Magnon bound states in an easy-axis Heisenberg ferromagnet of arbitrary dimensionality. Relation to magnetic solitons

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It is shown that several magnons in an easy-axis Heisenberg ferromagnet having a large spin S at the lattice site and large single-ion anisotropy produce (if their number $N \ll 2S$) on one- and two-dimensional lattices bound states at low values of the mass-center quasimomentum. On a three-dimensional lattice the bound states are formed by $N_c < N_1 < N_2 < ...$magnons. The upper bound of N_c is estimated. The quantum results for dimensionalities $d = 2$ and 3 are stronger than the quasiclassical results for dynamic solitons.

1. When other than linear approximations are used in solid-state theory, it becomes necessary to consider not a gas of independent elementary excitations but a system of interacting quasiparticles. The quantum-mechanical aspect of this problem is that the stationary states of such a system are not free-motion states, but scattering states and bound states (BS). The latter are observed in experiments on neutron and photon scattering and play an important role in low-temperature thermodynamics of low-dimensionality systems. They are also closely related to solitons, which are nonlinear excitations in crystals.

The best known problem is that of N magnons in an easy-axis Heisenberg ferrogmagnet. The Hamiltonian is

$$H=-\frac{1}{2}\sum_{m,n} J_{m,n}(S_m{}^zS_n{}^z+\sigma S_m{}^-S_n{}^+)-D\sum_n (S_n{}^z)^2, \quad (1)$$

where $J_{m,n} = J_{m-n} = J_\rho$ are exchange integrals satisfying the condition

$$J_\rho \geqslant 0,\ J_\rho < C_1 \exp(-C_2|\rho|) \quad (2)$$

(here and elsewhere C_i with $i = 1,2,...$ are positive constants and ρ is a cylindrical coordinate); $S_m^\pm = S_m^x \pm iS_m^y$, where $S_m^{x,y,z}$ are the projections of a spin operator, of value S, localized on the mth site of a d-dimensional primitive cubic lattice; $0 \leqslant \sigma \leqslant 1$, and $D \geqslant 0$ are the exchange- and single-ion anisotropy constants, respectively. The ground state χ_0 corresponds to the maximum projection of the total spin on the z axis. Following the ground state are states corresponding to the projections $S_{\max} - N$, $N = 1, 2,...$ of the total spin on the z axis—the so-called N-magnon states. Spin extensions can be regarded as quasiparticles, and the Hamiltonian (1) describes transport of the quasiparticles from site to site and their interaction. The number of quasiparticles is conserved in this model.

The state of an N-magnon system is described in the coordinate representation by a wave function ψ_n, where $n = \{n_1...n_N\}$ is the probability amplitude of the spin extensions being located on sites $n_1...n_N$ (if s spin extensions are located on site m, the number m is repeated s times in the sequence $\{n_1...n_N\}$. For the N-Magnon wave function ψ_n one obtains an N-particle Schrödinger equation on the lattice,[1] similar to the usual Schrödinger equation in continuous space.[2]

The most complete results for the model (1) were obtained for the dimensionality $d = 1$. For $S = 1/2$ and $D = 0$ there exists a single BS of N magnons at all values of the mass-center quasimomentum $\mathbf{K}$.[3–5] The two-dimensional problem has also been completely solved[6–8]; in particular, for $d = 2$ there exist from one to three BS, depending on the values of K, σ, and D. For $d = 3$ there are from zero to four BS, with no BS existing at $d = 3$ for small values of $|\mathbf{K}|$ and for low anisotropy. Bound states for $N = 3$ were considered in Refs. 9–12 (the Faddeev equations were solved numerically): at small values of $\mathbf{K}$ and $d = 2$, BS exist for all values of σ and D, while for $d = 3$ they exist only if the anisotropy is large enough. For fixed, N, σ, and D and for almost all "transport matrices" J_ρ there exists at $d = 3$ a value S_0 such that for all $S > S_0$, i.e., for low anisotropy, there are no BS of $2,...,N$ magnons.[13] Note that in the limiting case of the Ising model, when $\sigma = 0$ and the spin extensions are immobile, the BS are easily obtained and exist for all N and d.

