_{ni}/\Delta$ near the ith front (in a layer 2Δ thick) and to zero between the fronts. Equation (12) takes then the form

$$\frac{\partial \bar N}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar\varkappa_{\alpha\beta}\frac{\partial \bar N}{\partial r_\beta}-\bar u_\alpha\frac{\partial \bar N}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar N}{\partial p}\frac{\partial \bar u_\alpha}{\partial r_\alpha}+\frac{1}{3p^2}\frac{\partial}{\partial p}p^{1-\gamma}\int_0^p p'^{(\gamma+2)}\frac{\partial \bar N}{\partial p'}dp'\,\overline{\sum_i\Delta u_{ni}\delta(z_i)}. \tag{13}$$

3. AVERAGING OVER LARGE-SCALE MOTIONS OF THE MEDIUM

For further averaging of (13) over regions with dimensions of the order of the main turbulence scale, we introduce the notation

$$\left\langle\overline{\sum_i\Delta u_{ni}\delta(z_i)}\right\rangle=\frac{1}{\tau_{sh}},\quad \frac{\partial \bar u_\alpha}{\partial r_\alpha}=\Psi(\mathbf r,t),\quad \langle\Psi\rangle=0,$$

$$\overline{\sum_i\Delta u_{ni}\delta(z_i)}-\frac{1}{\tau_{sh}}=\varphi(\mathbf r,t),\quad \langle\varphi\rangle=0, \tag{14}$$

$$\hat P=\frac{p}{3}\frac{\partial}{\partial p},\quad \hat L=\frac{1}{3p^2}\frac{\partial}{\partial p}p^{1-\gamma}\int_0^p dp'p'^{(\gamma+2)}\frac{\partial}{\partial p'},$$

where the angular brackets denote the averaging in question. It is easy to verify that the operators $\hat P$ and $\hat L$ commute. We introduce next a new distribution function $\bar f(\mathbf r,p,t)$ connected with $\bar N$ by the relation

$$\bar N=\exp\left(\frac{\hat L t}{\tau_{sh}}\right)\bar f. \tag{15}$$

The equation for $\bar f$ is

$$\frac{\partial \bar f}{\partial t}-\frac{\partial}{\partial r_\alpha}\bar\varkappa_{\alpha\beta}\frac{\partial \bar f}{\partial r_\beta}=\hat Q\bar f, \tag{16}$$

where

$$\hat Q=-\bar{\mathbf u}\nabla+\Psi\hat P+\varphi\hat L \tag{17}$$

is a random operator. The tensor $\bar\varkappa_{\alpha\beta}$ of the diffusion due to the small-scale turbulence also undergoes, generally speaking, random changes in space and in time. By virtue of the condition $uL/v\Lambda \gg 1$, however, the particles will be transported in space mainly by the motion of the medium (turbulent diffusion). Diffusion due to the small-scale field plays a minor role under these conditions. We therefore assume the tensor $\bar{\varkappa}_{\alpha\beta}$ in (16) to be constant and isotropic; $\bar{\varkappa}\varkappa_{\alpha\beta} = \delta_{\alpha\beta}$ is independent also of the momentum variable p.

We average Eq. (16) by the general approach we developed in Ref. 12. We assume that only harmonics with close wave vectors correlate in the spectrum of the random quantities $\bar{u}_\alpha$, Ψ, and φ. This means that we can separate a macroscopically small wave-number interval Δk and assume that the harmonics pertaining to this interval are uncorrelated with all the remaining ones. We denote the corresponding contributions to the velocity fields by $\delta\bar{u}$, $\delta\Psi$, and $\delta\varphi$, where

$$\delta\Psi = \int_{-\infty}^{\infty} d\omega \int_{\Delta k} d\mathbf{k}\Psi_{\mathbf{k},\omega} e^{i(\mathbf{kr}-\omega t)}, \tag{18}$$

etc. When averaged, the Fourier components of the turbulent quantities satisfy the usual relations for homogeneous turbulence:

$$\langle \Psi_{\mathbf{k},\omega}\Psi_{\mathbf{k}',\omega'}\rangle = |\Psi|^2_{\mathbf{k},\omega}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'). \tag{19}$$

During the first stage we solve the following auxiliary problem: we average Eq. (16) in which we replace the total turbulent-velocity fields by $\delta\mathbf{u}$, $\delta\Psi$, and $\delta\varphi$. Since these quantities are too small (to the extent that Δk is small), this problem can be solved by perturbation theory.

