Interaction between gravitation and a non-Abelian matter multiplet in the framework of extended $N = 4$ supersymmetry

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The Lagrangian for the interaction between gravitation and a single matter multiplet is set up in the case of an arbitrary non-Abelian internal symmetry group G by means of a special procedure of reduction from a space with $D = 10$ dimensions. The resultant theory corresponds to the "minimal" version of $N = 4$ supergravity interacting with matter. In the conventional (nonminimal) version there appear beside the non-Abelian multiplet six auxiliary neutral vector $N = 4$ supermultiplets of matter, strongly (nonlinearly) mixed with the gravity supermultiplet. A gauge version of the $N = 4$ supersymmetry is also presented (in the framework of the minimal scheme), in which the global $SO(4)$ group acting in the space of the four supersymmetry generators is replaced by a local $SO(4)$ group. This local scheme corresponds to spontaneously broken $N = 4$ supersymmetry.

1. INTRODUCTION

The present work constitutes the completion of a cycle of papers (see Refs. 1 and 2) devoted to the construction of the interaction of gravity with matter in the framework of extended $N = 4$ supersymmetry. Such supersymmetry is the largest possible from among those which admit the inclusion of matter. The Lagrangian for the theory that describes matter with such a symmetry contains no free parameters (beside the one gauge coupling constant). Its second important property is its finiteness (the theory has no ultraviolet divergences). The theory has a significant flaw in that (in the absence of gravity) no mechanism exists for spontaneous breaking of the $N = 4$ supersymmetry, such breaking being required to describe the phenomenology. The hopes for the possible employment of such a theory in physics can be justified only in the event that the interaction with gravity would somehow guarantee a mechanism for spontaneous or explicit soft breaking of the $N = 4$ supersymmetry down to $N = 1$ (or to $N = 0$, i.e., to complete absence of supersymmetry). The study of this question naturally presumes the construction and analysis of the interaction of matter with gravity. This is the motivation for the present work.

In the conventional description of $N = 4$ supergravity, interacting with matter, which naturally arises in the reduction from a space of $D = 10$ dimensions (see Ref. 3), one encounters the problem of a large number of superfluous degrees of freedom—auxiliary neutral (under the internal symmetry group G) fields, which are strongly mixed with the components of the gravitational supermultiplet and make the analysis of the dynamics difficult. We emphasize that from the point of view of string theory, where the space with $D = 10$ dimensions has fundamental physical meaning, these neutral fields are not superfluous. But from the point of view in which the Minkowski space M_D $(D = 4)$ is considered fundamental there is, generally speaking, no basis for the introduction of such fields. Therefore in Minkowski space it is natural to start from a study of the simplest "minimal" scheme, where all superfluous degrees of freedom are absent. The construction of such a scheme is the aim of the present work.

The solution of this problem requires the application of artificial approaches, since it is not clear how to directly "disentangle" the strongly mixed gravity and matter degrees of freedom. In this work we make use of a special reduction scheme from the space M_D with ten dimensions $(D = 10)$ to M_4, in which it is possible while conserving supersymmetry to consistently turn the superfluous degrees of freedom into zero. The method was worked out in previous articles of the author on the example of pure supergravity[1,2] and gravity in interaction with an Abelian matter multiplet. (We emphasize that in the approach under discussion utilization of 10-dimensional supergravity is simply a matter of convenience. The fields in M_{10} carry no independent physical significance and therefore we are free to impose on them the restrictions needed for the solution of the posed problem.)

The same problem, but using different methods, was solved in Refs. 4–6. In Ref. 4 use was made of the supersymmetric generalization of the σ model for scalar fields on the manifold $O(n.6)/O(n) \otimes O(6)$, where n is the number of matter multiplets, with subsequent localization of certain subgroups of the $O(n.6)$ symmetry group. The beautiful approach used in Ref. 4 does not permit, however, the discussion of arbitrary internal symmetry groups G, but only admits special subgroups of $O(n.6)$.

The general case was considered in Refs. 5 and 6. Their starting point is the superconformal theory of interaction of matter and Weyl gravity in M_4 with assumed $SU(1.1)$ symmetry in the boson sector.[7] The passage to the interesting case of Poincaré gravity includes the imposition of a number of additional (nonlinear) conditions on the fields, which are explicitly solved only in special cases (for simplest groups G). To summarize, the establishment of a direct connection between Refs. 5 and 6 and our approach seems a difficult problem, since in our approach the Lagrangian is formulated explicitly in terms of independent fields. We shall show that certain examples of spontaneous breaking of $N = 4$ supersymmetry, in particular down to $N = 1$ (see Ref. 9), were discovered in Ref. 6, and in Refs. 8 and 9 based on the same approach.

The reduction procedure used in this paper is readily generalized, using methods developed in Ref. 10, to the case when the internal space $Q_6 = M_{10}/M_4$ is the group manifold of the $O(4)$ group. From the point of view of method our approach differs from that of Ref. 10 (and also from that of Refs. 11 and 12, which make use of a procedure close to that

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of Ref. 10) by a different interpretation of the third rank antisymmetric tensor that figures in the theory. As a result we obtain in M_4 a theory, in which a group $O(4)$ is localized, which acts in the space of four generators of the $N=4$ supersymmetry. In its results this theory corresponds to the inclusion of matter into the gauge version of $N=4$ supergravity, constructed by entirely different means in Ref. 13. The theory corresponds, generally speaking, to spontaneously broken $N=4$ supersymmetry. (The character of breaking depends on the choice of the symmetry group G in the matter sector.)

In Sec. 2 we fix the notation and their relation to the previous papers describing $N=1$ supergravity in interaction with matter in the space M_{10} (the Lagrangian for this system was obtained in Ref. 14). In Sec. 3 we describe the reduction procedure from M_{10} to M_4, in Sec. 4 we describe the bosonic part of the Lagrangian in M_4. In Sec. 5 the supersymmetry transformations are given, and in Sec. 6 we construct a gauge spontaneously broken version of $N=4$ supersymmetry. In Sec. 7 we state certain assertions relating to the spontaneous breaking of supersymmetry in the theory.

2. SUPERGRAVITY IN $D=10$ DIMENSIONS

The notation in this paper corresponds in the main to that used in previous papers,[1,2] but there are some changes which we now describe. World indices in M_{10} are denoted by letters with hats: $\hat{M}, \hat{N}, \hat{P}, \ldots$ The same notation is used for tangent indices: $\hat{A}, \hat{B}, \hat{C}, \ldots$ The matter multiplet contains the fields $A_{\hat{M}}$ (gluon field) and Λ (gluino field); $A_{\hat{M}}$ and Λ lie in the algebra G: $A_{\hat{M}}=(A_{\hat{M}})^\alpha T_\alpha$ and $\Lambda=\Lambda^\alpha T_\alpha$, where T_α are Hermitian generators of the group G in the fundamental representation, normalized by the condition $\mathrm{Tr}(T_\alpha T_\beta)=\delta_{\alpha\beta}$. The gravity multiplet includes the graviton $V^{\hat{A}}_{\hat{M}}$, the Majorana-Weyl left spinor-vector gravitino $\Psi_{\hat{M}}$, the Majorana-Weyl right spinor X, the antisymmetric gauge tensor field $A_{\hat{M}\hat{N}}$, and the scalar φ.

The gluon field tensor has the form

$$F_{\hat{M}\hat{N}}=2\partial_{[\hat{M}}A_{\hat{N}]}+ig[A_{\hat{M}},A_{\hat{N}}]. \tag{2.1}$$

The covariant field tensor, corresponding to the potentials $A_{\hat{M}\hat{N}}$, has the form (see Ref. 14)

$$F_{\hat{M}\hat{N}\hat{P}}=3\partial_{[\hat{M}}A_{\hat{N}\hat{P}]}-3k\,\mathrm{Tr}\,(A_{[\hat{M}}F_{\hat{N}\hat{P}]}-{}^2/_3 igA_{[\hat{M}}A_{\hat{N}}A_{\hat{P}]}). \tag{2.2}$$

