Slowing of a magnetic monopole in matter

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We have developed a theory of the interaction of a monopole with a material medium, based upon a hydrodynamical quantum-mechanics formulation that makes the fundamental role played by one-dimensional vortex line solutions particularly clear. We have found linear response functions for the effect of a monopole on perfect and viscous charged fluids, classical and quantum plasmas, superconductors, systems of oscillators, among others, and we have determined monopole energy losses in such media. The limits of applicability of the linear theory of monopole slowing are discussed.

1. INTRODUCTION

Among the predictions of fundamental physical theory as it exists today, two that are of special significance are the existence of an extremely heavy magnetic monopole and the decay of the proton. These predictions may be traced back to the realm of superhigh energies (above 10^{14} GeV), which play a key role in the theory. The detection of proton instability or a magnetic monopole would therefore present us with a unique source of information on such energies.[1)]

Searches for magnetic monopoles have been conducted using both the methods traditionally employed in nuclear physics (looking for the products of their deceleration in matter) and those targeted specifically at monopoles (looking for a jump in the magnetic flux in a superconductor or the catalysis of rapid proton decay).[1] These should be based on a theory that is capable of describing both single-particle and collective interactions of monopoles with a medium consisting of conventional particles. The appropriate generalization of macroscopic electrodynamics is usually achieved by introducing monopole sources into the Maxwell equations while retaining the form of the equations for matter.

It was recently realized,[2] however, that the information conveyed by the conventional matter equations may in principle not suffice for a description of the effects of a magnetic monopole on a medium. Additional information is required concerning the reaction of the medium to the distinct effects of the transverse electric and magnetic fields, which are rigorously coupled to one another by Faraday's law in conventional electrodynamics, but are independent in the presence of a magnetic monopole.

The incompleteness of the available information is further complicated by the non-Hamiltonian nature of the motion of a charged particle in the field of a monopole. This is not a problem for classical media, but quantum media cannot be described in the usual way: the Schrödinger equation contains not the field strengths but the potentials, which become meaningless when a monopole is present. The mathematical machinery then becomes much more complex: the Dirac method introduces a singular string, space becomes stratified in the Wu-Yang method, and so on.[3] The complexity of these approaches, however, makes their practicality far from obvious.

At the same time, as pointed out in Ref. 4, it is also possible to apply Madelung's simple and elegant formulation of quantum mechanics to the electrodynamics of magnetic monopoles, wherein the Schrödinger equation is supplanted by a set of hydrodynamic equations that include a special "quantum" force and the Lorentz force. Generalizing the Madelung approach to the magnetic monopole case is not particularly difficult, with the charge quantization condition

$$eg=2\pi N\hbar \tag{1.1}$$

once again providing a unique solution. That solution, however, pertains not to the wave function, as in Dirac's method, nor to the potential, as in the Wu-Yang method, but to the equations embodying the superposition principle. From here on, e is the electrical and g the magnetic charge, and N is an integer (which we take to be unity); we use Heaviside units, and the speed of light is also taken to be unity.

The significant advantage of the hydrodynamic formulation is that it is capable of providing a physical picture of the passage of a magnetic monopole through a medium. Apenko[5] has recently underscored the crucial role played in the electrodynamics of magnetic monopoles by the quasi-one-dimensional structure that they trail behind them, induced by the vortical electric field of the moving monopole, which produces a spiraling motion of the electrons in the medium. In hydrodynamic language, the counterpart of this structure is the generalized vorticity (which for brevity we simply call the vorticity) of a quantum fluid,

$$\mathbf{\Gamma}=\operatorname{rot}\mathbf{v}+\frac{e}{m}\mathbf{B}, \tag{1.2}$$

which is localized along the monopole trajectory that has been cleared out by the current (m is the mass of particles making up the medium, $\mathbf{v}$ is the fluid velocity, and $\mathbf{B}$ is the magnetic field strength).

In contrast to the nonphysical Dirac string (the price we pay for retaining the language of potentials), the vortex line solution is real. In particular, this is made manifest by the fact that a monopole expends part of its energy in producing it. Hence, magnetic monopoles will be slowed down by a medium even if there is no dissipation or emission of quasiparticles, and the energy lost by a monopole per unit length in this way will be independent its velocity. Curiously enough, this property was even mentioned by Fermi,[6] who related it to the inductive nature of the electric field associated with a moving magnetic monopole.

According to (1.1), the charge on a magnetic monopole is 137/2 times the charge on the electron, which would seem to completely preclude applying linear electrodynamics—which yields the lowest-order (Born) approximation for interaction with the medium—to monopoles. It turns out,

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however, that at least in the treatment of energy losses from magnetic monopoles, the linear approximation does have a rather wide range of applicability (with certain stipulations). This is ultimately related to the fact that the vorticity current is linear in the monopole charge.

In the present paper we formulate a mechanism for dealing with the macroscopic electrodynamics of a magnetic monopole in applications concerned with monopole energy losses. We consider a homogeneous, isotropic, nongyrotropic medium made up of nonrelativistic, spinless particles that interact in a self-consistent fashion. It is assumed that the medium itself does not contain magnetic monopoles, which act solely as external sources. It can be shown that all of these constraints can be circumvented.