The discrete Schrödinger equation is particularly simple in the case of a large spin and a strong single-ion anisotropy:

$$S \gg 1,\quad N \ll 2S,\quad D \gg J,\quad \sigma=1,\quad \text{где}\quad J=\sum_\rho J_\rho. \quad (3)$$

Neglecting in the Hamiltonian small (in norm) terms of order $\max\{N/2S, J/D\}$, the equation for the wave function of the N-magnon BS ψ_n takes under condition (3) the form

$$(H\psi)_n=S\sum_{i=1}^{N}\sum_\rho J_\rho\psi(n\backslash n_i)(n_i+\rho)^- -D\Big(\sum_{i<j}\delta_{n_i,n_j}\Big)\psi_n=E\psi_n, \quad (4)$$

where E is the energy of the BS and $(n/n_i)(n_i+\rho) = \{n_i...(n_i+\rho)...n_N\}$. In this case we have a model of N bosons on a lattice, interacting via two-particle contact attraction potential. This is the simplest of all nontrivial models of the problem of several quasiparticles on a lattice,[14] and was treated by numerical and approximate methods.[14–15] In the momentum representation, the state of the system is described by the wave function

$$\varphi(\mathbf{k})=\sum_n \psi_n \exp\Big(i\sum_{j=1}^{N}\mathbf{k}_j\mathbf{r}_{n_j}\Big),$$

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where $\mathbf{r}_{nj}$ is the coordinate of site j, the lattice period is set equal to unity, $\mathbf{k} = \{\mathbf{k}_1 ... \mathbf{k}_N\}$, $\mathbf{k}_1 \in T^d$, T^d is a d-dimensional torus—the Brillouin zone in the case of a primitive cubic lattice, and $\mathbf{k}_i$ is the quasimomentum of the ith magnon. The equation for the wave function of an N-magnon BS in the momentum representation is

$$(H\varphi)(\mathbf{k}) = \Big(\sum_{j=1}^{N} \varepsilon(\mathbf{k}_j)\Big)\varphi(\mathbf{k}) - D\sum_{i<j}\int_{(T^d)^2} \delta(\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_i' - \mathbf{k}_j')\varphi \times (\mathbf{k}_1 \ldots \mathbf{k}_i' \ldots \mathbf{k}_j' \ldots \mathbf{k}_N)\, d\mathbf{k}_i' d\mathbf{k}_j', \quad (5)$$

$$\varepsilon(\mathbf{k}) = -SJ + D(1-2S) - \sum_{\rho} J_\rho \exp[i\mathbf{k}\rho],$$

where $\varepsilon(\mathbf{k})$ is the magnon kinetic energy.

Dynamic magnetic solitons, which appear in continuous classical models of an easy-axis Heisenberg ferromagnet,[16] are linear superpositions of magnon BS, as shown by the results of semiclassical quantization of soliton solutions.[16–18] It follows from the semiclassical results that stable dynamic solitons, and consequently also N-magnon BS at $\mathbf{K} = 0$ exist for all N in the case $d = 1$, for $N > N_2 \sim D/SJ$ in the case $d = 2$, and for $N > N_3 \sim (D/SJ)^{3/2}$ in the case $d = 3$. We shall show that the quantum-mechanical approach yields stronger results concerning the existence of BS than the semiclassical approaches, which obviously contradict the known exact result for $d = 2$ and $N = 2$.

2. We begin with the two-particle problem. After separating the mass-center motion, the Hamiltonian takes in the momentum representation the form

$$(H_\mathbf{K}\varphi_2)(\mathbf{k}) = \varepsilon_\mathbf{K}(\mathbf{k})\varphi(\mathbf{k}) - D\int_{T^d} \varphi_2(\mathbf{k})\,d\mathbf{k}, \quad (6)$$

where

$$\varepsilon_\mathbf{K}(\mathbf{k}) = \varepsilon(\mathbf{k}/2+\mathbf{k}) + \varepsilon(\mathbf{k}/2-\mathbf{k}).$$

We consider a high symmetry-case of the general position:

$$\varepsilon_\mathbf{K}(\mathbf{k}) = (2m)^{-1}\mathbf{k}^2 + o(\mathbf{k}^2), \quad |\mathbf{k}| \ll 1,$$

and choose the energy origin such that $\varepsilon_\mathbf{K}(0) = 0$. We obtain a sufficient condition for the existence of BS with energy $E < 0$ by choosing a trial wave function $\varphi_2(\mathbf{k})$ such that $\langle\varphi_2|H_K|\varphi_2\rangle < 0$. The existence of a BS will follow then from the minimax principle.[2]

