Nonlinear response of cylindrical superconductor in intermediate state to current change

A. I. Bezuglyĭ and V. A. Shklovskiĭ

Kharkov Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR
(Submitted 6 July 1989)
Zh. Eksp. Teor. Fiz. **96**, 1906–1918 (November 1989)

The problem of the nonlinear response of a cylindrical superconductor to a change in the total current flowing through it is solved analytically on the basis of a macroscopic description of the electrodynamics of the intermediate (I) state (the Andreev equations). The total current varies in time in a specified (generally arbitrary) way, remaining above the critical current. A nonlinear integral equation is derived for the time dependence of the position of the N/I boundary. It is the motion of this boundary which is responsible for the nonlinearity of the response. Two types of electromagnetic responses of the sample are studied: the response to an abrupt change in the transport current and the response to small sinusoidal oscillations in this current. For the latter case, the frequency dependence and current dependence of two specifically nonlinear effects amenable to experimental test are derived and analyzed. These measurable quantities are (1) the amplitude of the second harmonic of the response and (2) the renormalization of the constant component of the voltage across the sample.

1. INTRODUCTION

When the superconductivity of a cylindrical sample is destroyed by a direct current $J > J_c$, where J_c is the critical current, in the absence of an external magnetic field, the sample goes into a resistive state of such a nature that the region of the intermediate (I) state, of radius a, is covered by a normal-metal layer whose thickness d increases with increasing reduced current $i \equiv J/J_c$ (Fig. 1). If the current through the sample begins to vary with the time in a specified way [e.g., $i(\tau) = i_0 + i_\sim$, where $i_\sim$ is an alternating increment of frequency Ω], while remaining above the critical current at all times [$i(\tau) > 1$], and if the intermediate state is able to change in structure over times $\tau_i \ll \Omega^{-1}$, then one might ask how the well-known "normal" skin effect would be altered in this situation by the presence of a core of radius a in the intermediate state. Qualitative differences are possible by virtue of two new circumstances: the nonlinearity of the problem and the specific features of the response of the intermediate state to a change in current.

The nonlinearity of the response [as a measure of which we can adopt the potential difference across the sample, as in the case of a conductor in its normal (N) state; more on this below] results from the presence of a movable N/I boundary. The amplitude of the displacements of this boundary turns out to be related to the magnitude of the current change in a nonlinear and nonlocal way.

Let us assume, however, that $i_\sim$ is of such a nature that the response can be treated as linear (see below for some estimates). It is then obvious from physical considerations that if the depth of the skin layer for $i_\sim$ in the N metal satisfies $\delta_N(\Omega) < d$ then the presence of the intermediate state will have essentially no effect on the impedance of the sample. If $\delta_N(\Omega) > d$, on the other hand, then the ordinary skin effect changes in such a way that the $i_\sim$ dissipation averaged over the period of the current oscillation (in contrast with the i_0 dissipation) is zero, as we showed in Ref. 2. As a result, in the case of a linear response the dynamic resistance of the sample will be the same as that of a hollow cylinder of an N metal with a wall thickness d and an outside diameter equal to the diameter of the sample. In particular, the onset of a plateau along the i scale which is characteristic of the dynamic resistance $R_d(i,\Omega)$ occurs when the relation $\delta_N(\Omega) \sim d$ holds. It was suggested in this connection in Ref. 2 that the frequency dispersion $R_d(i)$ be measured for a determination of d which would be independent of the structure of the intermediate state.

The description of the intermediate state which we used in the present paper is based on an analysis of the time-dependent equations for the macroscopic dynamics of the intermediate state which were proposed by Andreev[1)] (Ref. 3). Our paper is organized in the following way. In Sec. 2 we discuss the original system of equations in the I and N phases and the boundary conditions on these equations. In Sec. 3 we discuss the distinctive electrodynamics of the I state for displacements of the N/I boundary which are not small. In Sec. 4 we derive a general analytic solution for the problem of the nonlinear response of a superconducting cylinder in the intermediate state to a current change, in terms of the solutions of a nonlinear integral equation (found in the same section of the paper) for the displacements of the N/I boundary. In Sec. 5 we discuss the linear response of an N/I boundary to small oscillations or to a jump in the transport current. In the last Sec. 6 we use second-order perturbation theory to study nonlinear electromagnetic responses which are amenable to experimental observation.

2. STATEMENT OF THE PROBLEM

A constant current $i_0 > 1$ flows up to the time $\tau = 0$ through a cylindrical sample of radius R_0 with a normal-state conductivity σ. The radius of the intermediate state which corresponds to this current is a_0. At $\tau > 0$, the total current through the sample then varies in accordance with the specified law $i(\tau) > 1$, with the result that the radius of the intermediate state (i.e., the position of the N/I boundary) varies in accordance with some law $a(\tau)$ which we do not know at this point.

We are to find that electric field at the surface of the sample, $e(\tau)$, in terms of which we can express the measurable alternating voltage across the sample, $\mathscr{E}(\tau)$, by means of the formula[1]

 0038-5646/89/111076-07$04.00

$$\mathscr{E}=\frac{1}{c^2}\mathscr{L}_e\frac{di}{d\tau}+e(\tau)L, \tag{1}$$

where $\mathscr{L}_e$ is the so-called external inductance of the conductor[1] (an inductance which does not depend on the state of the conductor), L is the length of the conductor, and c is the velocity of light.

In our time-dependent problem, only the axial components of the electric field e and of the current density j, and also the azimuthal projections of the magnetic field h and the induction b, are nonzero (and depend on only the distance ρ to the axis of the sample and the time τ), by virtue of the geometry of the problem and our use of a macroscopic description of the intermediate state (i.e., a description averaged over the N–S layers).