2. BASIC EQUATIONS

Maxwell's equations for a magnetic monopole in a material medium are

$$\operatorname{rot}\mathbf{B}-\dot{\mathbf{E}}=\mathbf{j}+\mathbf{j}^i, \qquad (2.1a)$$
$$\operatorname{div}\mathbf{E}=\rho+\rho^i, \qquad (2.1b)$$
$$\operatorname{rot}\mathbf{E}+\dot{\mathbf{B}}=-\tilde{\mathbf{j}}, \qquad (2.1c)$$
$$\operatorname{div}\mathbf{B}=\tilde{\rho}. \qquad (2.1d)$$

Here ρ and $\mathbf{j}$ are the charge and current densities (the lack of an index denotes an external charge; a tilde denotes an external magnetic monopole; an index i indicates a quantity that has been induced within the medium by an external agent; there are no induced monopole quantities since there are no monopoles in the matter making up the medium). Monopoles are clearly the source of a longitudinal (subscript l) magnetic field $\mathbf{B}_l$ and the "annihilator" of the Faraday induction law: the transverse (subscript t) electric field $\mathbf{E}_t$ comes not just from a varying magnetic field but from the motion of magnetic monopoles as well.

The material relations that supplement (2.1) in the weakly interacting case are linear, and take the form[2]

$$\rho^i=(1-\varepsilon)\operatorname{div}\mathbf{E}, \quad \mathbf{j}_t^i=(\tilde{\varepsilon}-1)\dot{\mathbf{E}}_t+(1-1/\tilde{\mu})\operatorname{rot}\mathbf{B} \qquad (2.2)$$

(j_l^i is related to ρ^i through the continuity equation). The Fourier components of ε, $\tilde{\varepsilon}$, and $\tilde{\mu}$, which are referred to below as the response functions (see Ref. 2 for further details) depend on the frequency ω and wave vector $\mathbf{k}$ of the external influence. In particular, we obtain from (2.1) and (2.2) the expression for the magnetic field,

$$\mathbf{B}=\{i[\mathbf{kj}]+i\omega\tilde{\varepsilon}\tilde{\mathbf{j}}-i\mathbf{k}\tilde{\rho}/\tilde{\mu}\}/\sigma, \qquad (2.3)$$

where

$$\sigma=k^2/\tilde{\mu}-\omega^2\tilde{\varepsilon}. \qquad (2.4)$$

In the absence of magnetic monopoles, only the quantity (2.4), rather than $\tilde{\varepsilon}$ and $\tilde{\mu}$ individually, has any physical meaning, describing as it does the response of the medium to the unique [by virtue of (2.1c) with $\tilde{\mathbf{j}}=0$] transverse field $\mathbf{E}_t$, $\mathbf{B}_t$. There is therefore a family of equivalent pairs of $\tilde{\varepsilon}$ and $\tilde{\mu}$ corresponding to a given value of (2.4). The two most commonly used possibilities are

$$\tilde{\varepsilon}=\varepsilon, \quad \tilde{\mu}=\mu, \qquad (2.5a)$$
$$\tilde{\varepsilon}=\varepsilon_t=\varepsilon+(1-1/\mu)k^2/\omega^2, \quad \tilde{\mu}=1, \qquad (2.5b)$$

where ε and ε_t are the longitudinal and transverse dielectric constants, and μ is the magnetic permeability.

A magnetic monopole changes the situation radically, rendering $\mathbf{E}_t$ and $\mathbf{B}_l$ independent (by a choice of external sources, either one can be forced to zero). Then $\tilde{\varepsilon}$ and $\tilde{\mu}$ become characteristics of the medium in their own right, describing its response to these fields. This is patently clear from (2.3): with no monopoles, the magnetic field depends only on σ, while if any are present, it also depends on $\tilde{\varepsilon}$ and $\tilde{\mu}$ individually. This also means that additional information about the medium is required—the value of either $\tilde{\varepsilon}$ or $\tilde{\mu}$—in monopole electrodynamics. In any event, the usual choice, (2.5a), is unjustified and is almost always wrong (see Section 7 below).

One exception is the motion of a magnetic monopole in a hollow channel in some medium, where the channel is large in cross section compared with the medium's spatial dispersion radius.[5] This can be shown to be the case by making use of (2.1) to write (2.2) in the form

$$\mathbf{j}_t^i=(\varepsilon-1)\dot{\mathbf{E}}+(1-1/\mu)\operatorname{rot}\mathbf{B}+(1/\tilde{\mu}-1/\mu)\operatorname{rot}\Big(\int_{-\infty}^{t}dt'\,\tilde{\mathbf{j}}\Big).$$

The last term, which is responsible for the nonadherence to (2.5a), describes a ring current localized near the monopole trajectory. That current results from the solenoidal electric field of the moving magnetic monopole (see Introduction); it is obviously not present when a monopole moves through a channel.

To summarize, then, the central issue in the linear electrodynamics of magnetic monopoles is the determination of the three linear response functions of the medium, ε, $\tilde{\varepsilon}$, and $\tilde{\mu}$ (or ε, σ, and either $\tilde{\varepsilon}$ or $\tilde{\mu}$), two of which are already known from conventional electrodynamics. If it does become necessary to transcend the linear theory, one must start with a nonlinear generalization of the equations (2.2). In the hydrodynamic formulation adopted below, we carry out that generalization using the standard expressions for the charge density and (fluid) flux density in conjunction with the continuity and nonlinear Euler equations.

3. CLASSICAL MEDIA

The field strengths $\mathbf{E}$ and $\mathbf{B}$, respectively, are physically defined by the velocity-independent and velocity-independent terms of the Lorentz force in the equation of motion for a classical (heavy) test particle,

$$m\dot{\mathbf{v}}=e(\mathbf{E}+[\mathbf{vB}]). \qquad (3.1)$$

This same equation describes the motion of a particle inside a classical medium in the self-consistent field approximation, wherein besides the fields due to external sources, $\mathbf{E}$ and $\mathbf{B}$ incorporate the mean fields produced by all other particles in the medium.