Putting

$$\bar{f}(\mathbf{r}, p, t) = \Phi(\mathbf{r}, p, t) + \delta f(\mathbf{r}, p, t), \ \langle \bar{f}\rangle = \Phi, \ \langle \delta f\rangle = 0, \tag{20}$$

we obtain from (16) the set of equations

$$\frac{\partial \Phi}{\partial t} - \varkappa\Delta\Phi = \langle \hat{Q}\delta f\rangle, \quad \frac{\partial \delta f}{\partial t} - \varkappa\Delta\delta f = \hat{Q}\Phi, \tag{21}$$

from which we determine the fluctuating increment to the distribution function:

$$\delta f(\mathbf{r}, p, t) = \int d\mathbf{r}'\, dt'\, G_0(\mathbf{r}-\mathbf{r}', t-t')\hat{Q}(\mathbf{r}', p, t')\Phi(\mathbf{r}', p, t'). \tag{22}$$

Here $G_0(\boldsymbol{\rho},\tau)$ is the Green's function of a diffusion equation with small-scale (or molecular) diffusion coefficient:

$$G_0(\boldsymbol{\rho}, \tau) = (4\pi\varkappa\tau)^{-3/2}\exp(-\rho^2/4\varkappa\tau). \tag{23}$$

Averaging $\langle \hat{Q}\delta f\rangle$ yields an equation for the averaged distribution function $\Phi(\mathbf{r}, p, t)$:

$$\frac{\partial \Phi}{\partial t} - (\varkappa_{\alpha\beta} + \delta\chi_{\alpha\beta})\nabla_\alpha\nabla_\beta\Phi = \frac{1}{p^2}\frac{\partial}{\partial p}p^4\delta D\frac{\partial \Phi}{\partial p} + \delta A\hat{L}^2\Phi + \delta B\hat{L}(1+2\hat{P})\Phi. \tag{24}$$

The turbulent-diffusion tensor $\delta\chi_{\alpha\beta}$ and the coefficients δA, δB, and δD that determine the acceleration rate are

$$\delta\chi_{\alpha\beta} = \int G_0(\boldsymbol{\rho}, \tau)\langle \delta\bar{u}_\alpha(\mathbf{r}, t)\delta\bar{u}_\beta(\mathbf{r}', t')\rangle d\boldsymbol{\rho}\, d\tau + \ldots,$$

$$\delta D = \frac{1}{9}\int G_0(\boldsymbol{\rho}, \tau)\langle \delta\Psi(\mathbf{r}, t)\delta\Psi(\mathbf{r}', t')\rangle d\boldsymbol{\rho}\, d\tau,$$

$$\delta A = \int G_0(\boldsymbol{\rho}, \tau)\langle \delta\varphi(\mathbf{r}, t)\delta\varphi(\mathbf{r}', t')\rangle d\boldsymbol{\rho}\, d\tau, \tag{25}$$

$$\delta B = \int G_0(\boldsymbol{\rho}, \tau)\langle \delta\Psi(\mathbf{r}, t)\delta\varphi(\mathbf{r}', t')\rangle d\boldsymbol{\rho}\, d\tau,$$

where $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$ and $\tau = t - t'$. Terms due to compressibility and containing the quantities $\delta\Psi$ and $\delta\varphi$ have been omitted from the expression for the turbulent-diffusion tensor $\chi_{\alpha\beta}$ (see Ref. 12).

Case of weak acceleration. After solving the auxiliary problem by perturbation theory, we proceed to average Eq. (16) which contains the total velocity fields. In view of the assumed condition $uL/v\Lambda \gg 1$ (strong long-wave turbulence), perturbation theory cannot be applied to the total velocity field. We renormalize the kinetic coefficients by the scheme proposed in Ref. 12. We consider the case when the averaged kinetic equation retains the form (24) obtained by perturbation theory, but with different (renormalized) coefficients:

$$\frac{\partial \Phi}{\partial t} - \chi_{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi = \frac{1}{p^2}\frac{\partial}{\partial p}p^4 D\frac{\partial \Phi}{\partial p} + A\hat{L}^2\Phi + B\hat{L}(1+2\hat{P})\Phi. \tag{26}$$

The differential form of the operator acting on the coordinates implies a smooth distribution function Φ. This condition can be met by averaging over scales $R \gg L$, where L is the principal scale of the turbulence. The initial particle distribution should also be sufficiently smooth. The term $\chi_{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi$ can then be regarded as the leading term of the expansion in the parameter $L/R \ll 1$. The next terms must be retained if the total diffusion tensor $\chi_{\alpha\beta}$ (which takes into account both the small-scale scattering and the large-scale velocity eddies of the medium) vanishes for some reason.