At this point it is convenient to switch to dimensionless quantities, by expressing all distances in units of R_0, all times in units of $T=4\pi\sigma R_0^2/s^2$, the magnetic field in units of the critical field H_c, and the electric field in units of $E_c=cH_c/2\pi\sigma R_0$. [In this case $\mathscr{L}_e$ in (1) is expressed in units of $4R_0$.]

It can then be shown that in the I phase ($0<\rho<a$) the system of Andreev equations for the intermediate state [see Eq. (15) in Ref. 3] takes the form (x_n is the concentration of the N phase)

$$\frac{\partial e}{\partial\rho}=\frac{1}{2}\frac{\partial x_n}{\partial\tau},\quad e=\frac{x_n}{2\rho},\quad h=1,\quad b=x_n,\quad j=1/2\rho. \tag{2}$$

It is easy to see that the problem of solving system (2) reduces to that of solving a single linear equation for the field:

$$\frac{\partial e}{\partial\tau}-\frac{1}{\rho}\frac{\partial e}{\partial\rho}=0. \tag{3}$$

In the N phase ($a<\rho<1$) the quasisteady Maxwell's equations also reduce to a single equation for the field h,

$$\frac{\partial h}{\partial\tau}=\frac{\partial^2 h}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial h}{\partial\rho}-\frac{h}{\rho^2} \tag{4}$$

and the quantity e is related to h by

$$2e=\frac{1}{\rho}\frac{\partial}{\partial\rho}(h\rho)=\frac{h}{\rho}+\frac{\partial h}{\partial\rho}. \tag{5}$$

It can be shown that at the N/I boundary [$\rho=a(\tau)$] the boundary conditions are of the usual form, despite the motion of this boundary:

$$e_I(a)=e_N(a),\quad h_N(a)=h_I(a)=1. \tag{6}$$

Noting that we have $e_I(a)=a/2$ by virtue of (2), we find from (6) and (2) that the following condition must also hold at the N/I boundary:

$$(\partial h^N/\partial\rho)_{\rho=a(\tau)}=0. \tag{7}$$

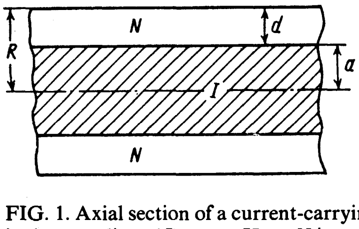

FIG. 1. Axial section of a current-carrying cylindrical superconductor in its intermediate (I) state. Here N is a normal-metal shell, and a is the radius of the intermediate state.

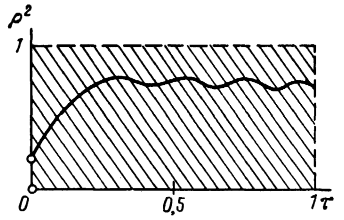

FIG. 2. Sloping straight lines—characteristics of Eq. (3) in the coordinates (ρ^2,τ); wavy line—a line (which intersects each characteristic only once) on which boundary conditions can be specified.

In Sec. 4 we will show how this condition can be utilized to derive a nonlinear integral equation for $a(\tau)$ if $i(\tau)$ is known.

3. ELECTRODYNAMICS OF THE I PHASE

The electric field in the I phase $e(\rho,\tau)$, satisfies a linear equation of the hyperbolic type, (3), which has wave solutions (which converge on the center) of the type $e\propto\exp(i\varphi)$, where

$$\varphi=\omega\tau+\varkappa^2\rho^2/2, \tag{8}$$

and ω and $\varkappa$ are related by the dispersion relation $\omega=\varkappa^2$. The phase velocity at which these waves, converging on the center ($\delta\rho<0$), propagate can be found from the condition that φ remain constant upon small deviations of ρ and τ. The result is $v_f\equiv\delta\rho/\delta\tau=-1/\rho$. If the motion of the interface, $a(\tau)$, is given, then to construct solutions of Eq. (3) we should use its characteristics, the equation for which is $d\rho/d\tau=-1/\rho$. We find

$$\rho^2=\rho_1^2+2(\tau_1-\tau), \tag{9}$$

where τ_1 and ρ_1 are integration constants. The solution of (3) which we are seeking and which satisfies boundary condition (6), $e(\rho=a(\tau))=[2a(\tau)]^{-1}$, then takes the form

$$e(\rho,\tau)=[2a(\tau_1)]^{-1}, \tag{10}$$

where $\tau_1=\tau_1(\rho,\tau)$ is found from Eq. (9) with $\rho_1=a(\tau_1)$.

As we know, the boundary conditions on a partial differential equation can be specified on only those lines in the (ρ,τ) space which intersect each of characteristics (9) of Eq. (3) only once.

Figure 2 shows one such possible line in terms of the coordinates ρ^2,τ, which render characteristics (9) straight lines (the sloping straight lines). It then follows from Fig. 2 that for $\dot a>0$ the magnitude of the velocity is arbitrary, but the velocity of the backward motion ($\dot a<0$) cannot be arbitrary. It is bounded by the requirement

$$da^2/d\tau\geqslant-2. \tag{11}$$

The physical meaning of this condition can be seen from an analysis for the concentration of the normal phase, $x_n(\rho,\tau)$ [see (2) and (10)]:

$$x_n=\rho/a(\tau_1). \tag{12}$$

In the I phase we must have $0<x_n<1$. Let us now examine the possibility that an N phase will appear inside the I phase (i.e., the case $x_n>1$). For this purpose we calculate the derivative $\partial x_n/\partial\rho$:

$$\partial x_n/\partial\rho = [1+a\dot{a}(1-x_n^2)]/a(1+\dot{a}a). \quad (13)$$

Here, as in (10), we have $a = a(\tau_1)$ and $a' \equiv da/d\tau_1$. From (13) we find a result which we have already mentioned: If $a' > 0$, then the quantity x_n in the I phase ($x_n < 1$) increases monotonically with increasing ρ, reaching $x_n = 1$ at $\rho = a(\tau)$. If $a' < 0$, on the other hand, then we have

$$(\partial x_n/\partial\rho)_{\rho=a(\tau)} = [a(1+a\dot{a})]^{-1}$$

at the boundary, so that the I phase is always stable $[x_n(\rho - 0) < 1]$ near the boundary if the condition $1 + aa' > 0$ holds. The latter condition is obviously equivalent to condition (11).