If one of the external sources is a magnetic monopole, the canonical formalism no longer applies to (3.1) (see Introduction). In fact, from Lagrange's equations,

$$\mathbf{p}=\nabla_v L, \quad \dot{\mathbf{p}}=\nabla_x L$$

(L is the Lagrangian, $\mathbf{p}$ is the canonical momentum), we have

$$\operatorname{rot}_v\mathbf{p}=0, \quad \operatorname{rot}_x\mathbf{p}=0, \quad \operatorname{div}_x\mathbf{p}=\operatorname{div}_v\dot{\mathbf{p}}, \quad \operatorname{rot}_x\mathbf{p}=-\operatorname{rot}_v\dot{\mathbf{p}}. \qquad (3.2)$$

The first and third of these equations, together with (3.1), yield

$$\frac{d}{dt}\operatorname{div}\mathbf{p}=0,$$

whereupon, given the initial condition $\mathbf{p}=m\mathbf{v}$ as $t\to-\infty$, the difference $\mathbf{p}-m\mathbf{v}$ depends solely on the coordinates. The remaining conditions (3.2) then lead to Maxwell's equations (2.1c, d) with vanishing right-hand sides. A magnetic monopole thus destroys the Lagrangian (or Hamiltonian) character of the equations of motion of a classical charged particle.

This is a rather general conclusion, and it applies to quantum dynamics as well. Moreover, in no way does it invalidate the classical equations of motion incorporating field strengths (recall the example of non-Hamiltonian dissipative dynamics), so a description of the effects of a magnetic monopole on a classical medium should encounter no difficulties. Furthermore, such a medium requires no further information about the response functions, which are given by Eqs. (2.5b) (see Ref. 2).

To prepare the way for a consistent treatment of the quantum mechanical problem, let us rewrite Eq. (3.1) in hydrodynamic form (see Ref. 7). Going to Euler variables with a stationary observer, and introducing the density $n(\mathbf{x},t)$ and velocity $\mathbf{v}(\mathbf{x},t)$ at a fixed point in space, we can easily derive the Euler equation and the equation of continuity:

$$\dot{\mathbf{v}}+(\mathbf{v}\nabla)\mathbf{v}=\frac{e}{m}(\mathbf{E}+[\mathbf{v}\mathbf{B}]),\quad \dot{n}+\operatorname{div}(n\mathbf{v})=0. \tag{3.3}$$

In general, when the correlation between particles (in other words, collisions) is important, a classical medium can be described by a kinetic equation, and the equations of hydrodynamics follow, to some approximation. For a barotropic fluid, these equations differ from (3.3) only by the addition of the quantity $-\nabla w$ on the right-hand side of the Euler equation, where $w(n)$ is the enthalpy. To take viscosity into account, it is necessary to add to that equation the quantity

$$\nu\Delta\mathbf{v}+(\zeta+\nu/3)\nabla\operatorname{div}\mathbf{v}, \tag{3.4}$$

where ν and ζ are the viscosity coefficients.

The hydrodynamic equations for the induced charge density and charged-fluid flux density (see Section 2) take the form

$$\rho^i=en,\quad \mathbf{j}^i=en\mathbf{v}, \tag{3.5}$$

where we have assumed that values unperturbed by the external field have been subtracted. Linearizing the hydrodynamic equations—i.e., working in the acoustic approximation—we can then go on to find the linear response functions of the medium (see Section 7 below).

If the hydrodynamic approximation is inapplicable, then the analogous linearization must be carried out for the kinetic equation that determines the monochromatic distribution function $f(\mathbf{x},t)$. The equations for the induced charge and flux densities are then

$$\rho^i=e\int d\mathbf{v}\,f,\quad \mathbf{j}^i=e\int d\mathbf{v}\,\mathbf{v}f. \tag{3.6}$$

4. QUANTUM MEDIA

The Schrödinger equation of quantum mechanics,

$$\left[i\hbar\frac{\partial}{\partial t}-(i\hbar\nabla+e\mathbf{A})^2/2m-e\varphi\right]\psi=0 \tag{4.1}$$

includes potentials φ and $\mathbf{A}$ defined by

$$\mathbf{E}=-\nabla\varphi-\dot{\mathbf{A}},\quad \mathbf{B}=\operatorname{rot}\mathbf{A}, \tag{4.2}$$

which leave out magnetic monopoles [see Eqs. (2.1c, d)]. The Madelung formulation does not share this shortcoming; one makes the replacement

$$\psi=n^{1/2}\exp(iS/\hbar),\quad \nabla S=m\mathbf{v}+e\mathbf{A}. \tag{4.3}$$

Substitution of (4.3) into (4.1) yields the equations of hydrodynamics,

$$\dot{\mathbf{v}}+(\mathbf{v}\nabla)\mathbf{v}=-\nabla w+\frac{e}{m}(\mathbf{E}+[\mathbf{v}\mathbf{B}]),\quad \dot{n}+\operatorname{div}(n\mathbf{v})=0, \tag{4.4}$$

where the "quantum" potential

$$w=-\frac{\hbar^2\Delta n^{1/2}}{2m^2n^{1/2}} \tag{4.5}$$

plays the role of the enthalpy. The expressions for the charge and flux densities remain unchanged [Eq. (3.5)].

A quantum object can be completely described in the language of the probability density n and the velocity $\mathbf{v}$ associated with the probability flux density. In particular, the phase difference in the wave function that arises when a split current merges is given by

$$\delta S=\left[\int_L d\mathbf{l}-\int_{L'} d\mathbf{l}\right](m\mathbf{v}+e\mathbf{A})=m\Phi,$$

where L and L' are the current paths, Σ is the surface that caps the paths, and

$$\Phi=\int_\Sigma d\mathbf{S}\left(\operatorname{rot}\mathbf{v}+\frac{e}{m}\mathbf{B}\right)=\int_\Sigma d\mathbf{S}\,\mathbf{\Gamma} \tag{4.6}$$

is the flux of vorticity (1.2) through that surface. In a simply-connected medium in the absence of magnetic monopoles, $\mathbf{\Gamma}=0$, as can be seen directly from (4.3).[2)]

When monopoles are present—when the fields cannot be represented as in (4.2) and the Schrödinger equation no longer makes immediate physical sense—the hydrodynamic equations (4.4), like the classical equations (3.3), continue to be useful. It is in fact Eqs. (4.4) that provide the basis for our subsequent development.[4] Both later in this section and again in Section 6 we shall return to the problem of justifying those equations.