The Lagrangian for the theory, describing the interaction of supergravity with one matter supermultiplet in M_{10} in the framework of $N=1$ supersymmetry, has the form[14]

$$L=L_g+L_m, \tag{2.3}$$

where L_g coincides in the absence of matter with the Lagrangian of $N=1$ supergravity, while L_m goes over in the absence of gravity into the $N=1$ supersymmetric matter Lagrangian. We shall present L_g and L_m accurate up to terms of fourth order in the fermion fields:

$$\begin{aligned}V^{-1}L_g=&-(1/4k^2)R-(i/2)\overline{\Psi}_{\hat{M}}\Gamma^{\hat{M}\hat{N}\hat{P}}D_{\hat{N}}(\omega)\Psi_{\hat{P}}\\&+(i/2)\overline{X}\Gamma^{\hat{M}}D_{\hat{M}}(\omega)X+(9/32k^2)\varphi^{-2}(\partial_{\hat{M}}\varphi)^2\\&+{}^1/_{12}\varphi^{-3/2}F_{\hat{M}\hat{N}\hat{P}}F^{\hat{M}\hat{N}\hat{P}}\\&+(3\sqrt{2}/8)\overline{\Psi}_{\hat{M}}\Gamma^{\hat{N}}\Gamma^{\hat{M}}X\varphi^{-1}(\partial_{\hat{N}}\varphi)\\&+(ik/24)\varphi^{-3/4}F_{\hat{P}\hat{Q}\hat{R}}(\overline{\Psi}_{\hat{M}}\Gamma^{\hat{M}\hat{N}\hat{P}\hat{Q}\hat{R}}\Psi_{\hat{N}}\\&-6\overline{\Psi}^{[\hat{P}}\Gamma^{\hat{Q}}\Psi^{\hat{R}]}+i\sqrt{2}\,\overline{\Psi}_{\hat{M}}\Gamma^{\hat{P}\hat{Q}\hat{R}}\Gamma^{\hat{M}}X).\end{aligned} \tag{2.4}$$

The term L_m has the form

$$\begin{aligned}V^{-1}L_m=&-{}^1/_4\varphi^{-3/4}F_{\hat{M}\hat{N}}F^{\hat{M}\hat{N}}+(i/2)\overline{\Lambda}\Gamma^{\hat{N}}D_{\hat{N}}(\omega)\Lambda\\&+(ik/24)\varphi^{-3/4}\overline{\Lambda}\Gamma^{\hat{M}\hat{N}\hat{P}}\Lambda F_{\hat{M}\hat{N}\hat{P}}\\&+(ik/2\sqrt{2})\varphi^{-3/8}\overline{\Lambda}\Gamma^{\hat{M}}\Gamma^{\hat{N}\hat{P}}(\Psi_{\hat{M}}+(i\sqrt{2}/12)\Gamma_{\hat{M}}X)F_{\hat{N}\hat{P}}.\end{aligned} \tag{2.5}$$

Here $V=\det(V^{\hat{A}}_{\hat{M}})$, R is the curvature scalar, and

$$D_{\hat{N}}(\omega)=(\partial_{\hat{N}}+{}^1/_4\omega_{\hat{N}\hat{A}\hat{B}}\Gamma^{\hat{A}\hat{B}}), \tag{2.6}$$

where $\omega_{\hat{N}\hat{A}\hat{B}}$ is the spin connection. Up to the agreed upon accuracy we have

$$\omega_{\hat{N}\hat{A}\hat{B}}=-{}^1/_2(\Omega_{\hat{N}\hat{A}\hat{B}}+\Omega_{\hat{B}\hat{N}\hat{A}}-\Omega_{\hat{A}\hat{B}\hat{N}}), \tag{2.7}$$

$$\Omega_{\hat{M}\hat{N}}{}^{\hat{A}}=\Omega_{\hat{M}\hat{B}\hat{C}}V_{\hat{N}}{}^{\hat{B}}\eta^{\hat{C}\hat{A}}=-\partial_{\hat{M}}V_{\hat{N}}{}^{\hat{A}}+\partial_{\hat{N}}V_{\hat{M}}{}^{\hat{A}}, \tag{2.8}$$

where $\eta^{\hat{A}\hat{B}}=(+,-,-,\ldots,-)$ is the flat metric in tangent space. In these equations k is the gravitational constant, $\Gamma^{\hat{A}}=\Gamma^{\hat{M}}V^{\hat{A}}_{\hat{M}}$ are 32×32 Dirac matrices in M_{10}, and $\Gamma^{\hat{M}\hat{N}\ldots\hat{Q}}=\Gamma^{[\hat{M}}\Gamma^{\hat{N}}\ldots\Gamma^{\hat{Q}]}$.

We present also the infinitesimal transformations, corresponding to the symmetries of the Lagrangian (2.1). The general-coordinate transformations have the form

$$\begin{aligned}&\delta_\xi V_{\hat{M}}{}^{\hat{A}}=\xi^{\hat{N}}(\partial_{\hat{N}}V^{\hat{A}}_{\hat{M}})+(\partial_{\hat{M}}\xi^{\hat{N}})V^{\hat{A}}_{\hat{N}},\\&\delta_\xi V_{\hat{A}}{}^{\hat{M}}=\xi^{\hat{N}}(\partial_{\hat{N}}V^{\hat{M}}{}_{\hat{A}})-(\partial_{\hat{N}}\xi^{\hat{M}})V^{\hat{N}}_{\hat{A}}\end{aligned} \tag{2.9}$$

and so forth. Here ξ are the parameters of the general-coordinate transformations.

The transformations from the gauge group G have the form

$$\begin{aligned}&\delta_\Omega A_{\hat{M}}=\partial_{\hat{M}}\Omega+ig[A_{\hat{M}},\Omega],\\&\delta_\Omega\Lambda=ig[\Lambda,\Omega],\\&\delta_\Omega A_{\hat{M}\hat{N}}=2k\,\mathrm{Tr}\,(A_{[\hat{M}}\partial_{\hat{N}]}\Omega),\end{aligned} \tag{2.10}$$

where Ω are the parameters of the transformations. There is also present invariance under Abelian gauge transformations that affect only the tensor field $A_{\hat{M}\hat{N}}$:

$$\delta_\eta A_{\hat{M}\hat{N}}=2\partial_{[\hat{M}}\eta_{\hat{N}]}, \tag{2.11}$$

where η are the parameters of the transformation. Lastly there is present Lorentz-type invariance with respect to the group $O(1.9)$, acting on the vector indices in the tangent space and the spinor indices:

$$\begin{aligned}&\delta_L V_{\hat{M}}{}^{\hat{A}}=L^{\hat{A}\hat{B}}V_{\hat{M}\hat{B}},\\&\delta_L X=-(i/2)L_{\hat{A}\hat{B}}\Sigma^{\hat{A}\hat{B}}X,\ \text{analogously for}\ \psi_{\hat{M}}\ \text{and}\ \Lambda,\end{aligned} \tag{2.12}$$

where $L_{\hat{A}\hat{B}}$ are parameters of the transformations, and $\Sigma^{\hat{A}\hat{B}}=(i/2)\Gamma^{\hat{A}\hat{B}}$ are the generators of the $O(1.9)$ group in the spinor representation.

As in Refs. 1 and 2, we make use of the Majorana representation for the Γ matrices in which all the matrices $\Gamma^{\hat{A}}$ are pure imaginary, while Γ_{11} (the analog of γ_5 in M_4) is real (γ_5, in contrast, is pure imaginary), and the Majorana spinors $\Psi_{\hat{M}}$, Λ and X are real and subject to the additional Weyl condition:

$$\Gamma_{11}\Psi_{\hat{M}}=\Psi_{\hat{M}},\quad \Gamma_{11}X=-X,\quad \Gamma_{11}\Lambda=\Lambda. \tag{2.13}$$

Lastly we present the transformations of the fields under $N = 1$ supersymmetry in M_{10}:

$$\delta_\varepsilon A_{\hat{M}} = -(i/\sqrt{2})\varphi^{3/4}(\bar{\varepsilon}\Gamma_{\hat{M}}\Lambda),$$
$$\delta_\varepsilon \Lambda = -(1/2\sqrt{2})\varphi^{-3/4}(\Gamma^{\hat{M}\hat{N}}\varepsilon)F_{\hat{M}\hat{N}},$$
$$\delta_\varepsilon V_{\hat{M}}^{\hat{A}} = -ik(\bar{\varepsilon}\Gamma^{\hat{A}}\Psi_{\hat{M}}) + L^{\hat{A}\hat{B}}V_{\hat{M}\hat{B}},$$
$$\delta_\varepsilon \log\varphi = -(2k\sqrt{2}/3)\bar{\varepsilon}X,$$
$$\delta_\varepsilon A_{\hat{M}\hat{N}} = \varphi^{3/4}(i\bar{\varepsilon}\Gamma_{[\hat{M}}\Psi_{\hat{N}]} + (1/2\sqrt{2})\bar{\varepsilon}\Gamma_{\hat{M}\hat{N}}X) - ik\sqrt{2}\varphi^{3/4}\operatorname{Tr}\bar{\varepsilon}\Gamma_{[\hat{M}}\Lambda A_{\hat{N}]},$$
$$\delta_\varepsilon X = (3\sqrt{2}i/8k)(\Gamma^{\hat{M}}\varepsilon)\partial_{\hat{M}}\log\varphi + (i/12\sqrt{2})\varphi^{-3/4}(\Gamma^{\hat{P}\hat{Q}\hat{R}}\varepsilon)F_{\hat{P}\hat{Q}\hat{R}} - (i/2)L_{\hat{A}\hat{B}}\Sigma^{\hat{A}\hat{B}}X,$$
$$\delta_\varepsilon \Psi_{\hat{M}} = (1/k)D_{\hat{M}}\varepsilon + {}^{1}/_{48}\varphi^{-3/4}(\Gamma^{\hat{P}\hat{Q}\hat{R}}{}_{\hat{M}} + 9\delta_{\hat{M}}^{[\hat{P}}\Gamma^{\hat{Q}\hat{R}]})\varepsilon F_{\hat{P}\hat{Q}\hat{R}} - (i/2)L_{\hat{A}\hat{B}}\Sigma^{\hat{A}\hat{B}}\Psi_{\hat{M}}. \quad (2.14)$$