Let us now calculate the dissipation of the energy of the alternating current, Q_I, in the I state $[0 < \rho < a(\tau)]$. Since we have $j_I = cH_c/4\pi r$ in dimensional units, we have

$$Q_I \equiv \int_0^{a(\tau)} 2\pi r dr\, jE = \frac{cH_cE_c}{2}\int_0^{a(\tau)} e(\rho,\tau)\,d\rho, \quad (14)$$

where $e(\rho,\tau)$ satisfies Eq. (3). Integrating (14) by parts, and using (3), we easily find

$$\int_0^{a(\tau)} e\,d\rho = 1 + \frac{1}{2}\frac{da^2}{d\tau} - \frac{d}{d\tau}\int_0^{a(\tau)} e(\rho,\tau)\rho^2\,d\rho. \quad (15)$$

It follows from (15) that if $a(\tau)$ is a periodic function of the time then by taking an average of the dynamic component of the dissipation, Q_I [the last two terms in (15)], we find zero. This result corresponds to the possibility, noted above, of the propagation of undamped waves $e(\rho,\tau)$ in the intermediate state. The absence of an average dynamic component of the dissipation in the I phase (in contrast with the situation in the N phase) can also be understood on the basis of the following comment: In the case of an ohmic conductor, both j_N and E_N in the expression $Q \propto jE$ oscillate, and they do so in phase, while in the I phase j_I is independent of the time, and only E_I oscillates. As a result we have $\overline{Q}_I \propto j_I \overline{E}_I = 0$. We thus have

$$\overline{Q}_I = cH_cE_c/2 = (cH_c)^2/8\pi\sigma R.$$

We also note that the so-called internal inductance per unit length of a cylindrical conductor,[1] defined by

$$\frac{\mathscr{L}_i J^2}{2} = \int_0^a 2\pi r\,dr\,\frac{BH}{4\pi}$$

for a cylinder of radius a in the intermediate state, is greater than that for the same conductor in the N state: Even in the case of a constant current we would have $\mathscr{L}_i^I = \frac{4}{3}\mathscr{L}_i^N$, where $\mathscr{L}_i^N = c^{-2}$. An important difference between $\mathscr{L}_i^I$ and $\mathscr{L}_i^N$ arises in the case of an alternating current. While $\mathscr{L}_i^N$ decreases with increasing frequency of the current, as a result of the skin effect, the quantity $\mathscr{L}_i^I$ remains essentially constant, because of the undamped penetration of the magnetic induction $b = x_n$ into the I phase during oscillations of the N/I boundary, as mentioned above.

4. NONLINEAR RESPONSE OF THE *N/I* BOUNDARY TO CURRENT OSCILLATIONS

It was shown in Sec. 3 that the electrodynamics of the I phase is completely determined once we know the law of motion of the N/I boundary, $a(\tau)$. In turn, the function $a(\tau)$ must be determined through a self-consistent solution (with allowance for the motion of the boundary) of the quasisteady Maxwell's equations in the N phase for $h(\rho,\tau)$, (4), on the interval $1 \leqslant \rho \leqslant a(\tau)$, with the boundary conditions

$$h[\rho=a(\tau),\ \tau]=1,\quad h(\rho=1,\ \tau)=h_0+h(\tau)=h_0(\tau) \quad (16)$$

for $\tau > 0$ and with the "static" initial condition ($\tau = 0$)

$$h_0(\rho) = {}^1/_2(\rho/a_0 + a_0/\rho). \quad (17)$$

The "driving force" $h(\tau)$ is related to the alternating increment in the transport current through the sample by the total-current law.[1]

The additional boundary condition in (7) will actually be used to determine $a(\tau)$. The unknowns are $h(\rho,\tau)$ and $a(\tau)$, in terms of which we can express the physical response $e(\rho = 1,\tau)$ with the help of (5).

We wish to emphasize that despite the overall linearity of Eq. (4) the problem of determining $h(\rho,\tau)$ in the face of finite oscillations of the N/I boundary is a nonlinear problem because of the presence of a movable boundary. A fairly general method for solving such problems has been presented by Grinberg.[5] Borrowing some ideas from his study, we will first derive a closed nonlinear integral equation for the coordinate of the boundary, $a(\tau)$. We will then write an expression for $h(\rho,\tau)$ in terms of the instantaneous $[a(\tau)$-dependent$]$ eigenfunctions of boundary-value problem (4), (16).

It is convenient to first rewrite Eq. (4), introducing a source $Q(\rho,\tau)$ in order to reformulate the entire problem with a homogeneous boundary condition at $\rho = 1$. Since the introduction of sources is ambiguous [and the equation for $a(\tau)$ must not depend on the particular way in which the sources are introduced], we will not be more specific about the nature of the sources at this point.