The leading considerations that permit the choice of the trial wave function are the following. Obviously, it is easiest to obtain a criterion for the existence of a weakly bound state. This means that the wave function is concentrated in the region of small quasimomenta. In addition, at large distances, where the potential does not act, the wave function should decrease just as the solution of the Helmholtz equation in empty space:

$$(-\Delta + \alpha^2)\varphi_2 = 0, \quad \varphi_2(\mathbf{k}) \propto (\mathbf{k}^2 + \alpha^2)^{-1}$$

for small $|\mathbf{k}|$ and $\alpha \to 0$. On this basis, we choose the trial wave function in the form

$$\varphi_2(\mathbf{k}) = \begin{cases} A_d(k_0, \alpha)(\mathbf{k}^2+\alpha^2)^{-1}, & |\mathbf{k}| < k_0, \\ 0, & |\mathbf{k}| > k_0, \end{cases} \quad (7)$$

where k_0 is a certain fixed value of the quasimomentum, $A_d(k_0, \alpha)$ is a normalization factor, and $d = 1$ or 2:

$$d=1:\quad A_1^2 \int_0^{k_0} \frac{d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} = A_1^2 \alpha^{-3} C_1^2,$$

$$A_1 = \alpha^{3/2}/C_1,\ C_1 \sim 1,\ \alpha \ll k_0,$$

$$A_2^2 \int_0^{k_0} \frac{2\pi\mathbf{k}\,d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} = A_2^2 \alpha^{-2} C_2, \quad d=2,$$

$$A_2 = \alpha/C_2,\ C_2 \sim 1,\ \alpha \ll k_0.$$

Let us calculate the matrix element

$$\langle\varphi|H_\mathbf{K}|\varphi\rangle = \int_{T^d} \varepsilon_\mathbf{K}(\mathbf{k})\varphi_2^2(\mathbf{k})\,d\mathbf{k} - D\int_{(T^d)^2} d\mathbf{k}\,d\mathbf{k}'\varphi_2(\mathbf{k})\varphi_2(\mathbf{k}') = A_d^2\left[\int_{\Omega_0} \frac{\varepsilon_\mathbf{K}(\mathbf{k})\,d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} - D\left(\int_{\Omega_0} \frac{d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)}\right)^2\right],$$

where $\Omega_0 = \{\mathbf{k} \in T^d : |\mathbf{k}| < k_0\}$, neglecting in the evaluation of the integrals the terms of high order of smallness in α:

$$d=1:\quad \int_{\Omega_0} \frac{\varepsilon_\mathbf{K}(\mathbf{k})\,d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} = \frac{2}{2m}\int_0^{k_0} \frac{\mathbf{k}^2 d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} = \frac{C_1'}{m\alpha} + O(1),$$

$$\left(\int_{\Omega_0} \frac{d\mathbf{k}}{\mathbf{k}^2+\alpha^2}\right)^2 = \frac{C_1''}{\alpha^2} + O(1), \quad C_1', C_1'' \sim 1.$$

$$d=2:\quad \int_{\Omega_0} \frac{\varepsilon_\mathbf{K}(\mathbf{k})\,d\mathbf{k}}{(\mathbf{k}^2+\alpha_2)^2} = \frac{2\pi}{2m}\int_0^{k_0} \frac{\mathbf{k}^3\,d\mathbf{k}}{(\mathbf{k}^2+\alpha^2)^2} = \frac{C_2'}{m}\ln\left(\frac{k_0}{\alpha}\right) + O(1),$$

$$\left(\int_{\Omega_0} \frac{d\mathbf{k}}{\mathbf{k}^2+\alpha^2}\right)^2 = C_2''\left[\ln\left(\frac{k_0}{\alpha}\right)\right]^2 + O(1), \quad C_2', C_2'' \sim 1 \quad (8)$$

$$\langle\varphi_2|H_\mathbf{K}|\varphi^2\rangle = \begin{cases} \dfrac{\alpha^2}{m}\tilde{C}_1 - D\alpha\tilde{\tilde{C}}_1 + O(\alpha^3), & d=1, \\ \dfrac{\alpha}{m}\ln\left(\dfrac{k_0}{\alpha}\right)\tilde{C}_2 - D\alpha\left[\ln\left(\dfrac{k_0}{\alpha}\right)\right]^2\tilde{\tilde{C}}_2 + O(\alpha), & d=2, \end{cases}$$

$O(x)$ denotes quantities of order x.

It is seen from (8) that if α is chosen small enough, a two-magnon BS exists for $d = 1$ and 2 at any value of $\mathbf{K}$.

An attempt to extend the variational method to include the case of a large number of magnons meets with the following difficulty. In the two-particle problem we know the location of the lower boundary of the continuous spectrum. In the N-particle problem, on the other hand, the criterion for the existence of a BS is the existence of a wave function φ_N such that

$$\langle\varphi_N|H_\mathbf{K}|\varphi_N\rangle < \Sigma_N(\mathbf{K}), \quad (9)$$