We note that the supersymmetry transformations in (2.14) are augmented by Lorentz rotations, to ensure that after reduction the supersymmetry transformations in M_4 have their standard form. The transformations (2.14) are written accurate to lowest order in the fermion fields. Since (see below) the Lorentz rotation parameters $L_{\hat{A}\hat{B}}$ will turn out to be quadratic in the fermion fields, they are relevant to the assumed accuracy in the variation $\delta_\varepsilon V_{\hat{M}}^{\hat{A}}$ only. (There are no obstacles of principle to the taking into account of terms of all orders in the fermion fields as they are contained in Ref. 14, however the resultant equations become considerably more complicated.)

3. REDUCTION FROM $D = 10$ DIMENSIONS TO THE MINKOWSKI SPACE

We assume that $M_{10} = M_4 \otimes Q_6$, with $\hat{M} = (\mu, M)$ and $\hat{A} = (\alpha, A)$. The Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ take on the values 0, 1, 2, 3 and refer to the Minkowski space M_4. The Latin indices take on the values 4, 5, ..., 9 (or 1, 2, ..., 6 depending on the context) and refer to the internal space Q_6. As before, indices from the middle of the alphabet are world indices, while those from the beginning of the alphabet refer to the (flat) tangent space. In the following x^μ are coordinates in M_4, while y^M are coordinates from Q_6.

We suppose first that all fields and parameters of symmetry transformations are independent of y^M. An exception is needed for $\xi^{\hat{M}} = (\xi^\mu, \xi^M)$ only. The conditions

$$\partial_N V_{\hat{M}}^{\hat{A}} = \partial_N A_{\hat{M}\hat{N}} = 0, \quad \partial_N \delta_\xi V_{\hat{M}}^{\hat{A}} = \partial_N \delta_\xi A_{\hat{M}\hat{N}} = 0$$

are consistent (see Ref. 15) provided that

$$\xi^\mu = \xi^\mu(x), \qquad \xi^M = a^M{}_N y^N + k\sqrt{2}\omega^M(x), \quad (3.1)$$

where a_N^M is a numerical 6×6 matrix from the group $SL(6.R)$. With respect to the world indices of Q_6 there remains, generally speaking, a global symmetry $SL(6.R)$, but in what follows we confine ourselves to the $O(6)$ subgroup. In what follows we often use the notation

$$\varkappa = k\sqrt{2}. \quad (3.2)$$

Let (for $a_N^M = 0$)

$$\xi^{\hat{M}} = (\xi^\mu(x), \varkappa\omega^M(x)), \quad \Omega = \Omega(x), \quad \eta_{\hat{M}} = (\eta_\mu(x), \eta_M(x)).$$

The functions $\xi^\mu(x)$ become parameters of general-coordinate transformations in M_4, while $\omega^M(x)$ become parameters of the Abelian $U(1)$ gauge group.[6] We set $\int d^6y = 1$ and use the same symbol for the gravitational constant in M_{10} and M_4. This makes it possible to omit writing out explicitly trivial dimensional factors in the relations between fields in M_{10} and M_4.

It is convenient to make the frame components $V_{Q_6}^{M_4}$ vanish: $V_M^\alpha = 0$, $V_A^\mu = 0$. This can be always accomplished with the help of an $O(1.9)$ rotation. Afterwards the selfconsistency conditions $\delta_\varepsilon V_M^\alpha = \delta_\varepsilon V_A^\mu = 0$ fix the components $L_{\alpha B} = -L_{B\alpha}$ of the rotation matrices in (2.14). And so we choose the frame $V_{\hat{M}}^{\hat{A}}$ in the form

$$\begin{pmatrix} V_\mu{}^\alpha(x) & V_\mu{}^A(x) \\ V_M{}^\alpha(x) & V_M{}^A(x) \end{pmatrix} = \begin{pmatrix} (\rho E)^{-1/2}e_\mu{}^\alpha & \varkappa\rho^{1/6}B_\mu{}^N E_N{}^A \\ 0 & \rho^{1/6}E_M{}^A \end{pmatrix}, \quad (3.3)$$

where e_μ^α is the frame in M_4, E_M^A is the frame in Q_6,

$$g_{\mu\nu} = e_\mu{}^\alpha e_{\nu\alpha}, \quad g_{MN} = E_M{}^A E_N{}^B \eta_{AB} \quad (3.4)$$

are the metric tensors in M_4 and Q_6, $\eta_{AB} = -\delta_{AB}$, and $E = \det E_M^A$. We note that

$$V = \det V_{\hat{M}}{}^{\hat{A}} = e(\rho E)^{-1}, \quad (3.5)$$

where $e = \det e_\mu^\alpha$.

It is convenient to take the scalar factor ρ in (3.3) in the form

$$\rho = \varphi^{-9/4} = E^{-3/4}\exp(-{}^{3}/_{2}kA). \quad (3.6)$$

Equation (3.6) is the definition of the new scalar field A. Such a definition ensures the correct normalization of the kinetic term of the A field and eliminates mixing terms of the type $\partial_\mu E \partial^\mu A$. (We emphasize that in M_4 the A field is a component of the gravity supermultiplet, while the scalar field E depends only on the matter scalar fields.)

In the following we describe in succession the reduction scheme for all the fields from M_{10} to M_4. Let

$$A_{\hat{M}} = \{A_\mu(x), A_M(x)\}, \quad \varkappa A_M = \phi_M, \qquad A_{\hat{M}\hat{N}} = \{A_{\mu\nu}(x), A_{\mu N}(x), A_{MN}(x)\}. \quad (3.7)$$

We consider first the $A_{\hat{M}\hat{N}}$ field. To reduce the number of independent fields in M_4 we impose the condition

$$A_{MN}(x) = 0. \quad (3.8)$$

Afterwards the selfconsistency condition $\delta_\varepsilon A_{MN} = 0$, where the variation δ_ε should be calculated according to (2.14), leads to the new condition:

$$Y = \Psi_B - [(i/2\sqrt{2})\Gamma_B X + E_B{}^N \operatorname{Tr}(\phi_N \Lambda)] = 0, \quad (3.9)$$

where $\Psi_B = V_B^{\hat{M}}\Psi_{\hat{M}}$, and ϕ_N is defined in (3.7). The new selfconsistency condition:

$$\delta_\varepsilon(Y) = 0,$$

leads to the new condition:

$$Z = \sqrt{2}A_{\mu M} - [B_\mu{}^N g_{MN} - \operatorname{Tr}(A_\mu \phi_M)] = 0. \quad (3.10)$$

We note that the condition (3.10) is invariant under all symmetries (2.9)–(2.11), where the gauge transformations of the B_μ^N fields should be compensated by transformations of the $A_{\mu M}$ fields according to (2.11).

Now the calculation of the variation $\delta_\varepsilon g_{MN}$ using the conditions (3.8)–(3.10) gives

$$\delta_\varepsilon g_{MN}=\mathrm{Tr}(\phi_M\delta_\varepsilon\phi_N+\phi_N\delta_\varepsilon\phi_M), \tag{3.11}$$

whence

$$g_{MN}=\eta_{MN}+\mathrm{Tr}(\phi_M\phi_N). \tag{3.12}$$

We note that the choice of the integration constant in (3.12) is dictated by the requirement that we have in the absence of gravity an $O(6)$-invariant, [or, what is equivalent, $SU(4)$-invariant] standard $N=4$ supersymmetric theory of matter. For this reason we have chosen for the indicated constant an $O(6)$-invariant tensor. (The overall numerical factor multiplying this tensor is fixed by the normalization of the kinetic terms in the Lagrangian.) As a result the initial global symmetry group $SL(6.R)$ is reduced to $O(6)$. (Other possibilities, however, exist also, see below.)