We thus replace the function h by means of the relation $h \equiv v + w$, introducing the new function $v(\rho,\tau)$, with homogeneous boundary conditions,

$$v(\rho=1,\ \tau)=0,\quad (\partial v/\partial\rho)_{\rho=a(\tau)}=0, \quad (18)$$

and with the auxiliary condition

$$v[\rho=a(\tau),\ \tau]=1-w[\rho=a(\tau),\ \tau]. \quad (19)$$

The function $w(\rho,\tau)$ must satisfy the two conditions

$$w(\rho=1,\ \tau)=h_0(\tau),\quad (\partial w/\partial\rho)_{\rho=a(\tau)}=0 \quad (20)$$

and is otherwise arbitrary. Introducing the notation

$$\hat{L} \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}, \quad (21)$$

we can then write an equation for $v(\rho,\tau)$ in the form

$$\hat{L}v + Q(\rho,\ \tau) = \partial v/\partial\tau, \quad (22)$$

where, as is easily seen, we have

$$Q(\rho,\ \tau) = \hat{L}w - \partial w/\partial\tau. \quad (23)$$

We turn now to a study of Eq. (22) with the three boundary conditions in (18) and (19). We denote by[5] $\varphi_\lambda(\rho)$ a function of the continuous parameter λ of such a nature that we have

$$\hat{L}\varphi_\lambda=\lambda\varphi_\lambda \quad (24)$$

and

$$\varphi_\lambda(\rho=1)=0,\quad (\partial\varphi_\lambda/\partial\rho)_{\rho=1}=-1. \quad (25)$$

It can be shown that for $\hat{L}$ as in (21) the explicit expression for this function is

$$\varphi_\lambda(\rho)=I_1(\lambda^{1/2})K_1(\lambda^{1/2}\rho)-K_1(\lambda^{1/2})I_1(\lambda^{1/2}\rho), \quad (26)$$

where I_1 and K_1 are modified Bessel functions.[6]

We now multiply Eq. (22) by $\rho\varphi_\lambda(\rho)$ and integrate it over ρ on the interval $[a(\tau),1]$. Introducing

$$\Phi(\lambda^{1/2},\tau)\equiv\int_{a(\tau)}^{1} v(\rho,\tau)\varphi_\lambda(\rho)\rho\,d\rho \quad (27)$$

and using the (easily verified) identity

$$\int_{a(\tau)}^{1}(\varphi\hat{L}v-v\hat{L}\varphi)\rho\,d\rho=\rho\left(\varphi\frac{\partial v}{\partial\rho}-v\frac{\partial\varphi}{\partial\rho}\right)\Big|_{a(\tau)}^{1} \quad (28)$$

and the relation

$$\int_{a(\tau)}^{1}\rho\frac{\partial v}{\partial\tau}\varphi_\lambda(\rho)\,d\rho=\frac{\partial\Phi}{\partial\tau}+[\rho\dot{\rho}v(\rho,\tau)\varphi_\lambda(\rho)]|_{\rho=a(\tau)}, \quad (29)$$

we find the following equations for Φ, using (24), (18), and (25):

$$\partial\Phi/\partial\tau-\lambda\Phi=Q(\lambda^{1/2},\tau)+D(\lambda^{1/2},\tau). \quad (30)$$

Here Q and D are known [by virtue of relations (23) and (19)] functions:

$$Q(\lambda^{1/2},\tau)\equiv\int_{a(\tau)}^{1}Q(\rho,\tau)\varphi_\lambda(\rho)\rho\,d\rho, \quad (31)$$

$$D(\lambda^{1/2},\tau)\equiv[\rho v(\rho,\tau)(\partial\varphi_\lambda/\partial\rho-\dot{\rho}\varphi_\lambda)]|_{\rho=a(\tau)}. \quad (32)$$

It is convenient to write a solution of Eq. (30) in the form

$$e^{-\lambda\tau}\Phi_\lambda(\tau)=\Phi_\lambda(0)+\int_0^\tau(Q+D)e^{-\lambda\tau'}\,d\tau'.$$

Setting $\tau\to+\infty$ here, we find the nonlinear integrodifferential equation for $a(\tau)$ which we have been seeking:

$$\Phi_\lambda(0)=-\int_0^\tau(Q+D)e^{-\lambda\tau'}\,d\tau', \quad (33)$$

where Q and D are given by (31) and (32), and $\Phi_\lambda(0)$ is given by (27) with $\tau=0$.

Before analyzing Eq. (33), we will derive one more expression for the unknown function $v(\rho,\tau)$, assuming for the time being that the function $a(\tau)$ is known. For this purpose we make the following substitution in expression (26) for $\varphi_\lambda(\rho)$:

$$\lambda^{1/2}\to\lambda_k^{1/2}\equiv-i\alpha_k/a(\tau). \quad (34)$$

It can then be shown that $\varphi_\lambda(\rho)$ becomes

$$\varphi_k(\rho)\equiv{}^1/_2\pi[J_1(\alpha_k\rho/a)Y_1(\alpha_k/a)-J_1(\alpha_k/a)Y_1(\alpha_k\rho/a)], \quad (35)$$

so that $\varphi_k(\rho=a(\tau))$ is an "instantaneous" eigenfunction of the operator $\hat{\mathscr{L}}$ in (21) on the interval $[a(\tau),1]$ with the boundary conditions

$$\varphi_k(\rho=1)=0,\qquad(\partial\varphi_k/\partial\rho)_{\rho=a(\tau)}=0, \quad (36)$$

if the a_k are the roots of the equation

$$J_1(\alpha_k/a)Y_1'(\alpha_k)-J_1'(\alpha_k)Y_1(\alpha_k/a)=0. \quad (37)$$

Here J_1 and Y_1 are the Bessel functions of the first and second kinds, respectively. Since the functions $\varphi_k(\rho)$ form a complete orthogonal set, expression (27) for $\Phi_\lambda(\tau)$, with substitution (34), can be thought of a series expansion of $v(\rho,\tau)$ in this set. If

$$\Phi_k(\tau)\equiv\int_{a(\tau)}^{1}v(\rho,\tau)\varphi_k(\rho,\tau)\rho\,d\rho=A_kv_k(\tau), \quad (38)$$

then $v_k(\tau)$ and the normalization constant $A_k(\tau)$ are determined by

$$v(\rho,\tau)\equiv\sum_{k=1}^{\infty}v_k(\tau)\varphi_k(\rho,\tau),\quad A_k(\tau)\equiv\int_{a(\tau)}^{1}\varphi_k^2(\rho,\tau)\rho\,d\rho. \quad (39)$$