where $\Sigma_N(\mathbf{K})$ is the lower boundary of the continuous spectrum. The continuous spectrum of the Hamiltonian (5) corresponds to scattering states and has a characteristic multichannel structure,[1,2,19,20]: it consists of a finite number of branches that are segments on the energy axis. Each branch corresponds to the scattering states of clusters of $n_1,...,n_s$ magnons, where $n_1 + ... + n_s = N$. The energy of such a state is $E_{n_1}(\mathbf{K}_1) + ... + E_{n_s}(\mathbf{K}_s)$, where $E_{n_i}(\mathbf{K}_i)$ is the energy of an n_i-magnon state and depends on $\mathbf{K}_i$. Assume that 2, 3,...,$N-1$ magnons form BS at small $|\mathbf{K}|$. Let $E_j(\mathbf{K})$, $j=2,...,N-1$ be the energies of the ground BS. It was shown in Ref. 1 that min $E_j(\mathbf{K}) = E_j(0)$, i.e., the energy of immobile space, a statement that is not trivial on a lattice. It follows hence that the lower edge of each branch of the continuous spectrum

$$\Sigma_{(n_1 \ldots n_s)} = E_{n_1}(0) + \ldots + E_{n_s}(0).$$

In addition, it follows from elementary variational considerations that the larger the number of bound magnons, the lower the energy of the ground BS: $E_{j_1}(0) < E_{j_2}(0)$, $j_1 > j_2$, meaning that the lower boundary of the continuous spectrum corresponds at $\mathbf{K}=0$ to a two-cluster breakup:

$$\Sigma_N(0) = E_{N_1}(0) + E_{N-N_1}(0)$$

at a certain N_1 [if $N_1 = 1$, then $E_1(0) = \varepsilon(0)$]. This means that the trial function φ_n must be chosen such that

$$\langle \varphi_N | H_{\mathbf{K}} | \varphi_N \rangle < \min_{N_1} [E_{N_1}(0) + E_{N-N_1}(0)].$$

The idea of choosing such a function was proposed in Ref. 19 for an N-particle Schrödinger equation in a continuous space. We extend this method to include the case of a lattice. Since, however, $\varepsilon(\mathbf{k})$ is not a quadratic function, this can be done only for small values of $|\mathbf{K}|$, inasmuch as for values that are not small the lower boundary of the continuous spectrum can correspond to an s-cluster breakup with $s > 2$, and the method of Ref. 19 is not valid.

3. We begin with the case $N=3$. Then $\Sigma_N(0) = E_2(0)$. Let $\Phi_{\mathbf{p}}(\mathbf{k})$ be the wave function of the ground two-magnon BS that is characterized by the value $\mathbf{p}$ of the mass-center quasi-momentum of two magnons. We choose the trial three-magnon function (at $\mathbf{K}=0$) in the form

$$\varphi_3(\mathbf{p}, \mathbf{k}) = \Phi_{\mathbf{p}}(\mathbf{k}) \varphi_2(-\mathbf{p}). \tag{10}$$

Using expression (5) for $H_{\mathbf{K}}$, we obtain

$$\begin{aligned}
\langle \varphi_3 | H_{\mathbf{K}} | \varphi_3 \rangle = & \int_{(T^d)^2} \varepsilon_{\mathbf{p}}(\mathbf{k}) \Phi_{\mathbf{p}}^2(\mathbf{k}) \varphi_2^2(\mathbf{p})\, d\mathbf{p}\, d\mathbf{k} \\
& + \int_{(T^d)^2} \varepsilon(-\mathbf{p}) \Phi_{\mathbf{p}}^2(\mathbf{k}) \varphi_2^2(\mathbf{p})\, d\mathbf{p}\, d\mathbf{k} \\
& - D \int_{(T^d)^3} \Phi_{\mathbf{p}}(\mathbf{k}) \Phi_{\mathbf{p}}(\mathbf{k}') \varphi_2^2(\mathbf{p})\, d\mathbf{p}\, d\mathbf{k}\, d\mathbf{k}' \\
& - D \int_{(T^d)^4} \Phi_{-\mathbf{k}_3}(2\mathbf{k}_1 + \mathbf{k}_3) \Phi_{-\mathbf{k}_3'}(-2\mathbf{k}_1' - \mathbf{k}_3') \varphi_2(\mathbf{k}_3) \varphi_2(\mathbf{k}_3') \\
& \times \delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_1' - \mathbf{k}_3')\, d\mathbf{k}_1\, d\mathbf{k}_1' d\mathbf{k}_3\, d\mathbf{k}_3' \\
& - D \int_{(T^d)^4} \Phi_{-\mathbf{k}_3}(-2\mathbf{k}_2 - \mathbf{k}_3) \Phi_{-\mathbf{k}_3'}(+2\mathbf{k}_2' + \mathbf{k}_3') \varphi_2(\mathbf{k}_3) \\
& \times \varphi_2(\mathbf{k}_3') \delta(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_2' - \mathbf{k}_3')\, d\mathbf{k}_2\, d\mathbf{k}_2'\, d\mathbf{k}_3'\, d\mathbf{k}_3 \\
= & \int_{T^d} d\mathbf{p} \varphi_2^2(\mathbf{p}) \Big\{ \Big[\int_{T^d} \Phi_{\mathbf{p}}^2(\mathbf{k}) \varepsilon_p(\mathbf{k})\, d\mathbf{k} \\
& - D \int_{(T^d)^2} \Phi_{\mathbf{p}}(\mathbf{k}) \Phi_{\mathbf{p}}(\mathbf{k}')\, d\mathbf{k}\, d\mathbf{k}' \Big] + \varepsilon(-\mathbf{p}) \\
& \times \Big(\int_{(T^d)} \Phi_{\mathbf{p}}^2(\mathbf{k})\, d\mathbf{k} \Big) \Big\} - D \int_{(T^d)^3} \varphi_2(\mathbf{k}_3) \varphi_2(\mathbf{k}_3') \\
& \times \Phi_{-\mathbf{k}_3}(2\mathbf{k}_1 + \mathbf{k}_3) \Phi_{-\mathbf{k}_3'}(2\mathbf{k}_1 + 2\mathbf{k}_3 - \mathbf{k}_3')\, d\mathbf{k}_1 d\mathbf{k}_3\, d\mathbf{k}_3' \\
& - D \int_{(T^d)^3} \varphi_2(\mathbf{k}_3) \varphi_2(\mathbf{k}_3') \Phi_{-\mathbf{k}_3}(-2\mathbf{k}_2 - \mathbf{k}_3) \\
& \times \Phi_{-\mathbf{k}_3'}(-2\mathbf{k}_2 - 2\mathbf{k}_3 + \mathbf{k}_3')\, d\mathbf{k}_2\, d\mathbf{k}_3 d\mathbf{k}_3'.
\end{aligned} \tag{11}$$