Retention of the same form of the acceleration [as in the perturbation theory (24)] implies smallness of acceleration over the correlation length L or else during the correlation time τ_c, i.e., the inequality $\Delta p \ll p$, where Δp is the change of the absolute value of the particle momentum over the correlation length. Note that we are dealing in this case with acceleration due to large-scale fluctuations of the quantities $\Psi(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$. The requirement that the acceleration be small does not pertain to the first-order effect due to the strong fronts [the term $\tau_{\rm sh}^{-1}\hat{L}\bar{N}$ in Eq. (13)], since this effect has been eliminated from Eq. (16) by using the transformation (15). This requirement that the coefficients A, B, and D be small can be met by a suitable choice of the spectral properties of Ψ and φ.

Assuming all the above conditions to be met we consider, in addition to the completely averaged equation (26), another equation in which the averaging is over all the field harmonics except those pertaining to the narrow wave-number interval Δk introduced above:

$$\frac{\partial \tilde{\Phi}}{\partial t} - \chi_{\alpha\beta}'\nabla_\alpha\nabla_\beta\tilde{\Phi} = \frac{1}{p^2}\frac{\partial}{\partial p}p^4 D'\frac{\partial \tilde{\Phi}}{\partial p} + A'\hat{L}^2\tilde{\Phi} + B'\hat{L}(1+2\hat{P})\tilde{\Phi} + \delta\hat{Q}\tilde{\Phi}. \tag{27}$$

Here $\tilde{\Phi}$ is a distribution function that must also be averaged over the amplitudes and phases of the turbulent eddies from the interval Δk; $\chi'_{\alpha\beta}$, D', A', and B' are kinetic coefficients that take into account the turbulent-velocity field with $\delta\bar{\mathbf{u}}$, $\delta\Psi$, and $\delta\varphi$ subtracted; and $\delta\hat{Q}$ is the operator (17) but containing only the non-averaged part of the velocity field. Since this operator is small, it can be averaged in Eq. (27) by perturbation theory using the procedure used already to derive (24).

As a result we obtain, as was to be expected, Eq. (26) in which

$$\chi_{\alpha\beta}=\chi_{\alpha\beta}'+\delta\chi_{\alpha\beta},\ D=D'+\delta D,$$
$$A=A'+\delta A,\ B=B'+\delta B. \tag{28}$$

The increments $\delta\bar{u}_\alpha$ due to the velocity field are given by Eqs. (25), in which G_0 should be replaced by a Green's function containing the complete diffusion tensor $\chi_{\alpha\beta}=\chi\delta_{\alpha\beta}$. In all other respects it agrees with (23) and does not take into account the particle acceleration within the correlation length L, in view of the smallness of this effect already suggested.

The complete kinetic coefficients are calculated by integrating the quantities $\delta\chi_{\alpha\beta}$, δA,... over all the wave numbers. We express the Fourier transform of the turbulent-velocity correlation tensor

$$K_{\alpha\beta}(\rho,\tau)=\langle\bar{u}_\alpha(\mathbf{r},t)\,\bar{u}_\beta(\mathbf{r}',t')\rangle=\int \tilde{K}_{\alpha\beta}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\rho-\omega\tau)}\frac{d\mathbf{k}\,d\omega}{(2\pi)^4} \tag{29}$$

in the form

$$\tilde{K}_{\alpha\beta}(\mathbf{k},\omega)=T(k,\omega)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)+S(k,\omega)\frac{k_\alpha k_\beta}{k^2}, \tag{30}$$

where T and S are scalar functions. The correlation function of the velocity divergence $\Psi(\mathbf{r},t)$ is expressed in terms of $S(k,\omega)$ as follows:

$$\langle\Psi(\mathbf{r},t)\,\Psi(\mathbf{r}',t')\rangle=\int\tilde{\Psi}(k,\omega)\,e^{i(\mathbf{k}\rho-\omega\tau)}\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}, \tag{31}$$
$$\tilde{\Psi}(k,\omega)=k^2S(k,\omega).$$

A complete description of the acceleration, however, requires according to (25), that two more spectral functions be introduced in addition to the functions T and S connected with the solenoidal and potential motions. The first, $\tilde{\varphi}(k,\omega)$, should describe the correlation of the velocity discontinuities on the shock fronts, and the second, which we designate by $\mu(k,\omega)$, should describe the mutual correlation of $\varphi(\mathbf{r},t)$ and $\Psi(\mathbf{r}',t')$. The introduction of these spectral functions is necessitated by the intermittent character of the particle distribution function whose description requires additional statistical information on the random velocity field (cf. the problem considered in Ref. 12).