The solution of Eq. (22) which we have been seeking is therefore

$$v(\rho,\tau)\equiv\sum_{k=1}^{\infty}\Phi_k(\tau)\varphi_k(\rho,\tau)/A_k(\tau), \quad (40)$$

where φ_k and A_k are given by (35) and (39), and

$$\Phi_k(\tau)\equiv e^{\lambda_k\tau}\left[\Phi_k(0)+\int_0^\tau\chi_k(\tau')e^{-\lambda_k\tau'}\,d\tau'\right]. \quad (41)$$

Here $\chi_k(\tau)\equiv Q_k(\tau)+D_k(\tau)$, and Q_k and D_k are found from (31) and (32) with the help of replacement (34).

If we are interested in only the steady state ($\tau\to\infty$) then the expression for $\Phi_k(\tau)$ simplifies, since the term $\exp(\lambda_k\tau)\Phi_k(0)$, which describes a transient process, disappears by virtue of the relation $\lambda_k=-[\alpha_k a(\tau)]^2<0$.

Expressions (40) and (41), along with expressions (39) and (35), thus in principle give us an analytic solution of our problem: From (33) we can find $a(\tau)$ as a function of the external driving force $h_0(\tau)$, and we can then find the response from (40).

Examining Eq. (33) for $a(\tau)$, we note that we can of course put it in a form which does not depend on the particular nature of the sources in (23). Since it is a somewhat complicated matter to prove this assertion in the general case, we will point out that in order to derive the final result it is more convenient to make this transition by first adding to (32) a term $v\rho(\partial\varphi/\partial\rho)_{\rho=1}=0$, which was omitted in the derivation of this expression, and then, after making the substitution $v=h-w$, setting $w=0$ [because with $w=0$ this term is no longer zero, since we have $h(\rho=1)=0$]. It is then convenient to rewrite expression (33) in the form

$$\int_{a_0}^{1}h(\rho,0)\varphi_\lambda(\rho)\rho\,d\rho=-\int_0^\infty[D_1(\tau)+D_2(\tau)]e^{-\lambda\tau}\,d\tau, \quad (42)$$

$$D_1(\tau)\equiv h_0(\tau)+\rho(\partial\varphi_\lambda/\partial\rho)_{\rho=a(\tau)},\quad D_2(\tau)\equiv-\rho\dot{\rho}\varphi_\lambda(\rho)|_{\rho=a(\tau)}. \quad (43)$$

Adopting static solution (17) as an initial condition, we have $\hat{L}h(\rho,0)=0$. Using identity (28) and the replacement $v\to h(\rho,0)$, we can then show that we have

$$\int_{a_0}^{1}h(\rho,0)\varphi_\lambda(\rho)\rho\,d\rho=-D_1(0)/\lambda. \quad (44)$$

Now (33) takes the form

$$\int_0^\infty [\dot{D}_1(\tau)+\lambda D_2(\tau)]e^{-\lambda\tau}\,d\tau=0. \qquad (45)$$

A further simplification results from the use of the time derivative of Eq. (24) at $\rho = a(\tau)$. We then finally find the following nonlinear integrodifferential equation for $a(\tau)$ (Ref. 2):

$$\int_0^\infty \frac{\dot{a}(\tau)}{a(\tau)}\varphi_\lambda[a(\tau)]e^{-\lambda\tau}\,d\tau=F(\lambda), \qquad (46)$$

$$F(\lambda)=-\int_0^\infty \dot{h}_0(\tau)e^{-\lambda\tau}\,d\tau, \qquad (47)$$

where $F(\lambda)$ is the Laplace transform of the known function $\dot{h}_0(\tau)$, and φ_λ is given by (26) with $\rho = a(\tau)$.

5. LINEAR RESPONSE OF THE *N/I* BOUNDARY

From Eq. (46) we easily find explicit expressions for the displacements of the N/I boundary in response to small oscillations or a jump in the transport current. Setting $a(\tau) = a_0 + a_1(\tau)$, where $a_1 \ll a_0$, in (46), we find a linear equation for $\dot{a}_1$:

$$\int_0^\infty \dot{a}_1(\tau)e^{-\lambda\tau}\,d\tau=\frac{a_0F(\lambda)}{\varphi_\lambda(a_0)}. \qquad (48)$$

The problem of determining $a_1(\tau)$ thus reduces to one of taking inverse Laplace transforms.

We first consider steady-state current oscillations $i(\tau) = i_0 + i_1 \exp(-i\Omega\tau)$. In this case we have $F(\lambda) = i\Omega i_1/(i\Omega + \lambda)$ and

$$\dot{a}_1(\tau)=\frac{i_1\Omega a_0}{2\pi}\int_{x-i\infty}^{x+i\infty}\frac{e^{\lambda\tau}\,d\lambda}{(i\Omega+\lambda)\varphi_\lambda(a_0)}. \qquad (49)$$

It can be shown that all the zeros of the function $\varphi_\lambda(a_0)$ in (49) are on the negative part of the real axis, at points $\lambda_n \equiv \zeta_n^2 < 0$, where ζ_n are simple zeros of the function

$$\varphi_\zeta(a)\equiv J_1(\zeta a)Y_1(\zeta)-J_1(\zeta)Y_1(\zeta a).$$

Accordingly, since we are interested in the asymptotic behavior of $\dot{a}_1$ as $\tau\to\infty$ (the steady state), it is sufficient to consider only the contribution from the pole $\lambda = -i\Omega$ to the integral in (49). We thus find

$$a_1(\tau)=-i_1a_0e^{-i\Omega\tau}/\varphi_{-i\Omega}(a_0). \qquad (50)$$

Let us analyze the dynamics of the N/I boundary in the limits of high and low frequencies, which correspond to strong and weak skin effects in a normal cylinder.[1] In the low-frequency limit ($\Omega \ll 1$), we will use the asymptotic expressions for the modified Bessel functions at small values of their argument in evaluating $\varphi_{-i\Omega}(a_0)$. Retaining the component which is linear in frequency, we find

$$a_1(\tau)=-\frac{2i_1a_0^2e^{-i\Omega\tau}}{1-a_0^2}\left\{1+\frac{i\Omega a_0^2}{2(1-a_0^2)}\right.$$
$$\left.\times\left[\ln a_0+\frac{1}{4}\left(\frac{1}{a_0^2}-a_0^2\right)\right]\right\} \qquad (51)$$