Note that the expression in the square brackets is equal to $\langle \Phi_p | H_k | \Phi_p \rangle = E_2(\mathbf{K})$. As to the second and third terms, inasmuch as the function $\varphi_2(\mathbf{k})$ is large at $|\mathbf{k}| \sim 0$ and decreases rapidly with increase of $|\mathbf{k}|$, we have

$$\int f(\mathbf{k}) \varphi_2(\mathbf{k})\, d\mathbf{k} \approx f(0) \int \varphi_2(\mathbf{k})\, d\mathbf{k}.$$

More accurately,

$$\begin{aligned}
& \int_{(T^d)^3} \Phi_{-\mathbf{k}_3}(2\mathbf{k}_1 + \mathbf{k}_3) \Phi_{-\mathbf{k}_3'}(2\mathbf{k}_1 + 2\mathbf{k}_3 - \mathbf{k}_3') \\
& \qquad \times \varphi_2(\mathbf{k}_3) \varphi_2(\mathbf{k}_3')\, d\mathbf{k}_1\, d\mathbf{k}_3\, d\mathbf{k}_3' \\
& = \int_{T^d} \Phi_0^2(\mathbf{k}_1)\, d\mathbf{k}_1 \Big(\int_{T^d} \varphi_2(\mathbf{k}_3)\, d\mathbf{k}_3 \Big)^2 + \begin{cases} O(\alpha^3), & d=1, \\ O(\alpha^2), & d=2. \end{cases}
\end{aligned}$$

A similar expression is given by the third term. Recognizing that

$$\int_{T^d} \Phi_{\mathbf{p}}^2(\mathbf{k})\, d\mathbf{k} = 1,$$

we have

$$\begin{aligned}
\langle \varphi_3 | H_0 | \varphi_3 \rangle = & \int_{T^d} d\mathbf{p} [E_2(\mathbf{p}) + \varepsilon(-\mathbf{p})] \varphi_2^2(\mathbf{p}) \\
& - 2D \Big(\int_{T^d} \varphi_2(\mathbf{k})\, d\mathbf{k} \Big)^2 \\
& + \begin{cases} O(\alpha^3), & d=1, \\ O(\alpha^2), & d=2. \end{cases}
\end{aligned} \tag{12}$$

Expressing $E_2(\mathbf{p})$ in the form

$$E_2(\mathbf{p}) = E_2(0) + \mathbf{p}^2/2M + o(\mathbf{p}^2)$$

(we consider only a high-symmetry case) we get, just as in the two-particle estimate,

$$\begin{aligned}
& \langle \varphi_3 | H_0 | \varphi_3 \rangle = E_2(0) \\
& + \begin{cases} B_1 \alpha^2 - B_2 D \alpha + O(\alpha^3), & d=1, \\ B_3 \alpha \ln\left(\dfrac{k_0}{\alpha}\right) - B_4 D \alpha \left[\ln\left(\dfrac{k_0}{\alpha}\right) \right]^2 + O(\alpha^2), & d=2 \end{cases}
\end{aligned} \tag{13}$$

($B_i > 0$. At a sufficiently small value of α we have $\langle \varphi_3 | H_0 | \varphi_3 \rangle < E_2(0)$, i.e., three magnons form a BS at $d=1$ and 2 and $\mathbf{K}=0$.