At this point one can be convinced of the important fact: when the Eqs. (3.8)–(3.11) are satisfied the selfconsistency condition $\delta_\varepsilon(Z)=0$, which we must impose to conserve supersymmetry, leads to no new constraints and is automatically satisfied. Thus the process of eliminating superfluous degrees of freedom ends at the third step starting from Eq. (3.8). [We emphasize that for any other choice of the starting condition instead of (3.8), the subsequent consistent application of the variations δ_ε will, generally speaking, result in the "extermination" of all interesting degrees of freedom for either matter or gravity.]

It is convenient to introduce in M_4 in place of A_μ a new gluon field, which transforms correctly under general-coordinate transformations. The standard procedure (see, in particular, Ref. 3) consists of starting from the flat components in M_{10}. We introduce

$$a_\mu=A_\mu-B_\mu{}^N\phi_N \tag{3.13}$$

as the gluon field. We note that $a_\mu=e_\mu^\alpha a_\alpha$, where

$$A_\alpha=(\rho E)^{1/2}a_\alpha, \tag{3.14}$$

with $A_\alpha=V_\alpha^{\hat M}A_{\hat M}$. In terms of the new field the relation (3.10) is rewritten in the form

$$\sqrt{2}A_{\mu N}=B_\mu{}^M\eta_{MN}-\mathrm{Tr}\,(a_\mu\phi_N). \tag{3.15}$$

We now go on to a discussion of the components $A_{\mu\nu}$ in (3.7). They describe in M_4 one degree of freedom—the pseudoscalar field $B(x)$. The contribution to the Lagrangian from that field is calculated with the help of the duality transformation (see, for example, Ref. 15; the calculational scheme, completely analogous to the case considered here, is presented in detail in Refs. 1 and 2, accurate up to some changes in the normalization of the fields). The result reduces to the following. We define the tensor field $f_{\alpha\beta\gamma}$, invariant under general-coordinate transformations in M_4:

$$F_{\alpha\beta\gamma}=(\rho E)^{3/2}f_{\alpha\beta\gamma},$$

where $F_{\alpha\beta\gamma}$ are the "flat" components of the initial tensor (2.2):

$$F_{\alpha\beta\gamma}=V_\alpha{}^{\hat M}V_\beta{}^{\hat N}V_\gamma{}^{\hat P}F_{\hat M\hat N\hat P}.$$

The scale factor $(\rho E)^{3/2}$ is extracted to ensure that the tensor with world indices of interest $f_{\mu\nu\sigma}$ contains no superfluous scale factors. In the end we obtain

$$f_{\mu\nu\sigma}=c_\mu{}^\alpha e_\nu{}^\beta e_\sigma{}^\gamma f_{\alpha\beta\gamma}=\Phi\equiv3\partial_{[\mu}A'_{\nu\sigma]}-3kB_{[\mu}{}^NB_{\nu\sigma]}{}^M\eta_{MN}$$
$$-3k\,\mathrm{Tr}\,(a_{[\mu}a_{\nu\sigma]}-{}^2/_3igka_{[\mu}a_\nu a_{\sigma]}), \tag{3.16}$$

where

$$A'_{\mu\nu}\equiv A_{\mu\nu}-2k\mathrm{Tr}(\phi_NB_{[\mu}{}^Na_{\nu]}), \tag{3.17}$$
$$B_{\mu\nu}{}^N=2\partial_{[\mu}B_{\nu]}{}^N, \tag{3.18}$$
$$a_{\mu\nu}=2\partial_{[\mu}a_{\nu]}+ig[a_\mu,a_\nu]. \tag{3.19}$$

As follows from (2.3), the tensor $f_{\mu\nu\sigma}$ enters into the Lagrangian in M_4 in the form

$$(e/12)\exp(-4kA)f_{\mu\nu\sigma}f^{\mu\nu\sigma}+ef_{\mu\nu\sigma}X^{\mu\nu\sigma}, \tag{3.20}$$

where $M^{\mu\nu\sigma}$ are fermionic terms. Adding to (3.20) the term with coupling:

$$-{}^1/_6B(x)\,\varepsilon^{\mu\nu\sigma\rho}\partial_\mu(f_{\nu\sigma\rho}-\phi), \tag{3.21}$$

where the field B plays the role of a Lagrange multiplier, one may view $f_{\mu\nu\sigma}$ as an independent variable. Here $\varepsilon^{\mu\nu\sigma\rho}$ is the completely antisymmetric tensor ($\varepsilon^{0123}=1$). Eliminating $f_{\mu\nu\sigma}$ with the help of the equations of motion we obtain

$$f_{\mu\nu\sigma}=-\exp\,(4kA)\,(e\varepsilon_{\mu\nu\sigma\rho}\partial^\rho B+6X_{\mu\nu\sigma}). \tag{3.22}$$

Upon substitution of this expression into the original Lagrangian with coupling we reproduce all terms containing the field B.

As a result we are left with the following independent boson fields in M_4: e_μ^α, B_μ^N, A, B (the boson part of the $N=4$ gravitational supermultiplet) and a_μ, ϕ_N (the boson part of the $N=4$ matter supermultiplet).

We consider now the reduction of the fermion fields. That aspect of the discussion is identical to that of Refs. 1 and 2 (and differs slightly from Ref. 3). We present only the final results. The spinor part of the $N=4$ gravitational supermultiplet in M_4 contains the Majorana spinor-vector gravitino ψ_μ and the Majorana spinor χ. The spinor part of the matter supermultiplet contains the Majorana spinor λ. All these spinors have in addition to the usual Dirac index in M_4 (which will not be written out explicitly) an internal index $J=1,\ldots,8$, with respect to which they transform like the spinor representation of the $O(6)$ group, that enters as a factor in $O(1.3)\otimes O(6)$. Since the initial 32-component spinors in M_{10} are subject to the $O(1.9)$-invariant condition (2.13), then the spinors that are in fact independent in M_4 turn out to be those which transform under the $O(4)$ group, and the corresponding internal index is $j=1,2,3,4$ (see below).

Upon neglecting the y dependence of the spinors, referred to M_{10}, we obtain the following relations:

$$X=-{}^1/_2(\rho E)^{1/4}\chi,$$
$$\Lambda=(\rho E)^{1/4}\lambda, \tag{3.23}$$
$$\Psi_\beta=(\rho E)^{1/4}\{\psi_\beta+\Gamma_B[(3i/4\sqrt{2})\chi-{}^1/_2\Gamma^A\lambda E^M{}_A\phi_M]\}.$$

Here $\Psi_\beta=V_\beta^{\hat M}\Psi_{\hat M}$ are the "flat" components of the gravitino in M_{10}, and $\psi_\beta=e_\beta^\mu\psi_\mu$ are the flat components of the gravitino in M_4. We emphasize that (3.23) agrees with the standard supertransformation of the frame in M_4:

$$\delta_\varepsilon e_\mu{}^\alpha = -ik\bar\varepsilon'\gamma^\alpha\psi_\mu, \qquad (3.24)$$

where ε' is the parameter of the supertransformations in M_4:

$$\varepsilon = (\rho E)^{-1/4}\varepsilon'. \qquad (3.25)$$

The relation (3.24) follows from (3.23) upon use of (2.14) if the components of the Lorentz rotation matrix are taken in the form

$$L^{\alpha\beta} = ik\bar\varepsilon\Gamma^{\alpha\beta}[-(3i/2\sqrt{2})X - {}^1/_2\Lambda E_A{}^N\phi_N],$$
$$L^{\alpha B} = ik\bar\varepsilon\Gamma^\alpha\Psi^{\prime B}. \qquad (3.26)$$

The value of $L^{\alpha B}$ in (3.26) agrees with the choice of the frame in the form (3.3). We note that so far the form of the elements L^{AB} has not been fixed. They are calculated from (2.14) for a concrete choice of the matrix E^A_M. One of the possible forms for E^A_M will be indicated below (in Sec. 4) after the introduction of appropriate notation.

In what follows we will no longer encounter the supersymmetry parameter ε from M_{10}. We shall only use the parameter ε' from M_4 and will drop the prime from its symbol.