Changing to Fourier transforms in Eqs. (25) and integrating over the entire range of wave numbers, we obtain for the kinetic coefficients the equations

$$\chi=\varkappa+\frac{1}{3}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+\chi k^2}-\frac{2k^2\chi S}{(i\omega+\chi k^2)^2}\right], \tag{32}$$

$$D=\frac{8}{9}\pi\chi\int_0^\infty k^6dk\int_0^\infty d\omega\frac{S(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}, \tag{33}$$

$$A=8\pi\chi\int_0^\infty k^4dk\int_0^\infty d\omega\frac{\tilde{\varphi}(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}, \tag{34}$$

$$B=8\pi\chi\int_0^\infty k^4dk\int_0^\infty d\omega\frac{\mu(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}. \tag{35}$$

It is recognized here that $\chi_{\alpha\beta}=\chi\delta_{\alpha\beta}$ holds for the present case of isotropic turbulence; in addition, it has been taken into account in the integrals that the spectral functions are even in the argument ω. To calculate the spatial-diffusion coefficient χ we must solve the transcendental equation (32), after which the calculation of the coefficients A, B, and D, which gives the acceleration rate, reduces to integration.

Returning to the complete distribution function $F=\langle\bar{N}\rangle$ with the aid of the transformation inverse to (15), we express the kinetic equation in the final form

$$\frac{\partial F}{\partial t}-\chi_{\alpha\beta}\nabla_\alpha\nabla_\beta F$$
$$=\left(\frac{1}{\tau_{sh}}+B\right)\hat{L}F+\frac{1}{p^2}\frac{\partial}{\partial p}p^4D\frac{\partial F}{\partial p}+A\hat{L}^2F+2B\hat{L}\hat{P}F. \tag{36}$$

A characteristic property of this equation is the presence of the integral operator $\hat{L}$, attesting to a strong acceleration of the particles near an individual shock front. The acceleration "in the mean" can also be small here if the fronts are produced infrequently enough.

Case of strong acceleration. If the particle-energy change during the correlation time of the turbulent velocities is not small, $\Delta p\gtrsim p$, the acceleration is described by an integral operator not only on the fronts but also between the fronts. The renormalized kinetic coefficients can nonetheless be calculated in this case, too, by modifying somewhat the scheme described above. The case of strong acceleration can be included in the general scheme because the acceleration operators $\hat{P}$ and $\hat{L}$ in the averaged equation (15) are homogeneous (they are invariant under a similarity transformation with respect to the momentum variable). The integral-equation kernel can therefore be represented after averaging as a function of the difference $\eta-\eta'$, where $\eta=\ln(p/p_0)$.

We seek the averaged equation for the complete distribution function $F(\mathbf{r},\eta,t)=\langle\bar{N}(\mathbf{r},p,t)\rangle$ in the form

$$\frac{\partial F}{\partial t}=\int_{-\infty}^{\infty}\chi_{\alpha\beta}(\eta-\eta')\,\nabla_\alpha\nabla_\beta F(\mathbf{r},\eta',t)\,d\eta'$$
$$+\left(\frac{\partial}{\partial\eta}+3\right)\int_{-\infty}^{\infty}D(\eta-\eta')F(\mathbf{r},\eta',t)\,d\eta', \tag{37}$$

where the operator

$$\frac{\partial}{\partial\eta}+3=\frac{1}{p^2}\frac{\partial}{\partial p}p^3 \tag{38}$$

ensures the vanishing of the second term when both parts of the equation are integrated over all the momenta. If the averaging is not complete and does not include the turbulent velocity-field harmonics pertaining to a narrow wave-number interval, we obtain in place of (37)

$$\frac{\partial\bar{F}}{\partial t}=\int_{-\infty}^{\infty}\chi_{\alpha\beta}'(\eta-\eta')\,\nabla_\alpha\nabla_\beta\bar{F}\,d\eta'$$
$$+\left(\frac{\partial}{\partial\eta}+3\right)\int_{-\infty}^{\infty}D'(\eta-\eta')\bar{F}\,d\eta'+\delta\hat{Q}\bar{F}. \tag{39}$$