Assume that at $\mathbf{K}=0$ there exist 4-,..., $(N-1)$-magnon BS and the lower boundary of the continuous spectrum corresponds to a breakup $N = N_1 + N_2$. If $N_1 = 1$, the proof is perfectly similar to that in the three-particle case. Let

$$\Phi_1(\mathbf{p}, \mathbf{k}_1 \ldots \mathbf{k}_{N_1-1}), \quad \Phi_2(-\mathbf{p}, \mathbf{q}_1 \ldots \mathbf{q}_{N_2-1})$$

be the wave functions of the N_1- and N_2-magnon ground BS, characterized by mass-center quasimomenta $\mathbf{p}$ and $-\mathbf{p}$ of the N_1 and N_2 magnons, respectively, and let

$$\{\mathbf{k}_1 \ldots \mathbf{k}_{N_1-1}\}, \qquad \{\mathbf{q}_1 \ldots \mathbf{q}_{N_2-1}\}$$

be the quasimomenta of the first,..., (N_1-1)st, (N_1+1)st,..., $(N-1)$st magnons. We seek the trial function of the N-magnon BS at $\mathbf{K}=0$ in the form

$$\varphi_N = \Phi_1(\mathbf{p}, \mathbf{k}_1 \ldots \mathbf{k}_{N_1-1})\Phi_2(-\mathbf{p}, \mathbf{q}_1 \ldots \mathbf{q}_{N_2-1})\varphi_2(\mathbf{p}). \quad (14)$$

It should be noted that if we have assumed the existence of BS at $\mathbf{K}=0$ and $N=s$, with energies not on the boundary of the continuous spectrum, it follows from the general premises of perturbation theory[2] that s-particle BS exist in a certain region of values $|\mathbf{K}| < k_0(s)$. Clearly, the constant k_0 in the definition of the function φ_2 can be chosen to have a certain value $k_0 < \min\{k_0(N_1), k_0(N_2)\}$. Using expression (5), we obtain

$$\langle \varphi_N | H_0 | \varphi_N \rangle$$

$$= \int_{(T^d)^{N-1}} \Big[\sum_{i=1}^{N_1-1} \varepsilon(\mathbf{k}_i) + \sum_{j=1}^{N_2-1} \varepsilon(\mathbf{q}_j) + \varepsilon\Big(\mathbf{p} - \sum_{i=1}^{N_1-1} \mathbf{k}_i\Big)$$

$$+ \varepsilon\Big(-\mathbf{p} - \sum_{j=1}^{N_2-1} \mathbf{q}_j\Big)\Big] \Phi_1^2(\mathbf{p}, \ldots)\Phi_2^2(-\mathbf{p}, \ldots)$$

$$\times \varphi_2^2(\mathbf{p})\, d\mathbf{p}\, d\mathbf{k}\, d\mathbf{q}$$

$$- D \sum_{\substack{i,j=1 \\ i\neq j}}^{N_1} \int_{(T^d)^{N+1}} \Phi_1(\mathbf{k}_1 \ldots \mathbf{k}_i \ldots \mathbf{k}_j \ldots \mathbf{k}_{N_1})$$

$$\times \Phi_1(\mathbf{k}_1 \ldots \mathbf{k}_i' \ldots \mathbf{k}_j' \ldots \mathbf{k}_{N_1})$$

$$\times \delta(\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_i' - \mathbf{k}_j')\Phi_2^2(-\mathbf{p}, \mathbf{q})\varphi_2^2(\mathbf{p})\, d\mathbf{k}\, d\mathbf{q}\, d\mathbf{k}_i' d\mathbf{k}_j'$$

$$- D \sum_{\substack{i,j=1 \\ i\neq j}}^{N_2} \int_{(T^d)^{N+1}} \Phi_2(\mathbf{q}_1 \ldots \mathbf{q}_i \ldots \mathbf{q}_j \ldots \mathbf{q}_{N_2})$$

$$\times \Phi_2(\mathbf{q}_1 \ldots \mathbf{q}_i' \ldots \mathbf{q}_j' \ldots \mathbf{q}_{N_2}) \times$$