The first term describes the adiabatic tracking of the instantaneous value of the current by the N/I boundary. The second is a nonadiabatic correction for the delay of the oscillations of the magnetic field near the N/I boundary with respect to the current oscillations.

At high frequencies the relationship between the thickness of the layer of normal metal covering the region of intermediate state, on the one hand, and the thickness of the skin layer in the normal cylinder, on the other, is important. If the N/I boundary is outside the skin layer ($1 - a_0 \gg \Omega^{-1/2}$), the amplitude of its oscillations is exponentially small:

$$a_1(\tau)=-2i_1a_0^{3/2}\Omega^{1/2}\exp[-(\Omega/2)^{1/2}(1-a_0)$$
$$+i(\Omega/2)^{1/2}(1-a_0)-i\pi/4-i\Omega\tau]. \qquad (52)$$

In the opposite limit, of a thin N layer ($1 - a_0 \ll \Omega^{-1/2} \ll 1$), we find

$$a_1(\tau)=-i_1e^{-i\Omega\tau}[1+{}^1/_6 i\Omega(1-a_0)^2]/(1-a_0), \qquad (53)$$

which is the same as the result in the limit $a_0\to 1$ in (51), which describes a nearly adiabatic situation. The reason for this agreement is that in each case the thickness of the skin layer is significantly greater than that of the normal layer; i.e., the magnetic field near the N/I boundary differs only slightly from the field at the surface.

It can be shown that the condition for the applicability of the linear-response expressions in (50)–(53) is the satisfaction of the inequalities $a_1 \ll a_0$ for $a_0 \lesssim 1$ and $a_1 \ll 1 - a_0$ for $a_0 \approx 1$. The latter inequality holds if $i_1 \ll (1 - a_0)^2$.

By analyzing the linear response of the N/I boundary to a jump in the current we can determine both the nature and the duration of the transients which arise in the system after an abrupt change in the transport current. If the current changes abruptly by an amount $i_1 \ll i_0$ at the time $\tau = 0$, the displacement of the N/I boundary can be written in the form

$$a_1(\tau)=-\frac{2i_1a_0^2}{1-a_0^2}-i_1a_0\sum_{n=1}^\infty\frac{\exp(-|\lambda_n|\tau)}{\lambda_n[\partial\varphi_\lambda(a_0)/\partial\lambda]_{\lambda=\lambda_n}}, \qquad (54)$$

where λ_n are the roots of the function $\varphi_\lambda(a_0)$. The duration of the transients is thus on the order of $|\lambda_i|^{-1}$ and is the time scale for the diffusion of the magnetic field over the thickness of the N layer.

An explicit expression for $a_1(\tau)$ can be found when the N layer is thin ($1 - a_0 \ll 1$), and the curvature of the layer is inconsequential. In this case we have

$$\varphi_\lambda(a_0)=\mathrm{sh}[\lambda^{1/2}(1-a_0)]/\lambda^{1/2},\ \lambda_n=-\pi^2n^2/(1-a_0)^2. \qquad (55)$$

Substitution of these expressions into (54) yields

$$a_1(\tau)=-\frac{i_1}{1-a_0}\left\{1+2\sum_{n=1}^\infty(-1)^n\exp\left[-\frac{\pi^2n^2\tau}{(1-a_0)^2}\right]\right\}$$
$$=-\frac{i_1}{1-a_0}\theta_3\left(\frac{1}{2},\frac{\pi\tau}{(1-a_0)^2}\right), \qquad (56)$$

where $\theta_3(\vartheta,\varkappa)$ is the theta function.[7]

The series in (56) converges poorly at small values $\tau \ll (1 - a_0)^2$. To find an asymptotic expression for $a_1(\tau)$ to describe the initial stage of the transients, we make use of a known formula from the theory of the theta functions[7]:

$$\theta_3(\vartheta/\varkappa i,\ 1/\varkappa)=\varkappa^{1/2}\exp(\pi\vartheta^2/\varkappa)\theta_3(\vartheta,\ \varkappa). \qquad (57)$$

Retaining the first two terms in the series expansion of the function $\theta_3(\vartheta/\varkappa i, 1/\varkappa)$, we find the following result for $\tau \ll (1 - a_0)^2$:

$$a_1(\tau)=-2i_1(\pi\tau)^{-1/2}\exp[-(1-a_0)^2/4\tau]. \qquad (58)$$

Expression (58) also holds when the change in the current occurs over a finite time (t_0). In this case, the range of applicability of this expression is limited by the inequalities $t_0 \ll \tau \ll (1-a_0)^2$.

We might add that in the case of rapid changes in the current the condition for stability of the motion of the N/I boundary [condition (11)] may be violated. In this sense, the case of a thin N layer (in which the current J_0 is close to the critical value) is a dangerous one. Analysis shows that the motion of the N/I boundary will definitely be stable under the inequality $i_1 \ll (1-a_0)^3$, which is more stringent than the condition for the applicability of the linear response, $i_1 \ll (1-a_0)^2$.