We present for completeness the expressions for the spinors in M_4, using the $O(4)$ notation:

$$\lambda^J = \frac{1}{\sqrt{2}}\begin{pmatrix}\lambda^j \\ -i\gamma_5\lambda^j\end{pmatrix}, \text{ analogously for } \psi_\mu \text{ and } \varepsilon;$$
$$\chi^J = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi^j \\ i\gamma_5\chi^j\end{pmatrix}. \qquad (3.27)$$

Here $j = 1, \dots, 4$; $J = 1, \dots, 8$.

Everything is now ready to obtain from (2.3) and (2.14) the desired Lagrangian and the $N=4$ supersymmetry transformations in M_4. For simplicity we confine ourselves here to the boson part of the Lagrangian only.

4. THE BOSON LAGRANGIAN IN MINKOWSKI SPACE

For the calculation of the Lagrangian in M_4 it is convenient to use the Einstein term in (2.3) in the form

$$-(1/4k^2)VR = (V/16k^2)[(\Omega_{\hat A\hat B\hat C})^2 - 2\Omega^{\hat A\hat B\hat C}\Omega_{\hat C\hat A\hat B} - 4(\Omega_{\hat A\hat C}{}^{\hat C})^2], \qquad (4.1)$$

where the $\Omega_{...}$ are defined in (2.8). This familiar relation is accurate apart from a total derivative. In the parametrization used by us the nonzero components of the tensor $\Omega_{...}$ are equal to

$$\Omega_{\alpha BC} = -\Omega_{B\alpha C} = -e_\alpha{}^\mu(\rho E)^{1/2}(E_B{}^N\partial_\mu E_{NC} + \eta_{BC}\partial_\mu\log\rho^{1/6}),$$
$$\Omega_{\alpha\beta C} = -\varkappa(\rho E)\rho^{1/6}B^N_{\mu\nu}E_{NC}e^\mu_\alpha e^\nu_\beta, \qquad (4.2)$$
$$\Omega_{\alpha\beta\gamma} = (\rho E)^{1/2}[\Omega'_{\alpha\beta\gamma} - \eta_{\gamma[\alpha}\partial_{\beta]}\log(\rho E)],$$

where Ω' is the analogous to (2.8) tensor in M_4:

$$(\Omega')_{\mu\nu}{}^\alpha = -\partial_\mu e_\nu{}^\alpha + \partial_\nu e_\mu{}^\alpha.$$

By an equation similar to (4.1) Ω' defines the curvature scalar R' in M_4. (Since the R from M_{10} will not be encountered any more in the following, we shall use the standard notation R for the curvature scalar in M_4.) Now, following some algebraic transformations, we may write the desired boson Lagrangian in M_4:

$$L_B = L_e + L_s^{(g)} + L_s^{(m)} + L_v, \qquad (4.3)$$

where L_e is the standard Einstein term: $L_e = -(e/4k^2)R$, and $L_s^{(g)}$ contains the scalars from the gravitational multiplet:

$$e^{-1}L_s^{(g)} = {}^1/_2(\partial_\mu A)^2 + {}^1/_2\exp(4kA)(\partial_\mu B)^2. \qquad (4.4)$$

The $L_s^{(m)}$ term contains the matter scalars:

$$e^{-1}L_s^{(m)} = (1/4k^2)\{-\mathrm{Tr}(\nabla_\mu\phi_M)^2 + [\mathrm{Tr}(\phi_M\nabla_\mu\phi_N)]^2\} - U_s. \qquad (4.5)$$

The scalar potential equals

$$U_s = (g^2/4k^4) \times \exp(2kA)\{-{}^1/_4\,\mathrm{Tr}[\phi_M,\phi_N]^2 + {}^2/_3[\mathrm{Tr}(\phi_{[M}\phi_N\phi_{P]})]^2\}. \qquad (4.6)$$

In these equations (as well as everywhere in what follows) the squares of all tensor quantities should be calculated with the appropriate metric tensor, for example $\phi^2_M = \phi_M g^{MN}\phi_N$ etc., where $g^{MN} = (g^{-1})_{MN}$, the tensor g_{MN} is defined in (3.4), (3.12). The derivatives in (4.5) are defined by the relation

$$\nabla_\mu\phi_N = \partial_\mu\phi_N + ig[a_\mu, \phi_N]. \qquad (4.7)$$

We remind the reader that ϕ_N and a_μ lie in the algebra G.

It is convenient to introduce a new notation in order to write in a compact way the term L_v, which contains the contribution of the vector fields. We shall view ϕ_M as a matrix $\phi_{M\alpha}$, where α is the index of the selfconjugate representation of the group G. Then ϕ^T will be the matrix with elements $\phi_{\alpha M}(=\phi_{M\alpha})$. The quantity $B^N_{\mu\nu}$ is to be viewed now as an element of the column matrix $B_{\mu\nu}$, and the quantity $(a_{\mu\nu})_\alpha$ as an element of the row matrix $a_{\mu\nu}$. With the help of this notation the matrices g_{MN} and g^{MN} are written in the form

$$g = -1 + \phi\phi^T,$$
$$g^{-1} = -1 - \phi\frac{1}{1-\phi^T\phi}\phi^T. \qquad (4.8)$$

In this notation the term $L_s^{(m)}$ has the form

$$e^{-1}L_s^{(m)} = \frac{1}{4k^2}\mathrm{Tr}\Big[\frac{1}{1-\phi^T\phi}(\nabla_\mu\phi)^T(\nabla^\mu\phi) + \frac{1}{1-\phi^T\phi}(\nabla_\mu\phi)^T\phi \times\frac{1}{1-\phi^T\phi}\phi^T(\nabla_\mu\phi)\Big] - U_s. \qquad (4.9)$$

The term L_v has the form

$$e^{-1}L_v = -{}^1/_4\exp(-2kA)\Big(B_{\mu\nu}B^{\mu\nu} + a_{\mu\nu}\frac{1+\phi^T\phi}{1-\phi^T\phi}a^{\mu\nu} + 2B_{\mu\nu}\phi\frac{1}{1-\phi^T\phi}\phi^TB^{\mu\nu} + 4B_{\mu\nu}\phi\frac{1}{1-\phi^T\phi}a^{\mu\nu}\Big) + {}^1/_2kB(\tilde a_{\mu\nu}a^{\mu\nu} - \tilde B_{\mu\nu}B^{\mu\nu}). \qquad (4.10)$$

In this equation

$$\tilde B_{\mu\nu} = {}^1/_2e\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}, \quad \tilde a_{\mu\nu} = {}^1/_2e\varepsilon_{\mu\nu\rho\sigma}a^{\rho\sigma}. \qquad (4.11)$$

We present also another useful representation of L_v, employing to this end the complex variables

$$S = \exp(-2kA) - 2ikB,$$
$$a_{\mu\nu}{}^+ = {}^1/_2(a_{\mu\nu} + i\tilde a_{\mu\nu}), \qquad (4.12)$$
$$B_{\mu\nu}{}^+ = {}^1/_2(B_{\mu\nu} + i\tilde B_{\mu\nu}).$$

In terms of these variables L_v takes the form

$$e^{-1}L_v=\frac{1}{4}S(a^{+}{}_{\mu\nu}{}^2-B^{+}{}_{\mu\nu}{}^2)$$

$$-\frac{1}{4}(S+S^{+})(a^{+}{}_{\mu\nu}+B^{+}{}_{\mu\nu}\phi)\frac{1}{1-\phi^T\phi}(a^{+\mu\nu}+\phi^TB^{+\mu\nu})$$

$$+\text{э. c.}=-\frac{1}{4}(C_B^{+\mu\nu}B^{+}{}_{\mu\nu}+G_a^{+\mu\nu}a^{+}{}_{\mu\nu})+\text{H.a.}, \qquad (4.13)$$

where

$$G_B{}^{+}=-(2/e)\,\delta L_v/\delta B^{+}, \quad G_a{}^{+}=-(2/e)\,\delta L_v/\delta a^{+}.$$

In this way L_v in (4.13) is represented in the standard form[16] with very broad dynamical symmetry in the equations of motion.

Lastly, to conclude this section, we present in the chosen notation one of the possible forms for the frame E_M^A:

$$E_{MA}=\delta_A{}^NW_{NM}, \quad E^{MA}=\delta_N{}^AW^{NM}, \qquad (4.14)$$

where $W^{MN}=(W^{-1})_{MN}$ and the matrix W may be chosen in the form

$$W=-1+\phi\frac{1}{1+H^{1/2}}\phi^T, \quad W^{-1}=-1-\phi H^{-1/2}\frac{1}{1+H^{1/2}}\phi^T,$$

where $H=1-\phi^T\phi$.