Here the operator $\delta\hat{Q}$ contains only the nonaveraged part of the velocity field. The operators $\hat{P}$ and $\hat{L}$ should be expressed in terms of the variable η. We take next the Fourier transform of Eq. (39) with respect to the variable η:

$$F_s(\mathbf{r},t)=\int_{-\infty}^{\infty} F(\mathbf{r},\eta,t)\exp(i\eta s)\,d\eta. \tag{40}$$

Equation (39) takes the form

$$\frac{\partial F_s}{\partial t}=\bar{\chi}_{\alpha\beta}'(s)\nabla_\alpha\nabla_\beta F_s-(is-3)\bar{D}'(s)F_s$$
$$-\delta\bar{u}_\alpha\nabla_\alpha F_s-\frac{is}{3}\delta\Psi F_s-\frac{is(is-3)}{3(is-2-\gamma)}\delta\varphi F_s. \tag{41}$$

We average the last equation over the ensemble of realizations of $\delta\bar{u}_\alpha$, $\delta\Psi$, and $\delta\varphi$ by perturbation theory, using the smallness of these quantities. The result is the equation

$$\frac{\partial F_s}{\partial t}=[\bar{\chi}_{\alpha\beta}'+\delta\bar{\chi}_{\alpha\beta}(s)]\nabla_\alpha\nabla_\beta F_s-(is-3)[\bar{D}'(s)-\delta\bar{D}(s)]F_s, \tag{42}$$

which is the Fourier transform of (37). The contributions to the kinetic coefficients from $\delta\bar{u}_\alpha$, $\delta\Psi$, and $\delta\varphi$ are given by

$$\delta\bar{\chi}_{\alpha\beta}(s)=\int d\rho\,d\tau\,\delta K_{\gamma\beta}(\rho,\tau)\frac{\partial G_s}{\partial\rho_\gamma}(\rho,\tau)\rho_\alpha$$
$$-\frac{is(is-3)}{18}\int d\rho\,d\tau\frac{\partial^2\delta K_{\gamma\delta}}{\partial\rho_\gamma\,\partial\rho_\delta}G_s(\rho,\tau)\rho_\alpha\rho_\beta$$
$$+\frac{is(is-3)(2is-3)}{18(is-2-\gamma)}\int d\rho\,d\tau\langle\delta\Psi(\mathbf{r},t)\delta\varphi(\mathbf{r}',t')\rangle G_s(\rho,\tau)\rho_\alpha\rho_\beta$$
$$+\frac{1}{2}\left(\frac{is}{3}\right)^2\frac{(is-3)^2}{(is-2-\gamma)^2}$$
$$\times\int d\rho\,d\tau\langle\delta\varphi(\mathbf{r},t)\delta\varphi(\mathbf{r}',t')\rangle G_s(\rho,\tau)\rho_\alpha\rho_\beta, \tag{43}$$

$$\delta\bar{D}(s)=\frac{is}{9}\int d\rho\,d\tau\frac{\partial^2\delta K_{\alpha\beta}}{\partial\rho_\alpha\partial\rho_\beta}G_s(\rho,\tau)$$
$$+\frac{is(3-2is)}{9(is-2-\gamma)}\int d\rho\,d\tau\langle\delta\Psi(\mathbf{r},t)\delta\varphi(\mathbf{r}',t')\rangle G_s(\rho,\tau)$$
$$-\left(\frac{is}{3}\right)^2\frac{is-3}{(is-2-\gamma)^2}\int d\rho\,d\tau\langle\delta\varphi(\mathbf{r},t)\delta\varphi(\mathbf{r}',t')\rangle G_s(\rho,\tau). \tag{44}$$

Here

$$\delta K_{\alpha\beta}(\rho,\tau)=\langle\delta\bar{u}_\alpha(\mathbf{r},t)\delta\bar{u}_\beta(\mathbf{r}',t')\rangle,\ \rho=\mathbf{r}-\mathbf{r}',\ \tau=t-t';$$

$G_s(\rho,\tau)$ is the exact Green's function of the problem and is defined as

$$G_s(\rho,\tau)=(4\pi\bar{\chi}(s)\tau)^{-3/2}\exp\left\{-\frac{\rho^2}{4\bar{\chi}(s)\tau}-(is-3)\bar{D}(s)\tau\right\}. \tag{45}$$