6. NONLINEAR ELECTROMAGNETIC RESPONSE OF A CYLINDER IN THE INTERMEDIATE STATE TO A CHANGE IN CURRENT

As applications of the approach developed above for describing the dynamics of the N/I boundary, we will discuss in this section of the paper two problems involving the nonlinear electromagnetic response of a cylinder in the intermediate state: (1) the response to small-amplitude sinusoidal oscillations of the current; (2) the response to a slight jump in the transport current.

Calculating the response of a given order requires a systematic calculation of the responses of lower orders, as we know. We will accordingly restrict the discussion to the response of second order, first analyzing the linear response, although the linear response (in contrast with the subsequent responses) does not involve the dynamics of the N/I boundary.

Let us examine the case in which the current which is flowing, $i(\tau)$, contains both a direct component i_0 and a small alternating component

$$i_1(\tau) = i_1 \exp(-i\Omega\tau) \qquad (i_1 \ll i_0).$$

The equation for the alternating component of the magnetic field in the N phase ($h_1 \propto i_1$) has the form of diffusion equation (4) with the boundary conditions

$$h_1(1,\tau) = i_1 e^{-i\Omega\tau}, \quad h_1(a_0,\tau) = 0.$$

This situation corresponds to the problem of the skin effect in a hollow cylinder with an inner radius a_0 (Ref. 2). The solution of the problem is expressed in terms of modified Bessel functions:

$$h_1(\rho,\tau) = i_1 e^{-i\Omega\tau}[K_1(xa_0)I_1(x\rho) - I_1(xa_0)K_1(x\rho)]/\varphi_{-i\Omega}(a_0), \tag{59}$$

where $x = (-i\Omega)^{1/2}$. The dynamic impedance per unit length of the cylinder, normalized to the resistance of a normal cylinder, is

$$Z = -i\Omega\mathscr{L}_e/4L + {}^1/_2\{1 + x[K_1(xa_0)I_1'(x) - I_1(xa_0)K_1'(x)]/\varphi_{-i\Omega}(a_0)\}. \tag{60}$$

In the limit of a weak skin effect ($\Omega \ll 1$) we have

$$Z = -\frac{i\Omega\mathscr{L}_e}{4L} + \frac{1}{1-a_0^2}\left[1 - \frac{i\Omega a_0^2}{4}\left(\frac{1}{2a_0^2} - \frac{3}{2} - \frac{2a_0^2}{1-a_0^2}\ln a_0\right)\right]. \tag{61}$$

As $a_0 \to 1$, the real part of the impedance increases without bound, while the contribution of the internal inductance to the imaginary part of Z tends toward zero along with the volume of the N layer.

In the high-frequency case ($\Omega \gg 1$), two limiting situations are possible. If the skin thickness is small in comparison with the thickness of the N region ($\Omega^{-1/2} \ll 1 - a_0$), we have the familiar situation of a strong skin effect, and the expression for the impedance is that given in Ref. 1. In the opposite limit, the length scale of the variation in the magnetic field due to the skin effect is far greater than the thickness of the N layer, so we go back to expression (61) with $a_0 \approx 1$.

We turn now to a calculation of the second-order response. The magnetic field $h_2(\rho,\tau)$, which is proportional to i_1^2, satisfies Eq. (4). A nontrivial boundary condition (at $\rho = a_0$) can be found by expanding the first of equations (16) in the small amplitude of the alternating current, i_1. The result is $h_2(a_0,\tau) = a_1^2(\tau)/2a_0^2$. The second boundary condition is the trivial condition $h_2(1,\tau) = 0$. It follows from the formulation of the problem of the second-order response that the field $h_2(\rho,\tau)$ can be written as the sum of constant and alternating components:

$$h_2(\rho,\tau) = \bar{h}_2(\rho) + \mathrm{Re}[h_2(\rho)e^{-2i\Omega\tau}].$$

A boundary-value problem can be formulated for each of the components. We omit the calculations and proceed immediately to the results. For the amplitude of the alternating component of the magnetic field we have

$$h_2(\rho) = (i_1^2/4)[K_1(2^{1/2}x\rho)I_1(2^{1/2}x) - I_1(2^{1/2}x\rho)K_1(2^{1/2}x)]/\varphi_{-i\Omega}^2(a_0)\varphi_{-2i\Omega}(a_0). \tag{62}$$

From (5) and (62) we find an expression for the second harmonic of the electric field at the surface of the cylinder:

$$e_2(1,\tau) = -[i_1^2/8\varphi_{-i\Omega}^2(a_0)\varphi_{-2i\Omega}(a_0)]e^{-2i\Omega\tau}. \tag{63}$$

At low frequencies ($\Omega \ll 1$), this expression has the asymptotic form

$$e_2(1,\tau) \approx -\frac{i_1^2 a_0^3}{(1-a_0^2)^3}e^{-2i\Omega\tau} \times\left\{1 + \frac{2ia_0^2\Omega}{1-a_0^2}\left[\ln a_0 + \frac{1}{4}\left(\frac{1}{a_0^2} - a_0^2\right)\right]\right\}. \tag{64}$$

Analysis of $e_2(\rho,\tau)$ shows that the amplitude of the alternating component of the electric field reaches a maximum at $\rho = a_0$, and it decreases in the direction toward the surface of the cylinder because of the skin effect. In the high-frequency case ($\Omega \gg 1$), under the condition $\Omega(1-a_0)^2 \gg 1$, the quantity $e_2(1,\tau)$ thus turns out to be exponentially small:

$$e_2(1,\tau) \propto \exp[-(2^{1/2}+1)\Omega^{1/2}(1-a_0)].$$

If, on the other hand, the condition $\Omega(1-a_0)^2 \ll 1$ holds at high frequencies, we find

$$e_2(1,\tau) \approx -[i_1^2/8(1-a_0)^3][1 + {}^2/_3 i\Omega(1-a_0)^2]e^{-2i\Omega\tau}. \tag{65}$$