5. SUPERSYMMETRY TRANSFORMATIONS

To construct the supersymmetry transformations in M_4 use should be made of the relations (2.14), having taken into account the connection between the fields in M_{10} and M_4 established in Sec. 3. The result looks rather unwieldy. To simplify it as much as possible it is convenient to introduce new variables. For the components of the antisymmetric second rank tensor—

$$\begin{aligned} &f_{\mu\nu}=a_{\mu\nu}+B_{\mu\nu}{}^N\phi_N,\\ &\varkappa f_{\mu N}=\partial_\mu\phi_N+ig[a_\mu,\phi_N],\\ &\varkappa^2 f_{MN}=ig[\phi_M,\phi_N]. \end{aligned} \qquad (5.1)$$

For the components of the antisymmetric third rank tensor—

$$\begin{aligned} &\sqrt{2}f_{\mu\nu N}=g_{MN}B_{\mu\nu}{}^M-2\,\mathrm{Tr}(\phi_Nf_{\mu\nu}),\\ &\varkappa\sqrt{2}f_{\mu MN}=\mathrm{Tr}(\phi_M\nabla_\mu\phi_N-\phi_N\nabla_\mu\phi_M),\\ &\varkappa^2\sqrt{2}f_{MNQ}=-4ig\,\mathrm{Tr}(\phi_{[M}\phi_N\phi_{Q]})=-2ig\,\mathrm{Tr}(\phi_M,\phi_Q]). \end{aligned} \qquad (5.2)$$

The here introduced tensors $f_{...}$ are related as follows to the flat (with respect to the M_4 indices) components of the original tensors $F_{\hat{M}\hat{N}}$ and $F_{\hat{M}\hat{N}\hat{Q}}$:

$$F_{\alpha\beta}=(\rho E)e^\mu{}_\alpha e^\nu{}_\beta f_{\mu\nu}, \quad F_{\alpha N}=(\rho E)^{1/2}e^\mu{}_\alpha f_{\mu N}, \quad F_{MN}=f_{MN}. \qquad (5.3)$$

Similarly for the third rank tensor [compare also Eq. (3.15)]:

$$\begin{aligned} &F_{\alpha\beta N}=(\rho E)e^\mu{}_\alpha e^\nu{}_\beta f_{\mu\nu N},\\ &F_{\alpha MN}=(\rho E)^{1/2}e^\mu{}_\alpha f_{\mu MN}, \quad F_{NMQ}=f_{MNQ}. \end{aligned} \qquad (5.4)$$

We are now in a position to write down the supersymmetry transformations in application to the components of the gravitational and matter $N=4$ supermultiplets in M_4. For the gravitational supermultiplet

$$\delta_\varepsilon A=\bar\varepsilon\Big[\frac{1}{\sqrt2}\chi+\frac{i}{2}\mathrm{Tr}(\hat\phi\lambda)\Big],$$

$$\delta_\varepsilon B=\frac{1}{\sqrt2}\exp(-2kA)\bar\varepsilon\gamma_5\Big[i\chi-\frac{1}{\sqrt2}\mathrm{Tr}(\hat\phi\lambda)\Big],$$

$$\delta_\varepsilon e_\mu{}^\alpha=-ik\bar\varepsilon\gamma_\alpha\psi_\mu,$$

$$\delta_\varepsilon B_\mu{}^N=-\frac{i}{\sqrt2}\exp(kA)E^N{}_B\bar\varepsilon\Big\{\Gamma^B\psi_\mu-\frac{i}{\sqrt2}\gamma_\mu\Gamma^B\chi+\gamma_\mu\mathrm{Tr}\Big[\Big(\phi^B+\frac12\Gamma^B\hat\phi\Big)\lambda\Big]\Big\},$$

$$\begin{aligned}\delta_\varepsilon\psi_\mu={}&\frac1k D_\mu{}'\varepsilon-\frac{i}{2}\exp(2kA)\gamma_5\varepsilon\partial_\mu B+\frac14\Gamma^{AB}\varepsilon f_{\mu MN}E^M{}_AE^N{}_B\\ &-\frac{1}{4\sqrt2}\exp(-kA)\gamma^{\nu\sigma}\gamma_\mu\Gamma^B\varepsilon E^M{}_B[g_{MN}B_{\nu\sigma}{}^N-\mathrm{Tr}(f_{\nu\sigma}\phi_M)]\\ &+\frac{1}{24}\exp(kA)\gamma_\mu\Gamma^{ABC}\varepsilon f_{MNP}E^M{}_AE^N{}_BE^P{}_C,\end{aligned}$$

$$\begin{aligned}\delta_\varepsilon\chi={}&-\frac{i}{\sqrt2}\gamma_\mu\varepsilon\partial_\mu A+\frac{1}{\sqrt2}\exp(2kA)\gamma^\mu\gamma_5\varepsilon\partial_\mu B\\ &-\frac{i}{2\sqrt2 k}\gamma_\mu\varepsilon\partial_\mu\log E\\ &-\frac{i}{6\sqrt2}[\exp(kA)E^M{}_AE^N{}_BE^P{}_Cf_{MNP}\Gamma^{ABC}\varepsilon+3E^N{}_BE^P{}_Cf_{\mu NP}\gamma^\mu\Gamma^{BC}\varepsilon\\ &+3\exp(-kA)E^P{}_Cf_{\mu\nu P}\gamma^{\mu\nu}\Gamma^C\varepsilon].\end{aligned} \qquad (5.5)$$

For the matter supermultiplet

$$\delta_\varepsilon\phi_M=-ik\bar\varepsilon\Gamma_B\lambda E_M{}^B,$$

$$\delta_\varepsilon a_\mu=\frac{i}{\sqrt2}\exp(kA)\bar\varepsilon\Big\{-\gamma_\mu\lambda+\hat\phi\psi_\mu-\frac{i}{\sqrt2}\gamma_\mu\hat\phi\chi+\phi_B\gamma_\mu\mathrm{Tr}\Big[\Big(\phi^B+\frac12\Gamma^B\hat\phi\Big)\lambda\Big]\Big\},$$

$$\begin{aligned}\delta_\varepsilon\lambda={}&-\frac{1}{2\sqrt2}[\exp(kA)E^M{}_AE^N{}_Bf_{MN}\Gamma^{AB}\varepsilon+2E^N{}_Bf_{\mu N}\gamma^\mu\Gamma^B\varepsilon\\ &+\exp(-kA)f_{\mu\nu}\gamma^{\mu\nu}\varepsilon].\end{aligned} \qquad (5.6)$$

Here we introduced the following notation:

$$\begin{aligned}&\phi_B=E^M{}_B\phi_M, \quad \hat\phi=\phi_B\Gamma^B,\\ &D_\mu{}'\varepsilon=D_\mu\varepsilon+{}^1/_4(E^N{}_A\partial_\mu E_{NB})\Gamma^{AB}\varepsilon.\end{aligned} \qquad (5.7)$$

The supersymmetry algebra is closed in the following sense:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]=i\delta_\xi+\delta_\Omega+\delta_\omega+\delta_L, \qquad (5.8)$$

where δ_ξ is a general-coordinate transformation, δ_Ω is a gauge transformation from the group G (where a_μ are the corresponding gauge fields), δ_ω is a gauge transformation from the group $U(1)^6$ (B_μ^N are the corresponding gauge fields), and δ_L are Lorentz rotations from the group $O(1.3)$ (see Secs. 2 and 3). The parameters of these transformations are as follows:

$$\xi^\mu=\xi_{21}{}^\mu, \quad \Omega=-\xi_{21}{}^\mu a_\mu-\frac{i}{\varkappa}\exp(kA)\phi_B\xi_{21}{}^B,$$

$$\omega^N=-i\xi_{21}{}^\mu B_\mu{}^N+\frac{i}{\varkappa}\exp(kA)E^N{}_B\xi_{21}{}^B,$$

$$L^{\alpha\beta}=-\frac{i\varkappa}{4}\exp(-kA)E^M{}_B\bar\varepsilon_2\times(\gamma^{\alpha\beta\nu\sigma}+2\eta^{\nu[\alpha}\eta^{\beta]\sigma})\Gamma^B\varepsilon_1[g_{MN}B_{\nu\sigma}{}^N-\mathrm{Tr}(\phi_Mf_{\nu\sigma})], \qquad (5.9)$$

where $\xi_{21}^{\mu} = \bar{\varepsilon}_2 \gamma^{\mu} \varepsilon_1$, $\xi_{21}^{B} = \bar{\varepsilon}_2 \Gamma^B \varepsilon_1$. We note that it is difficult to obtain the variation δ_ε directly from (2.14). Its form is reconstructed from the supersymmetry algebra.