The quantities $\bar{\chi}(s)$ and $D(s)$ are the Fourier transforms of the integral kernels of Eq. (37). We obtain for them, by integrating (43) and (44) over the entire wave-number interval, a system of two transcendental equations:

$$\bar{\chi}(s)=\varkappa+\frac{1}{3}\int\frac{dk\,d\omega}{(2\pi)^4}\left\{\frac{2T+S(k,\omega)}{i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2}\right.$$
$$\left.-\frac{2k^2\bar{\chi}(s)S(k,\omega)}{[i\omega+(is-3)\bar{D}(s)+\chi(s)k^2]^2}\right\}$$
$$+\int\frac{dk\,d\omega}{(2\pi)^4}\bar{\chi}(s)\frac{\left[i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)\frac{k^2}{3}\right]}{[i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2]^3}$$
$$\times\left\{k^2\frac{is(is-3)}{9}S(k,\omega)-\frac{is(is-3)}{9(is-2-\gamma)}\mu(k,\omega)\right.$$
$$\left.+\frac{s^2(is-3)^2}{9(is-2-\gamma)^2}\bar{\varphi}(k,\omega)\right\}; \tag{46}$$

$$\bar{D}(s)=\frac{is}{9}\int\frac{dk\,d\omega}{(2\pi)^4}[-k^2S(k,\omega)+(3-2is)\mu(k,\omega)(is-2-\gamma)^{-1}$$
$$-(is-3)is\bar{\varphi}(k,\omega)(is-2-\gamma)^{-2}]/$$
$$[i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2]+is/3(is-2-\gamma)\tau_{sh}. \tag{47}$$

The integration constants are chosen so that Eq. (37) goes over into (36) for weak acceleration. The desired quantities $\bar{\chi}(s)$ and $\bar{D}(s)$ can be obtained by numerically solving the system (46) and (47). In the case of weak acceleration, $\bar{\chi}(s)$ no longer depends on s and Eq. (46) takes the form (32). Equation (47) becomes

$$\bar{D}(s)=-isD-\left(\frac{is}{3}\right)^2\frac{is-3}{(is-\gamma-2)^2}A$$
$$+\frac{is}{3(is-\gamma-2)}\left(1-\frac{2is}{3}\right)B$$
$$+\frac{1}{\tau_{sh}}\frac{is}{3(is-\gamma-2)}, \tag{48}$$

where D, A, and B are given by Eqs. (33)–(35).

4. ENERGY SPECTRA OF ACCELERATED PARTICLES.

Let us calculate the stationary particle spectra of particles accelerated by an ensemble of shock waves and large-scale turbulent motions, with allowance for particles leaving the acceleration region. In the case of weak acceleration over the correlation length of a large-scale velocity field, the starting point is Eq. (36), which we express in the form

$$\frac{F}{\tau_e}=\left(\frac{1}{\tau_{sh}}+B\right)\hat{L}F+\frac{1}{p^2}\frac{\partial}{\partial p}p^4D\frac{\partial F}{\partial p}+A\hat{L}^2F$$
$$+2B\hat{L}\hat{P}F+\frac{q_0}{p_0^2}\delta(p-p_0). \tag{49}$$

We have added to the equation a source of monoenergetic particles, and replaced the diffusion term by F/τ_e, where τ_e is the time of departure of the particles from the acceleration region. Note that in the determination of the inhomogeneous distribution of the particles we arrive, by separation of variables, at the very same equation for the eigenfunction that depends on p, with $1/\tau_e$ replaced by an eigenvalue determined by the boundary conditions. As regards the strong acceleration on the shock fronts and the associated intermittance effects, they are fully preserved in Eq. (49).

It is convenient to determine the particle spectra by transforming with respect to the variable $\eta=\ln(p/p_0)$. From (49) we obtain the Fourier transform F_s:

$$F_s = \frac{q_0}{\dfrac{1}{\tau_s} + (is-3)\bar{D}(s)}, \tag{50}$$

where $\bar{D}(s)$ is defined in (48).