For the problem with a constant component of the magnetic field, the equation for $\bar{h}_2(\rho)$ is the same as (4) with $\partial\bar{h}_2/\partial_\tau = 0$. The nontrivial boundary condition (at $\rho = a_0$) is written in the form

$$\bar{h}_2(a_0) = i_1^2/4|\varphi_{-i\Omega}(a_0)|^2,$$

while at the surface of the cylinder we have $\bar{h}_2(1) = 0$. The solution of this problem allows us to renormalize the constant electric field at the surface of the cylinder:

$$\bar{e}_2(1) = -\frac{a_0}{4(1-a_0^2)}\frac{i_1^2}{|\varphi_{-i\Omega}(a_0)|^2}. \tag{66}$$

Let us examine the specific frequency dependence of $\bar{e}_2(1)$. At $\Omega \ll 1$, it takes the form of a correction which is quadratic in the frequency:

$$\bar{e}_2(1) = -\frac{a_0^3 i_1^2}{(1-a_0^2)^3}\left\{1+\frac{\Omega^2}{4}\left[\frac{a_0^4 \ln^2 a_0}{(1-a_0^2)^2}-\frac{a_0^2}{4}+\frac{(1-a_0^2)^2}{48}\right]\right\}^{-1} \quad (67)$$

The expression in square brackets in (67) is a monotonically decreasing function of a_0. An increase in the frequency results in a sharp decrease in $\bar{e}_2(1)$ at $\Omega \sim (1-a_0)^{-2}$, while at frequencies $\Omega \gg (1-a_0)^{-2}$ the renormalization becomes exponentially small:

$$\bar{e}_2(1) \propto \exp[-(2\Omega)^{1/2}(1-a_0)].$$

As the frequency is increased, one can work from the sharp decrease in both $\bar{e}_2(1)$ and the amplitude of the second harmonic, $e_2(1,\tau)$, to experimentally determine the thickness of the normal layer covering the region of the intermediate state.

We turn now to the last question, of the electromagnetic response of this system to a jump in the current:

$$i(\tau) = i_0 + i_1\theta(\tau), \quad i_1 \ll i_0.$$

The increment in the magnetic field which is linear in i_1 satisfies Eq. (4) with the boundary conditions

$$h_1(1,\tau) = i_1\theta(\tau), \qquad h_1(a_0,\tau) = 0$$

and the initial condition $h_1(\rho,\tau=0) = 0$. It is a straightforward matter to derive a solution $h_1(\rho,\tau)$ by means of Laplace time transforms:

$$h_1(\rho,\tau) = \frac{i_1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{d\lambda}{\lambda\varphi_\lambda(a_0)} e^{\lambda\tau}[K_1(\lambda^{1/2}a_0)I_1(\lambda^{1/2}\rho) - I_1(\lambda^{1/2}a_0)K_1(\lambda^{1/2}\rho)]. \quad (68)$$

The singularities of the integrand in (68) are poles at $\lambda = 0$ and $\lambda = \lambda_n$, where λ_n are the roots of the function $\varphi_\lambda(a_0)$, $\lambda_n < 0$. The contribution from the $\lambda = 0$ pole gives us the limiting value $h_1(\rho,\infty)$, which is reached at the end of the transients. The duration of the transients is on the order of $|\lambda_1|^{-1}$. In the case of a thin N layer $(1-a_0 \ll 1)$ we can use the properties of the theta functions and write $h_1(\rho,\tau)$ as a series which converges well under the condition $\tau \ll |\lambda_1|^{-1}$. We see from this series that the transients constitute a diffusion of the magnetic field from the surface of the cylinder into the interior of the N layer.

From (5) and (68) we find the increment in the electric field which is linear in i_1:

$$e_1(\rho,\tau) = \frac{i_1}{4\pi i}\int_{x-i\infty}^{x+i\infty} \frac{d\lambda}{\lambda^{1/2}\varphi_\lambda(a_0)} e^{\lambda\tau}[K_1(\lambda^{1/2}a_0)I_0(\lambda^{1/2}\rho) + I_1(\lambda^{1/2}a_0)K_1(\lambda^{1/2}\rho)]. \quad (69)$$

In the case of a thin N layer, this increment can be expressed in terms of the function θ_3:

$$e_1(\rho,\tau) = \frac{i_1}{2(1-a_0)}\theta_3\left(\frac{1-\rho}{2(1-a_0)};\frac{\pi\tau}{(1-a_0)^2}\right)$$
$$= \frac{i_1}{2(\pi\tau)^{1/2}}\exp\left[-\frac{(1-\rho)^2}{4\tau}\right]\theta_3\left(\frac{(1-\rho)(1-a_0)}{2\pi\tau i};\frac{(1-a_0)^2}{\pi\tau}\right). \quad (70)$$

The experimental electric field at the surface of a cylinder under the condition $\tau \ll (1-a_0)^2$ has a square-root singularity, $e_1(1,\tau) = i_1/2(\pi\tau)^{1/2}$, since early in the process the current i_1 is concentrated in a narrow layer of thickness $\sim\tau^{1/2}$ near the surface of the cylinder. Under the condition $\tau \gg (1-a_0)^2$, the electric field $e_1(1,\tau)$ differs from its steady-state value by an exponentially small amount.

We note in conclusion that an analysis of the quadratic increment in the magnetic field shows that the duration of the transients is the same as that for the linear increment.

We are indebted to A. F. Andreev for a useful discussion of these results.

[1)]This macroscopic description of the intermediate state, in which thermodynamic considerations are not invoked, makes no distinction between possible microscopic realizations of the structure of the intermediate state: a London (static) structure, a Gorter (dynamic) structure, or structures of the more general type found by Andreev.[4]

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Translated by Dave Parsons