This concludes the construction of the $N=4$ supersymmetric theory describing the interaction of gravity with a matter multiplet. (There are precisely four supersymmetry transformations, to which correspond as parameters the four Majorana spinors ε^j, $j=1, \ldots, 4$.) In addition to the usual symmetry of gravity and matter in M_4 [general covariance with respect to world indices, local $O(1.3)$ group with respect to tangent and spinor indices in M_4, non-Abelian gauge internal symmetry G], the theory has a global $O(6)$ [or $SU(4)$] symmetry. Its origin is as follows.

The $O(6)$ symmetry in the indices M, N, ... arose as a remnant of the original $SL(6.R)$ (see above). In addition, the $O(6)$ symmetry in the indices A, B, ... and the spinor indices of the internal space Q_6 is contained as a factor in the original $O(1.3) \otimes O(6)$. The choice of the frame $E_M{}^A$ in the form ensuring the correct limit δ_M^A as $k \to O$ preserves as symmetry only the diagonal (global) subgroup $O(6)$ in the direct product of the two indicated $O(6)$ groups. Upon vanishing of all matter fields the theory goes over into the $SU(4)$-invariant version of $N=4$ supergravity constructed in Ref. 17. Upon vanishing of gravity (i.e. for $k \to 0$) the theory goes over into the standard $SU(4)$-invariant theory of matter with $N=4$ supersymmetry (see, for example, Refs. 15 and 18).

Next we discuss the possibility of localizing the $O(4)$ subgroup of the $O(6)$ symmetry group, which leads to a theory where in the limiting case of absence of matter we have the localized [under the $O(4)$ group] version of the $N=4$ supergravity constructed (from other considerations) in Ref. 13.

6. A (SPONTANEOUSLY BROKEN) GAUGE VERSION OF THE *N*=4 SUPERGRAVITY INTERACTING WITH MATTER

The construction of such a version is carried out by making use of a procedure (within the framework of a generalization of the reduction scheme discussed in Sec. 3) discussed in Ref. 10 and corresponding to taking into account of zero modes only in compactification to the manifold Q_6, which is in the case under discussion the manifold of the $O(4)$ group. (We differ from Ref. 10 only in the interpretation of the components F_{MNP} of the antisymmetric tensor.)

The construction of Ref. 10 as applied to our problem reduces to the following. In the reduction process all tensor fields (with respect to the indices M, N, ... from Q_6) depend on y. But in all cases (with the exception of the A_{MN} field, see below) this dependence is determined by the D-functions of some symmetry group S, acting on the world indices of the Q_6 space, i.e.

$$E_M{}^A \to E_N{}^A U^N{}_M(y), \qquad B_\mu{}^M \to (U^{-1}(y))^M{}_N B_\mu{}^N \tag{6.1}$$

etc. Moreover, the significant part of this y-dependence will cancel out in the Lagrangian in M_4 due to the pairing up of indices. But a special treatment is needed for terms with derivatives in the original Lagrangian in M_{10}. Since the originai Lagrangian contains only antisymmetric structures of the type $\partial_{\hat{M}} X_{\hat{N}} - \partial_{\hat{N}} X_{\hat{M}}$, where X is some field, the y-dependence may appear only in the form of the combination

$$(U^{-1})^M{}_P (U^{-1})^L{}_Q (\partial_L U^N{}_M - \partial_M U^N{}_L) = \varkappa^{-1} C^N{}_{PQ}. \tag{6.2}$$

Equation (6.2) is the definition of the tensor C^N_{PQ}. It is important that the combination (6.2) may be viewed as y-independent. Here the operators

$$L_Q = \varkappa (U^{-1})^N{}_Q \partial_N$$

are the generators of the algebra:

$$[L_M, L_N] = C^Q{}_{MN} L_Q, \tag{6.3}$$

and, correspondingly, the C^Q_{MN} are the structure constants of a certain subgroup S' of the group S.

We confine ourselves here to the consideration of compact semisimple groups and therefore assume that the Killing tensor $C^Q_{MP} C^P_{NQ}$ is proportional to $\eta_{MN} = -\delta_{MN}$ and, what is particularly important for the following, the tensor

$$C_{MNP} = \eta_{PL} C^L{}_{MN} \tag{6.4}$$

is totally antisymmetric. We emphasize that the following discussion is, generally speaking, valid also for all noncompact groups resulting from a restriction to a real form of the complex $O(4)$ algebra. This is because the tensor C_{MNP}, defined according to a relation of the type (6.4) but with a new Killing tensor $\bar{\eta}_{MN}$, is antisymmetric as before. We have, in particular, for the Lorentz group $O(1.3)$ that $\bar{\eta}_{MN} = \mathrm{diag}(-1, -1, -1, 1, 1, 1)$.

However in the case of noncompact groups there is no $O(6)$-symmetric limit corresponding to the absence of matter. In particular, it is precisely this desire that such a limit should exist that has kept us so far from considering the more general $O(4) = SU(2) \otimes SU(2)$-invariant tensor $\eta_{MN'} = \mathrm{diag}(-\lambda_1, -\lambda_1, -\lambda_1, -\lambda_2, -\lambda_2, -\lambda_2)$. The choice of the tensor $\eta_{MN'}$ in place of η_{MN} in (3.12) is allowed, but it leads to different versions of supergravity in M_4. Due to space limitations we do not study this possibility in this article.

Now with (6.1) and (6.2) taken into account the condition (3.8) is no longer selfconsistent. In particular it contradicts the equation of motion

$$R_{MN} = \varkappa^2 \eta_{MA} \delta_N{}^B F^{ACD} F_{BCD} + \ldots,$$

since now in the absence of all other fields we have $R_{MN} = (1/4\varkappa^2) . U^P_M U^Q_N \eta_{PQ} \neq 0$. But in our arrangement in M_4 there are no additional degrees of freedom connected with A_{MN}. The only possibility for resolving this contradiction is to set the field A_{MN} to depend on y only, and in such a way that the contribution from A_{MN} to the field tensor be a constant (accurate up to U^M_N factors). The only invariant constant third rank tensor is C_{MNP}. We therefore set

$$A_{MN} = A_{MN}(y) \neq 0, \quad F_{MNP} = (a/\varkappa^2) U^Q{}_M U^R{}_N U^S{}_P C_{QRS} + \ldots, \tag{6.5}$$

where a is some constant [the terms in F_{MNP} containing matter fields enter as before and are not shown explicitly in (6.5)]. The constant a is found in a selfconsistent manner from a variety of considerations. We briefly illustrate one of them.

The tensors $f_{\ldots}$, introduced by us in (5.1) and (5.2), are now proportional to U^M_N factors. It is therefore convenient to introduce new quantities:

$$f'_{\mu N} = (U^{-1})^R{}_N f_{\mu R}, \quad f'_{MN} = (U^{-1})^Q{}_M (U^{-1})^R{}_N f_{QR} \ \text{etc.} \tag{6.6}$$

(similar relations hold for other tensors $f_{...}$ and $f'_{...}$ with world indices in the Q_6 space). Namely the tensors $f'_{...}$ enter now the Lagrangian and the supersymmetry transformations. Their calculation gives

$$\varkappa f'_{\mu N}=\nabla_\mu(a,B)\phi_N,$$
$$\varkappa^2 f'_{MN}=(-\phi_Q C^Q{}_{MN}+ig[\phi_M,\phi_N]), \qquad (6.7)$$

and analogously

$$\sqrt{2}f'_{\mu\nu P}=g_{PM}B_{\mu\nu}{}^M-2\phi_P f_{\mu\nu},$$
$$\sqrt{2}\varkappa f'_{\mu NP}=\phi_N\nabla_\mu(a,B)\phi_P-\phi_P\nabla_\mu(a,B)\phi_N$$
$$+\sqrt{2}C^Q{}_{NP}[A_{\mu Q}+(1/\sqrt{2})\mathrm{Tr}(a_\mu\phi_Q)-a\eta_{MQ}B_\mu{}^M], \qquad (6.8)$$
$$\sqrt{2}\varkappa^2 f'_{MNP}=a\sqrt{2}C_{MNP}+\mathrm{Tr}(3\phi_{[M}C^Q{}_{NP]}\phi_Q-4ig\phi_{[M}\phi_N\phi_{P]}).$$

In these equations

$$\nabla_\mu(a,B)\phi_N=\partial_\mu\phi_N+ig[a_\mu,\phi_N]+\phi_Q C^Q{}_{MN}B_\mu{}^M,$$
$$B_{\mu\nu}{}^M=\partial_\mu B_\nu{}^M-\partial_\nu B_\mu{}^M-C^M{}_{PQ}B_\mu{}^P B_\nu{}^Q. \qquad (6.9)$$

The requirement of gauge invariance of $f'_{\mu NP}$ leads to the coupling

$$A_{\mu M}+(1/\sqrt{2})\mathrm{Tr}(a_\mu\phi_M)-a\eta_{MN}B_\mu{}^N=0. \qquad (6.10)$$

Comparison with (3.10) gives

$$a=1/\sqrt{2}. \qquad (6.11)$$

We emphasize that in the case under consideration, under the condition $\delta_\varepsilon A_{MN}=0$, Eqs. (3.9) and (3.10) exist as before and are consistent with supersymmetry.