The Euler equation (4.4) can be rewritten in the form (see Ref. 7)

$$\dot{\mathbf{v}}+\nabla(v^2/2+w)=\frac{e}{m}\mathbf{E}+[\mathbf{v}\mathbf{\Gamma}], \tag{4.7}$$

where the monopole source is clearly *in evidence as the* last term on the right-hand side. The longitudinal component of (4.7) yields the generalization of the Bernoulli equation,

$$v^2/2+w+U=\text{const},\quad U=\Delta^{-1}\left[-e^2\frac{\delta n}{m}+\operatorname{div}(\dot{\mathbf{v}}-[\mathbf{v}\mathbf{\Gamma}])\right], \tag{4.8}$$

where δn is the change in density induced by the external agent.

We can preserve the form of the Lorentz force in the presence of a magnetic monopole by assuming that a particle in the medium feels only the fields themselves, regardless of whether they originated with a conventional charge or a monopole. The net result is that the Lorentz force contains no field terms specifically relegated to magnetic monopoles (a longitudinal magnetic field, for instance). In fact, when monopoles are present, the analog of (4.2) becomes [see (1.2), (4.7)]

$$\mathbf{E}=-\nabla\varphi-\dot{\mathbf{A}}-\frac{m}{e}[\mathbf{v}\boldsymbol{\Gamma}], \quad \mathbf{B}=\operatorname{rot}\mathbf{A}+\frac{m}{e}\boldsymbol{\Gamma}, \tag{4.9}$$

where

$$\mathbf{A}=-\frac{m}{e}\mathbf{v}, \quad \varphi=-\frac{m}{e}(v^2/2+w).$$

We see from (4.9), then, that the Lorentz force actually depends only on $\mathbf{A}$ and φ.

Everything that we have said thus far refers to the case in which interactions among the particles of the medium are described by the self-consistent field method. The machinery becomes more complicated (but still retains the advantage of using the language of fields rather than potentials) when correlations between particles—pairwise, ternary, etc.—must be taken into consideration. In that event, the hydrodynamic quantities will depend on the coordinates of the pairs and triples of particles involved (a subject discussed in greater depth in Ref. 8).

5. VORTICITY

The results to be derived in the present section apply both to quantum media and to an ideal classical fluid. Taking the curl of Eq. (4.7) and making use of (1.2) and (2.1), we obtain the generalized Helmholtz equation[7] with a magnetic monopole as the source of vorticity:

$$\dot{\boldsymbol{\Gamma}}-\operatorname{rot}[\mathbf{v}\boldsymbol{\Gamma}]=-\frac{e}{m}\tilde{\mathbf{j}}. \tag{5.1}$$

Furthermore, we shall assume that the monopole moves at constant velocity $\mathbf{u}$ with respect to the medium (deceleration of the monopole due to energy losses can be neglected by virtue of its large mass):

$$\tilde{\rho}=g\delta(\mathbf{x}-\mathbf{u}t), \quad \mathbf{J}=\mathbf{u}\tilde{\rho}. \tag{5.2}$$

When the velocity $\mathbf{v}$ is constant, the solution of (5.1) takes the form

$$\boldsymbol{\Gamma}=\frac{eg}{m}(\mathbf{v}-\mathbf{u})\int_{-\infty}^{t}dt'\,\delta(\mathbf{x}-\mathbf{u}t'-\mathbf{v}(t-t')).$$

This represents an infinitely thin string (the continuous straight line in Fig. 1) proceeding along the monopole trajectory (the magnitude is $\int_{-\infty}^{t}dt'\tilde{\mathbf{j}}$, the dashed line in Fig. 1) cleared out by the liquid flow. It can be shown that even in the general case of variable velocity, the solution has the same meaning, which is expressed in terms of the law of motion for a parcel of "fluid" in the given velocity field. This is the analog of Thomson's theorem: the vorticity is carried along by the moving fluid.[7]

The foregoing may be verified by considering the vorticity flux (4.6) through a surface patch Σ moving with the fluid. A standard formula from vector analysis,

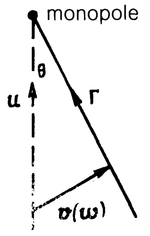

FIG. 1.

$$\frac{d}{dt}\int_{\Sigma} d\mathbf{S}\mathbf{a}=\int_{\Sigma} d\mathbf{S}(\dot{\mathbf{a}}-\operatorname{rot}[\mathbf{v}\mathbf{a}]+\mathbf{v}\operatorname{div}\mathbf{a}),$$

together with Eq. (5.1) gives the equation for the displaced flux along the monopole trajectory:[4]

$$d\Phi/dt=\frac{e}{m}\int_{\Sigma} d\mathbf{S}(\mathbf{v}-\mathbf{u})\tilde{\rho}.$$

Hence, by integrating over an infinitesimal time interval encompassing the instant at which the monopole crosses the surface Σ, we obtain the value of Φ itself after the surface has been crossed (obviously, before the crossing, $\Phi=0$):

$$\Phi=eg/m. \tag{5.3}$$

It is noteworthy that this quantity—a topological invariant—is linear in the magnetic monopole charge g.