Taking the inverse Fourier transform and changing over to the variable p, we obtain the following particle spectra:

$$F(p) = \frac{q_0\tau_{sh}}{a}\begin{cases} \dfrac{(x_1+\alpha)^2}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}\left(\dfrac{p}{p_0}\right)^{x_1}, & p\leqslant p_0, \\ \dfrac{(x_2+\alpha)^2}{(x_1-x_2)(x_2-x_3)(x_2-x_4)}\left(\dfrac{p}{p_0}\right)^{x_2} \\ +\dfrac{(x_3+\alpha)^2}{(x_1-x_3)(x_3-x_2)(x_3-x_4)}\left(\dfrac{p}{p_0}\right)^{x_3} \\ +\dfrac{(x_4+\alpha)^2}{(x_1-x_4)(x_4-x_2)(x_4-x_3)}\left(\dfrac{p}{p_0}\right)^{x_4}, & p\geqslant p_0, \end{cases} \tag{51}$$

where x_1, x_2, x_3, and x_4 are real roots of the fourth-degree polynomial $ax^4+bx^3+cx^2+dx+e$. The coefficients of the polynomial are connected with the normalized kinetic coefficients A, B, and D [defined by relations (32)–(35)] by the equations

$$\begin{cases} a=A_1+2B_1+9D_1, \\ b=6A_1+9B_1+27D_1+3+2\alpha(B_1+9D_1), \\ c=9(A_1+B_1+1)+3\alpha(18D_1+3B_1+3D_1\alpha+1)-9\varepsilon, \\ d=9\alpha(3\alpha D_1+B_1+1)-18\varepsilon\alpha, \\ e=-9\varepsilon\alpha^2, \end{cases} \tag{52}$$

where

$$A_1=\tau_{sh}A,\quad B_1=\tau_{sh}B,\quad D_1=\tau_{sh}D,\quad \varepsilon=\frac{\tau_{sh}}{\tau_s},\quad \alpha=\gamma+2.$$

If all the roots of the polynomial are real, three of them are of the same sign, and calculations of A_1, B_1, $9D_1\leqslant 0.1$, and $\varepsilon>0$ show that one root (x_1) is positive and three roots (x_2, x_3, x_4) are negative. The negative root with the smallest absolute value, x_2, determines the asymptotic behavior of the spectrum at large momenta $p\gg p_0$. If $\varepsilon\to 0$, then $x_2\to -3-0$ for all α that are meaningful in this problem ($\alpha\geqslant 4$). This corresponds to prolonged interaction of the particles with the shock fronts and with the large-scale motions, as a result of which a rather hard spectrum of accelerated particles is formed. In the opposite limiting case $\varepsilon\gg 1$ we have $x_2\to -\alpha$. The reason is that the time needed to leave the system is short and the particles manage to interact with only one front, so their spectrum is the same as near a single planar shock front. The root x_1, which determines the spectrum of the particles in the region of small momenta $p\ll p_0$ behaves like $x_1\sim\varepsilon$ at $\varepsilon\leqslant 1$.

A distinctive feature of the problem is the presence of solutions corresponding to spectra with nonmonotonic components in the momentum region $p\geqslant p_0$. In particular, if $A_1\sim 0.1$ or $B_1\sim 0.1$ and $\varepsilon\leqslant 1$, and the exponent satisfies $\alpha\geqslant 8$ (this corresponds to shock waves with Mach numbers of order $\sqrt{2}$) we have complex roots $x_3=x_4^*$ ($\mathrm{Re}(x_3)<0$). The real roots are $x_1>0$ and $x_2<0$, with $x_2>\mathrm{Re}(x_3)$. The particle spectrum takes then the form

$$F(p) = \frac{q_0\tau_{sh}}{a}\begin{cases} \dfrac{(x_1+\alpha)^2}{(x_1-x_2)|x_1-x_3|^2}\left(\dfrac{p}{p_0}\right)^{x_1}, & p\leqslant p_0, \\ \dfrac{(x_2+\alpha)^2}{(x_1-x_3)|x_2-x_3|^2}\left(\dfrac{p}{p_0}\right)^{x_2} \\ -2\,\mathrm{Re}\left[\dfrac{(x_3+\alpha)^2}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}\left(\dfrac{p}{p_0}\right)^{x_3}\right], \\ p\geqslant p_0. \end{cases} \tag{53}$$

At large momenta $p\gg p_0$ the spectrum is a superposition of a monotonic contribution on an oscillating one. The scale of the oscillations in momentum space is determined by $\mathrm{Im}(x_3)$. The condition $F(p)>0$ is, of course met here, since the positive term that describes the power-law spectrum dominates for $p\gg p_0$.

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