Thus we find that the entire distinction from the previous case consists in the appearance of structures, covariant under the action of the non-Abelian gauge group S' on the indices $M, N, \ldots$ [see (6.9)], and also in the appearance of certain additional terms, containing the tensor C_{MNQ} [see (6.8)]. In particular, in Eq. (4.2) in place of ordinary derivatives there appear now covariant derivatives

$$\nabla_\mu(B)E_N{}^A=\partial_\mu E_N{}^A+E_Q{}^A C^Q{}_{MN}B_\mu{}^M,$$
$$\nabla_\mu(B)E^N{}_A=\partial_\mu E^N{}_A-C^N{}_{PQ}B_\mu{}^P E^Q{}_A. \qquad (6.12)$$

But, further, there appear additional different from zero components:

$$\Omega_{ABC}=\varkappa^{-1}\rho^{-1/6}C^Q{}_{MN}E^M{}_A E^N{}_B E_{QC}, \qquad (6.13)$$

which will give rise to the appearance of new components of the spin connection.

The final result for the Lagrangian and the supersymmetry transformations consists of the following. In essence all the old equations are preserved but with ordinary derivatives of tensorial quantities (with respect to the indices M, $N, \ldots$) replaced by covariant derivatives [see (6.12) and (6.9)], with the Abelian tensor $B^N_{\mu\nu}$ replaced by the non-Abelian tensor from (6.9), and with all tensors $f_{...}$ replaced by the corresponding tensors $f'_{...}$ from (6.7) and (6.8). Further, in the variation $\delta_\varepsilon\psi_\mu$ in (5.5) one should make the replacement $f_{MNP}\to f'_{MNP}-(3/\sqrt{2}\varkappa^2)g_{Q[M}C^Q_{NP]}$. Moreover, due to the appearance of the new components (6.13) there will arise from the curvature scalar in M_{10} additional, as compared to (4.6), terms in the scalar potential in M_4 (see Ref. 10). As a result U_s takes the form

$$U_S=\varkappa^{-4}$$
$$\times\exp(2kA)(-{}^1/_8 C^L{}_{MN}(C^{L'}{}_{M'N'}g^{MM'}g^{NN'}g_{LL'}+2C^M{}_{LN'}g^{NN'})$$
$$+{}^1/_4\mathrm{Tr}(\varkappa^2 f'_{MN})^2-{}^1/_{24}(\sqrt{2}\varkappa^2 f'_{MNP})^2, \qquad (6.14)$$

where f'_{MN} and f'_{MNP} are defined in (6.7) and (6.8) with (6.11) taken into account. Also, as before, all squares of tensorial quantities should be evaluated with the help of the tensor g^{MN}:$\phi^2_M=\phi_N\phi_M g^{MN}$ etc.

We note that only the compact group $O(4)$ (or its subgroups) may be chosen for the non-Abelian group S' acting on the indices $M, N, \ldots$ (in the approach under consideration). The choice of a group of order six is conditioned by the presence of the six gauge fields B^N_μ. The requirement of compactness arises from the condition of positive-definiteness for the kinetic energy form of the B^N_μ fields. Thus we have in (4.10) in the case of the $O(4)$ group

$$B_{\mu\nu}B^{\mu\nu}=-\eta_{MN}B_{\mu\nu}{}^M B^{\mu\nu,N}.$$

For the noncompact extensions of $O(4)$ [i.e. for the $O(1.3)$, $O(2.2)$ and $O(4)^*$ groups] the replacement $\eta_{MN}\to\bar\eta_{MN}$ is needed, where $\bar\eta_{MN}$ is the corresponding sign-indefinite Killing tensor. This results in an inadmissible form for the kinetic energy. One could close one's eyes on this difficulty by admitting an indefinite metric in the gravitational sector, where the dynamics cannot be studied in any case. However the simplest ansatzes, corresponding to spontaneous breaking of supersymmetry (see below), result in this indefinite metric "penetrating" into the matter sector, which is inadmissible.

The detailed study of the U_s in (4.6) and (6.14) is a separate problem, which we have not yet considered in sufficient detail. However certain facts can be readily established. These are discussed in the next section.

7. THE POSSIBILITY OF SPONTANEOUS SUPERSYMMETRY BREAKING FOR A VANISHING COSMOLOGICAL CONSTANT

We consider first the version to which corresponds the scalar potential (4.6). It has "valleys" in the space of scalar fields in the directions where $[\phi_M,\phi_N]=0$. {We note that $2\mathrm{Tr}(\phi_{[M}\phi_N\phi_{P]})=\mathrm{Tr}(\phi_M[\phi_N,\phi_p])$.} We have no idea how the theory is stabilized in the direction of these valleys (this is a common problem for theories with extended supersymmetry; it is interesting that gravity does not remove it in the version under consideration).

The vacuum state for $\phi_M=0$ corresponds to a vanishing cosmological term and $N=4$ supersymmetry. The natural ansatz for spontaneous breaking, corresponding to the embedding of the group S' in G, consists of the choice $\phi^\alpha_M=v\delta^\alpha_M$, where v is some constant and α is the index of the selfconjugate representation of the group G. Evaluating U_s for this ansatz we obtain

$$U_S=\frac{g^2}{12\varkappa^4}\exp(2kA)(f_{\alpha\beta}{}^\gamma)^2\left[-\frac{v^4(3-v^2)}{(1-v^2)^3}\right], \qquad (7.1)$$

where $f^\gamma_{\alpha\beta}$ are the structure constants corresponding to the subgroup S' in G; contraction over the indices in evaluating the square of the tensor $f^\gamma_{\alpha\beta}$ is achieved with the help of the corresponding Killing tensor. For compact groups $(f^\gamma_{\alpha\beta})^2\neq0$ always. It is obvious that (for the simplest ansatz) spontaneous breaking of supersymmetry is impossible for a vanishing cosmological term, if noncompact groups are not considered. Further, one may obtain directly from an analysis of the relations (6.5) and (6.6) that for the ansatz being considered for the compact group S' [i.e. for $O(6)$ or its subgroups] it is not possible to achieve partial breaking of supersymmetry, in particular the case when $N = 4$ supersymmetry is broken down to $N = 1$ supersymmetry is not realized.

We move now to the more complicated case corresponding to the scalar potential (6.14). According to Ref. 19, the problems of spontaneous breaking of supersymmetry and of the existence of a cosmological term are solved simply on the basis of considerations of the variations δ_ε of the spinor fields (i.e. it is not necessary to turn to U_s). In particular, if we have for some ε_j of some vacuum configuration of scalar fields

$$\langle\langle \delta_\varepsilon \psi \rangle\rangle = 0, \quad \langle\langle \delta_\varepsilon \chi \rangle\rangle = 0, \quad \langle\langle \delta_\varepsilon \lambda \rangle\rangle = 0, \tag{7.2}$$

where the symbol $\langle\langle ... \rangle\rangle$ denotes the vacuum expectation value, then the corresponding supersymmetry is not broken and in that same configuration there exists a stationary point for U_s and a vanishing cosmological term (the converse is also true). One may verify, using the relations (5.5) and (5.6) (but with the replacement $f \to f'_{...}$ according to the prescription of Sec. 6), that for the group $S' = O(4)$ in the state where $\langle\langle \phi \rangle\rangle = 0$ the conditions (7.2) are not realized for any ε^j. In this way the $N = 4$ supersymmetry is spontaneously broken down to $N = 0$ (i.e. broken totally).

If one enters the state under consideration along the trajectory $\phi^\alpha_N = v\delta^\alpha_N$, then one can verify that $v = 0$ corresponds to an inflection point of the potential and a negative cosmological term (for an appropriate normalization of C^Q_{MN} it coincides with the cosmological term of Ref. 13). Unfortunately so far we have been unable to find for the group $S' = O(4)$ other, more interesting, solutions.

As was noted, the solutions corresponding to spontaneous partial breaking of $N = 4$ supersymmetry in the formulation of Ref. 6 and found in Refs. 8 and 9 require noncompact internal symmetry groups. The question of the physical admissibility of such solutions requires in any case further study.

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