The vorticity string is largely the same as the vortex line solutions found in superconductors, liquid helium, and the Higgs vacuum. If we set up a cylindrical coordinate coincident with the axis of an element of the string, we can use (1.1), (1.2), and (5.3) to find the leading azimuthal component of fluid velocity near the string (and far from its end) (see Introduction):

$$v_\varphi=eg/2\pi m\rho=\hbar/m\rho. \tag{5.4}$$

Inserting (4.5) and (5.4) into (4.8) and retaining the most singular terms near the string yields

$$(\Delta-1/\rho^2)n^{1/2}=0, \quad n\propto\rho^2. \tag{5.5}$$

The vanishing of the density at the string itself, which leads to the relations

$$n^{1/2}\boldsymbol{\Gamma}=0, \quad n\boldsymbol{\Gamma}=0, \tag{5.6}$$

is due to the expulsion of fluid by the centrifugal force attributable to the velocity (5.4).

We now give some further consideration to the limits of applicability of the asymptotic formulae (5.4) and (5.5). The first loses its pertinence at the "London depth" $1/\omega_p$ (c/ω_p in conventional units, where c is the speed of light), which determines the magnetic-field screening radius, where

$$\omega_p^2=e^2n_0/m \tag{5.7}$$

is the square of the plasma frequency and n_0 is the unperturbed density of the fluid. The applicability of (5.5) for the density is much more severely restricted to a length

$$\xi=(\hbar/m\omega_p)^{1/2}, \tag{5.8}$$

which comes about in conjunction with the electrostatic term containing δn in (4.8).[3)]

Here, without derivation, we also set down a modification of (5.4) and (5.5) that applies near the end of a string (the z coordinate is reckoned from the monopole along the string)

$$v_\varphi=\frac{\hbar}{2m\rho}(1-x), \qquad n\propto\rho^2 f(x), \tag{5.9}$$

where f may be expressed in terms of a hypergeometric function, and

$$x=z/[z^2+\rho^2(1-u^2)]^{1/2}.$$

It is then clear that (5.6) is valid over the entire length of the string, including its ends.[4)]

Spinning the fluid up to the velocity (5.4), the monopole imparts a kinetic energy Q (per unit length of the string) and momentum M (per particle of the medium). An elementary calculation shows that within the limits of applicability of (5.4),

$$Q=\frac{\pi\hbar^2 n_0}{m}\mathscr{L}=\frac{g^2\omega_p^2}{4\pi}L, \qquad M=\hbar, \tag{5.10}$$

where $\mathscr{L}$ is some logarithmic factor (see also Ref. 3).

6. THE SUPERPOSITION PRINCIPLE

The extension of the Madelung formulation to magnetic monopoles requires that one verify the validity of the principle of superposition of states, upon which the physical interpretation of the machinery of quantum mechanics rests. When there are no magnetic monopoles present, this is guaranteed by the equivalence of the equations of hydrodynamics and the linear Schrödinger equation (4.1); any pair of solutions $\psi_{1,2}$ [and pair of complex parameters $C_{1,2}=\psi_{1,2}\exp(i\alpha_{1,2})$] can be used to construct a third solution

$$\psi_3=C_1\psi_1+C_2\psi_2 \tag{6.1}$$

that satisfies the conditions imposed by the physical requirements.

In the language of hydrodynamics, we need to compare a pair of solutions of (4.4), $n_1,\mathbf{v}_1$ and $n_2,\mathbf{v}_2$, with a third solution $n_3,\mathbf{v}_3$. Combining (6.1) with (4.3) and writing

$$\theta=\alpha_2-\alpha_1+\frac{m}{\hbar}\int_L d\mathbf{l}(\mathbf{v}_2-\mathbf{v}_1), \qquad \boldsymbol{\sigma}=\frac{\hbar}{m}(n_1^{1/2}\nabla n_2^{1/2}-n_2^{1/2}\nabla n_1^{1/2}) \tag{6.2}$$

(the contour L follows the current point $\mathbf{x}$), we obtain

$$n_3=\gamma_1^2n_1+\gamma_2^2n_2+2\gamma_1\gamma_2(n_1n_2)^{1/2}\cos\theta,$$
$$n_3\mathbf{v}_3=\gamma_1^2n_1\mathbf{v}_1+\gamma_2^2n_2\mathbf{v}_2+\gamma_1\gamma_2\cdot[(n_1n_2)^{1/2}(\mathbf{v}_1+\mathbf{v}_2)\cos\theta+\boldsymbol{\sigma}\sin\theta]. \tag{6.3}$$

The uniqueness of these relations is implied by their independence of the choice of L in (6.2). The phase difference between two contours L and L',

$$\delta\theta=\frac{m}{\hbar}\left[\int_L d\mathbf{l}-\int_{L'} d\mathbf{l}\right](\mathbf{v}_2-\mathbf{v}_1)=\frac{m}{\hbar}(\Phi_2-\Phi_1), \tag{6.4}$$

vanishes in the absence of magnetic monopoles by virtue of the fact that $\Gamma=0$.

When monopoles are present, the condition for uniqueness in the preceding sense comes from (1.1). The phase θ actually enters into (6.3) only insofar as it affects the sign of the trigonometric functions, and when (1.1) is satisfied, the right-hand side of (6.4) is a multiple of 2π [see (5.3)].[4] It remains to show that the equations (6.3) indeed satisfy (4.4) if the latter are satisfied individually by $n_1,\mathbf{v}_1$ and $n_2,\mathbf{v}_2$. It is convenient here to use the Euler equation in the form (4.7), where the presence of a monopole is explicitly reflected by the Γ-dependent term, which can in fact give rise to quantities that are "dangerous" in the sense of interest to us here. First and foremost, these include the cross products $[\mathbf{v}_{1,2}\times\Gamma]$ and $[\sigma\times\Gamma]$ obtained from the expressions for $\mathbf{v}_{1,2}$ and $[\mathbf{v}_3\times\Gamma_3]$. These products are multiplied by $\sqrt{n}$ and n, and by virtue of (5.6) they drop out of the final result.