$$\times \delta(\mathbf{q}_i + \mathbf{q}_j - \mathbf{q}_i' - \mathbf{q}_j')\Phi_1^2(\mathbf{p}, \mathbf{k})\varphi_2^2(\mathbf{p})\, d\mathbf{k}\, d\mathbf{q}\, d\mathbf{q}_i' d\mathbf{q}'$$

$$- D \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_{(T^d)^{N+1}} \Phi_1(\mathbf{k}_1 \ldots \mathbf{k}_i \ldots \mathbf{k}_N)$$

$$\times \Phi_1(\mathbf{k}_1 \ldots \mathbf{k}_i' \ldots \mathbf{k}_N)\Phi_2(\mathbf{q}_1 \ldots \mathbf{q}_j$$

$$\ldots \mathbf{q}_N)\Phi_2(\mathbf{q}_1 \ldots \mathbf{q}_j' \ldots \mathbf{q}_N)\delta(\mathbf{k}_i + \mathbf{q}_j - \mathbf{k}_i' - \mathbf{q}')$$

$$\times \varphi_2^2(\mathbf{p})\, d\mathbf{k}\, d\mathbf{q}\, d\mathbf{k}_i' d\mathbf{q}_j'$$

$$= \int_{(T^d)^{N_1}} \{\langle \Phi_2 | H_{N_2} | \Phi_2 \rangle\} \varphi_2^2(\mathbf{p})\Phi_1^2(\mathbf{p}, \mathbf{k})\, d\mathbf{p}\, d\mathbf{k}$$

$$+ \int_{(T^d)^{N_2}} \{\langle \Phi_1 | H_{N_1} | \Phi_1 \rangle\} \varphi_2^2(\mathbf{p})\Phi_2^2(\mathbf{p}, \mathbf{q})\, d\mathbf{p}\, d\mathbf{q}$$

$$- N_1 N_2 D \Big(\int_{T^d} \varphi_2(\mathbf{p})\, d\mathbf{p} \Big)^2$$

$$= \int_{T^d} [E_{N_1}(\mathbf{p}) + E_{N_2}(-\mathbf{p})]\varphi_2^2(\mathbf{p})\, d\mathbf{p} - N_1 N_2 D \Big(\int_{T^d} \varphi_2(\mathbf{p})\, d\mathbf{p} \Big)^2. \quad (15)$$

In the second step we have used here the fact that in the first and third terms the sum of the integrals over $d\mathbf{k}(\mathbf{k} = \{\mathbf{k}_1 \ldots \mathbf{k}_{N_1-1}\})$ is $\langle \Phi_1 | H_{\mathbf{p}} | \Phi_1 \rangle = E_{N_1}(\mathbf{p})$, and in the second and fourth terms the sum of the integrals over $d\mathbf{q}$ is $\langle \Phi_2 | H_{-\mathbf{p}} | \Phi_2 \rangle = E_{N_2}(-\mathbf{p})$. In the calculation of the fifth term we have taken from under the integrals over $\mathbf{p}$ and $\mathbf{p}'$ the expressions

$$\Phi_1^2(0, \mathbf{k}_1 \ldots \mathbf{k}_{N_1-1})\Phi_2^2(0, \mathbf{q}_1 \ldots \mathbf{q}_{N_2-1}),$$

neglecting terms of higher order of smallness in α. In the third step we used the wave-function normalization condition. We have arrived at a two-particle estimate and found that an N-magnon BS exists at $\mathbf{K}=0$ and $d=1,2$ if there exist 2-, 3-,..., $(N-1)$-magnon BS at $d=1,2$ and at small $|\mathbf{K}|$. From the existence of an N-magnon BS at $\mathbf{K}=0$ follows its existence at small $|\mathbf{K}|$ by virtue of the general perturbation-theory theorems (it is easy to verify that the entire reasoning is independent of the dimensions of the existence region). The existence of such BS follows hence by induction.

The physical meaning of the method employed is simple. The BS is represented as an effective quasiparticle formed by intracluster interactions. Two such quasiparticles interact effectively, so that the interaction formed by the intercluster forces turns out to be attractive. Then, by the two-particle criterion, the two effective quasiparticles become bound, and an N-magnon BS exists by induction at $d=1,2$ and $\mathbf{K}=0$.