Hazardous quantities of another type,

$$\int_L d\mathbf{l}\,\boldsymbol{\tau}, \qquad \boldsymbol{\tau}=[\mathbf{v}_2\Gamma_2]-[\mathbf{v}_1\Gamma_1]$$

come about when the equations in (6.3) are differentiated with respect to time. Taking advantage of the demonstrated path-independence of (6.3), the contour can be deformed so that in general it has no points in common with the string if the point $\mathbf{x}$ lies outside the string, or, if it does not, so that the contour at a point of intersection is orthogonal to τ. Quantities of this second type therefore do not contribute to the result either.

7. LINEAR RESPONSE FUNCTIONS

Following the prescription referred to at the end of Section 3, we can now find all of the linear response functions of the medium needed for the electrodynamics of magnetic monopoles. This we shall do for a number of simple media of practical importance.

There exists a class of media for which $\tilde\varepsilon=\varepsilon_t$ and $\tilde\mu=1$ [see Eq. (2.5b)]. It includes the classical media (Section 3), and also quantum media at rest and in their unperturbed state (magnetic-field effects in such media are at worst second-order in the field, so $\tilde\mu=1$; see (4.4). Members of the class include the following media:

Classical media with no spatial dispersion are characterized by ε and μ independent of $\mathbf{k}$, and in accordance with (2.5b),

$$\varepsilon=\varepsilon(\omega), \qquad \tilde\varepsilon=\varepsilon(\omega)+[1-1/\mu(\omega)]k^2/\omega^2. \tag{7.1}$$

A classical fluid (ideal, at rest, homogenous) has the response function [see (5.7) and Section 3]

$$\varepsilon=1-\omega_p^2/(\omega^2-k^2s^2), \quad \tilde\varepsilon=1-\omega_p^2/\omega^2, \tag{7.2}$$

where $s^2=n\partial w/\partial n$ is the square of the speed of sound. For a viscous fluid (see [3.4)], ω^2 must be replaced by $\omega[\omega+i(4\nu/3+\zeta)k^2]$ in the expression for ε and by $\omega(\omega+i\nu k^2)$ in the expression for $\tilde\varepsilon$.

A superconducting condensate is described by (4.4) when the correlation length is small (type II superconductor, charged Bose fluid), and it has the response function

$$\varepsilon=1-\omega_p^2/(\omega^2-\hbar^2k^4/4m^2), \qquad \tilde\varepsilon=1-\omega_p^2/\omega^2. \tag{7.3}$$

A system of oscillators (immobile, randomly distributed in space) can serve as a model of a gas of diatomic molecules, lattice vibrations, and so forth under appropriate limiting conditions. It is then necessary to subtract the quantity $\Omega^2(\mathbf{x}-\mathbf{x}_0)$ (the restoring force) from the electric term of the Lorentz force in (4.4), where Ω is the natural frequency of the oscillator, and to subsequently average over the position $\mathbf{x}_0$ of the center of the oscillator. The velocity in an unperturbed medium is zero, while the density is given by the ground-state wave function of the oscillator. The solution takes the form

$$\varepsilon=1-\omega_p^2\sum_{n=0}^{\infty}\gamma_n,\quad \tilde{\varepsilon}=1-\frac{\omega_p^2}{\omega^2}\left(1+\Omega^2\sum_{n=0}^{\infty}(n+1)\gamma_n\right),\qquad(7.4)$$

where

$$\gamma_n=e^{-\alpha}\alpha^n/n!\,(\omega^2-(n+1)^2\Omega^2),\quad \alpha=\hbar k^2/2m\Omega.$$

The response function of a quantum medium with nonzero velocity in its unperturbed state cannot be calculated via conventional electrodynamics, and must be treated as a special case. We shall take up an example of such a medium below.

A plasma is a system of weakly interacting charged particles that can model a wide range of objects, from conventional hot plasma to the electron fluid in a metal or semiconductor. Introducing the polarizabilities α, $\tilde{\alpha}$, and α_t, we can write the corresponding response functions in the form

$$\begin{gathered}\varepsilon=1+\langle\alpha\rangle,\quad \alpha=-\omega_p^2/[(\omega-\mathbf{kw})^2-\hbar^2k^4/4m^2],\\ \tilde{\varepsilon}=1+\langle\tilde{\alpha}\rangle,\quad \tilde{\alpha}=-\omega_p^2/\omega(\omega-\mathbf{kw}),\end{gathered}\qquad(7.5)$$

where angular brackets denote an average over an ensemble with a given distribution over speed w. The value of α_t can be obtained from the relation

$$\omega^2\alpha_t=\omega(\omega-\mathbf{kw})\tilde{\alpha}+\frac{1}{2}[k^2w^2-(\mathbf{kw})^2]\alpha.\qquad(7.6)$$

As it turns out, this relation also holds for the general model of a medium each of whose particles is bound to a center of force, while the centers of force themselves are distributed in some prescribed manner over velocity (for example, an oscillator or atomic gas). In that model, α and α_t can be obtained from standard electrodynamics (they can be expressed in terms of retarded commutators of the charge density and current—ultimately, in terms of the Green's function of a particle in the field of a force center), determining the quantity $\tilde{\alpha}\neq\alpha_t$ using (7.6).

8. ENERGY LOSSES OF A MAGNETIC MONOPOLE

One of the most important applications of monopole electrodynamics is the calculation of the energy Q lost by a monopole per unit length of travel in some medium ($\mathbf{u}$ is the monopole velocity relative to the medium). Making use of (5.2) and the relation $\mathbf{B}(t,\mathbf{x})=\mathbf{B}(\mathbf{x}-\mathbf{u}t)$, the energy losses of a magnetic monopole in a stationary medium take the form[2]

$$Q=-\frac{1}{u}\int d\mathbf{x}\,\tilde{\mathbf{j}}\mathbf{B}=-g\mathbf{u}\mathbf{B}(0)/u.\qquad(8.1)$$

Here we must eliminate terms that do not depend on the parameters of the medium, and that contribute not to monopole energy losses but to the energy of the monopole itself. This would include the quantity $\mathbf{B}_l$, which is completely determined by the external source [see (2.1d)].

Substituting (2.3) into (8.1) yields

$$Q=\frac{g^2}{2\pi^2u^2}\int_0^{q_0}\frac{dk}{k}\int_0^{ku}d\omega\,\omega(k^2u^2-\omega^2)\,\mathrm{Im}(\tilde{\varepsilon}/\sigma),\qquad(8.2)$$

where q_0 is the lesser of the upper bound on the momentum transfer to the medium (for finite energy losses) and the limit obtained from recoil effects.[9] The general structure of (8.2) resembles that of (5.10) [see (1.1), (5.7)],

$$Q=\frac{g^2\omega_p^2}{4\pi}\Lambda,\qquad(8.3)$$

where the dimensionless quantity Λ depends on q_0, u, and the parameters of the medium.

Monopole energy losses contributing to the creation of a vortex string and independent of u (see Introduction) are related to the imaginary part of the function $\tilde{\varepsilon}$ in (8.2), which gives rise to a singularity of the form $1/\omega^2$. They are either equal to the total monopole losses [as in an ideal fluid or superconductor; see (7.2), (7.3)],

$$\Lambda=\ln(q_0/\omega_p),\qquad(8.4)$$

or some fraction of the latter [as in a system of oscillators; see (7.4)],

$$\Lambda=\ln\left[q_0\left(\frac{\hbar}{m\Omega}\right)^{1/2}\right]\qquad(8.5)$$

Equation (8.4) represents the limiting total loss in a viscous fluid in which $\nu\to0$, in a plasma with $u\gg\langle w\rangle$ [see (7.5)], in a system of oscillators with $u>(1+\omega_p^2/\Omega^2)^{-1/2}$, and in media with $\tilde{\mu}=1$ as $u\to1$ (the latter may be deduced from the Leontovich relations; see Ref. 9).

Losses ascribable to string formation lead to a pronounced difference between monopole and charge deceleration processes. In an ideal fluid or a superconductor, a slow charge will not be decelerated (the D'Alembert-Euler paradox), while a magnetic monopole will lose a fraction of its energy to string formation.

As a string spreads, the quantity Λ will begin to depend on the monopole velocity. This results from the deflection of monopole trajectories in media consisting of an ensemble of subsystems distributed over velocity; thus, for a plasma,

$$\Lambda=\frac{2}{3}u\langle w^{-1}\rangle\ln(q_0/\omega_p),\quad u\ll\langle w\rangle.\qquad(8.6)$$

Dissipation in the medium (which is also a direct source of energy loss) leads to string destruction as well. For a viscous fluid,

$$\Lambda=\ln R\ \ (R\gg1,\ R_0<1),\quad \Lambda\propto R^{2/3}\ \ (R\ll1),\qquad(8.7)$$

where the Reynolds number $R=u/\nu\omega_p\gg R_0=u/\nu q_0$ [Eq. (8.4) yields $R_0\gg1$].

One special source of magnetic monopole slowing is Cherenkov radiation, which is described by the imaginary part of $1/\sigma$ in (8.2). For a classical medium with no spatial dispersion, the result differs from the standard expression by an extra factor of $\mu(\omega)^2$ in something that looks like the Tamm-Frank equation.[2,5,10] The Cherenkov loss in a system of oscillators is given by

$$\Lambda=\ln\left[\left(\frac{m\Omega}{\hbar}\right)^{1/2}\Big/\bar{\omega}\right],$$

where $\overline{\omega}$ is the larger of ω_p and $\Omega(u^{-2}-1)^{1/2}$ [see (7.4)]. In such a medium, the total monopole losses [taking (8.5) into account] are

$$\Lambda = \ln(q_0/\overline{\omega}). \tag{8.8}$$

From everything that we have said, then, it is clear that magnetic monopole energy losses are essentially logarithmic, with a large argument of the logarithm. In evaluating the contribution made by nonlinear effects, this permits one to work in the logarithmic approximation (see Section 9 below).

9. NONLINEAR EFFECTS

When higher-order effects in the monopole interaction with the medium are taken into consideration (the charge g is large; see Introduction), the results obtained in Section 8 for the linear (Born) approximation may no longer hold. It turns out, however, that those results are in fact widely applicable.

One can get some idea of the role played by nonlinear effects from the dimensionless ratio η between the first- and second-order corrections to the velocity of the medium:

$$\eta \propto eg/md\,\langle|\mathbf{u}-\mathbf{w}|\rangle \propto \hbar/md\,\max(u,w) \tag{9.1}$$

[see (1.1), (2.1), (4.4)]. Here d is a typical scale length of the medium, $\mathbf{w}$, as before, is the velocity of the medium/ensemble subsystem, and $\mathbf{u}-\mathbf{w}$ is the velocity of the monopole relative to the medium. Therefore, in a classical medium, where the denominator of (9.1) is basically the classical action, the Born approximation is valid. This is a direct consequence of the fact that g vanishes as $\hbar$ goes to zero (see (1.1)).

The quantity $V=\hbar/md$ in (9.1) is essentially the characteristic velocity of a quantum medium (in a nonrelativistic medium, $V\ll 1$). Rewriting (9.1) in the form

$$\eta \propto V/\max(u,w), \tag{9.2}$$

we see that the Born approximation holds in a quantum medium under two conditions: a) the temperature of the medium is high enough, $T>mV^2$ (such a medium still remains a quantum system when $dn_0^{1/3}>1$); b) the magnetic monopole moves rapidly enough, $u>V$ (this is in accord with the result obtained by Apenko[5] for a system of oscillators).

It remains to examine the case of a slow magnetic monopole in a relatively cool quantum medium. We remarked in Section 4 that independent of the monopole's velocity, its longitudinal magnetic field has no effect on the medium. The validity of the Born approximation would then seem to follow directly for $u\to 0$, as the transverse fields, which are inductive for a magnetic monopole, should at first glance vanish in that limit. This, however, is not true, because of a peculiarity of the response function $\tilde{\varepsilon}$ at small $\omega=\mathbf{k}\cdot\mathbf{u}$ [see (2.3), (5.2), and Section 7].[5)] But as we noted in Section 8, this feature is responsible for the "string" part of the monopole energy losses, which are associated with the formation of a vortex string and are independent of the monopole velocity. The Born approximation is therefore only applicable to the remaining "non-string" part of the energy losses; for that part, the convergence of the Born series is the same as for a conventional particle carrying a charge (137/2) ue.

The issue of the applicability of the Born approximation to the string part of the losses is particularly simple for a structureless (no bound entities) medium that is stationary in the absence of a magnetic monopole—a superconducting condensate, for example. In that event, the vorticity equals its Born-approximation value, proportional to g [see (5.3)] and pointing in the same direction as the monopole velocity (the only direction physically distinguished by the system). This preordains the applicability of Born-type formulas to such media [see (8.3) and (8.4)].

Specifically, from (2.1) and (3.3), we obtain an expression for the field $\mathbf{B}(\mathbf{x}-\mathbf{u}t)$:

$$\mathbf{B}(\mathbf{x}) = \Box^{-1}[e\,\mathrm{rot}(n\mathbf{v}) - g(\nabla - \mathbf{u}(\mathbf{u}\nabla))\delta(\mathbf{x})], \tag{9.3}$$

where $\Box=(\mathbf{u}\nabla)^2-\Delta$. In a cylindrical coordinate system with the axis directed along $\mathbf{u}$,

$$Q = \frac{eg}{4\pi}\int d\mathbf{x}\, nv_\varphi \frac{d}{d\rho}[z^2+\rho^2(1-u^2)]^{-1/2} \tag{9.4}$$

[see (8.1)]. Making use of (1.2), (9.3), and the azimuthal symmetry of the problem, v_φ reduces to the vorticity of the fluid

$$v_\varphi = -(\Box + e^2n/m)^{-1} d\Gamma_z/d\rho,$$

which is in agreement with (5.4) far from the end of the string, and falls off exponentially for $z,\rho>\omega_p^{-1}$. The density, moreover, is equal to the constant value n_0 far from the string, and it drops to zero for $\rho<\xi$ [see (5.8)]. With the foregoing in mind [along with the fact that the neighborhood of a magnetic monopole injects a nonlogarithmic contribution into (9.4); see (5.9)], Eqs. (8.3) and (8.4) actually follow from (9.4). The only reservation here—but an important one—is that $1/\xi$ must be included among the quantities competing in the determination of the upper limit q_0 (see Section 8). The net result is a nonlogarithmic energy dependence of monopole losses in the ultrarelativistic regime characteristic of a charged particle. Then and only then do nonlinear effects become manifest—the primordial value of the exponent 2 in (5.5) was $|eg|/\pi\hbar$ [see (1.1)].

All of the above is true for all of the subsystems comprising a structureless medium/ensemble (a plasma, for example) with one difference—the vorticity is now directed along $\mathbf{u}-\mathbf{w}$. This gives rise to the additional factor (see Fig. 1)

$$\langle\cos\theta\rangle = \langle(u^2-\mathbf{u}\mathbf{w})/u|\mathbf{u}-\mathbf{w}|\rangle,$$

and we revert to the Born equations (8.3), (8.4), and (8.6).

The presence of coupled entities within the medium (oscillators, atoms, and so forth) that have typical scale lengths d and velocities V (see above) may alter the situation. For the Born approximation to be applicable, it is then necessary that $\eta\ll 1$ for those entities [see (9.2)]. When that condition is not satisfied, nonlinear effects may come to the fore.

We see, then, that the relatively broad applicability of the Born equations of Section 8 to magnetic *monopole* energy losses is based on the quantum smallness of the monopole charge, its proportionality to a topological invariant of the string [Eq. (5.3)], and the logarithmic behavior of the losses when the argument of the logarithm is large (proportional to the speed of light).

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1) Direct experimental studies are unthinkable, even far in the future: the energy of accelerators currently in the planning stage is at most 10^5 GeV. It would also be well not to place too much hope in cosmology, since no measurement can ever be made without some independent understanding of the design of the measuring instrument. In the meantime, our data on the "cosmological instrument"—the structure of the universe—must be dredged out of the same meager source from which we expect to obtain fundamental information. Furthermore, our picture of the universe these days changes just about as fast as our ideas about the microworld.

2) In the Madelung formulation, which contains no potentials at all, the groundlessness of assertions appealing to the Bohm-Aharonov effect about the special role potentials play in quantum mechanics is especially clear.

3) Note that (5.8) has no bearing on the behavior of a single quantum particle, for which there is no electrostatic term (self-action). Equation (5.8) is also invalid in a superconductor, where apart from the condensate described by a coherent wave function, there is also a normal, charged fluid that fills in the density depression (the normal crust on a vortex filament). In that case, ξ is determined by the correlation length of the superconductor.

4) These equations imply the equivalence of current-induced and genuine magnetic dipoles; there is no medium within the latter (which consist of a monopole—antimonopole pair connected by a string, the product of a pair-creation event).

5) This is exactly why the magnetic field of a magnetic monopole exhibits the Meissner effect in a superconductor: the longitudinal and transverse components of the field cancel, no matter what the monopole velocity.[2]

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