4. We turn now to the case $d=3$. A small number of magnons does not become bound at large values of S.[13] Let us show that a sufficiently large number can be bound and let us estimate this number. We assume that at $\mathbf{K}=0$ there are no BS of 2,...,$N-1$ magnons. The lower boundary of the continuous N-particle spectrum is then

$$\Sigma_N(0) = N \min_{\mathbf{k}} \varepsilon(\mathbf{k}) = -NSJ.$$

The energy origin is chosen here such that

$$\varepsilon(0) = -S \sum_{\rho} J_\rho = -SJ, \quad J_0 = 0.$$

We choose the N-magnon trial function in the form

$$\Phi_N(n_1 \ldots n_N) = \prod_{i=1}^{N} \delta_{n_i,0}, \quad (16)$$

i.e., all the spin extensions are on one site. Using expression (4), we get

$$\langle \Phi_N | H | \Phi_N \rangle = -DN(N-1)/2. \quad (17)$$

The criterion for the appearance of an N-magnon BS is

$$\langle \Phi_N | H | \Phi_N \rangle \langle \Sigma_N(0) < -NSJ,$$
$$N > (SJ/D + 1). \quad (18)$$

The estimate (18) shows that a BS appears at a certain $N_c \sim SJ/D$ (this is an upper-bound estimate). The reasoning in Ref. 19, which can be used without change for the case of a lattice, makes it possible to prove that there exists an infinite sequence of natural numbers $N_c < N_1 < N_2 < \ldots$ such that N_c-, N_1-, ... -magnon BS exist.

5. It is of interest to compare the results with the soliton solutions of the classical models and with their semiclassical interpretation. The quantum-mechanical treatment undertaken in the present paper predicts the existence of weakly bound states, and consequently, in accord with the generally accepted treatment, also of solitons, which are not revealed by the semiclassical theory.

How is this explained? It can be assumed that the energy of two-dimensional BS is exponentially small at $N < N_2$ (this is known exactly for $N = 2$), and therefore the semiclassical theory does not "catch" such BS. Another treatment is also possible. The transition from the quantum many-particle problem on a lattice to the classical continual nonlinear problem is connected with a transition from a lattice to a continuum for weakly localized states also with use of the mean-field approximation and the semiclassical approach. This means that if we wish to consider the exact problem of N spin extensions within the framework of this very same approximation, we must solve the problem of binding one magnon in a field of $N-1$ magnons in a continuous space and in the semiclassical approximation (and pair interaction of magnons, to be specific,—a spherical potential well of unit radius and of depth D). The problem of the discrete levels of a particle in an external field $V(\mathbf{r})$, i.e., of the number of BS in the semiclassical approximation, has been solved in Ref. 21, Ch. VII) for $d = 3$, but the solution can be easily extended to the cases $d = 1$ and 2. The number of levels is

$$p = C_d m^{d/2} \int [-V(\mathbf{r})]^{d/2} d\mathbf{r}, \quad C_d \sim 1,$$

i.e., if

$$m^{d/2} \int [-V(\mathbf{r})]^{d/2} d\mathbf{r} \leqslant 1$$

there are no BS. In our case the potential $V(\mathbf{r})$ is produced by $N-1$ magnons, and since the BS are weakly localized, the potential wells corresponding to the magnons do not overlap.

In the case $d = 2$ we have then

$$\int -V(\mathbf{r}) d\mathbf{r} = 4\pi D(N-1),$$

i.e., the criterion for the onset of BS is

$$mD(N-1) \geqslant 1.$$

Since $m \sim (SJ)^{-1}$, it follows that $N_2 \gtrsim (SJ/D + 1)$ coincides with the result for solitons.

In the case $d = 3$

$$\int [-V(\mathbf{r})]^{3/2} d\mathbf{r} = \frac{4\pi}{3} D^{3/2}(N-1),$$

$$N_3 \geqslant \left(\frac{SJ}{D}\right)^{3/2} + 1$$

also coincides with the result for solitons. The agreement between the quantum and classical results for $d = 1$ is apparently due to the exact integrability of the considered one-dimensional models.[16]

6. We note in conclusion that the above proofs of the existence of BS can be easily used for the following cases.

1) Special cases of dispersion laws: $\varepsilon(\mathbf{k}) \propto \mathbf{k}^{2n}$ or $E_j(\mathbf{K}) \propto \mathbf{K}^{2n}$ for small $|k|$ and $|\mathbf{K}|$, and $n = 2,3,\ldots$. The trial two-particle function must be chosen here in the form $\varphi_2(k) \propto (k^{2n} + \alpha^2)^{-1}$. Then,

$$\langle \varphi_2 | H_\mathbf{k} | \varphi_2 \rangle \sim \begin{cases} \alpha^{-(2-1/n)} - D\alpha^{-2(2-1/n)}, & d=1, \\ \alpha^{-(2-2/n)} - D\alpha^{-2(2-2/n)}, & d=2, \end{cases} \tag{19}$$

and at a sufficiently small value of α we have $\langle \varphi_2 | H_k | \varphi_2 \rangle < 0$. The rest of the proof is similar to that given above.

2) The non-symmetric case:

$$\varepsilon(\mathbf{k}) \propto \sum_{i=1}^{d} \frac{\mathbf{k}_i^2}{2m_i}, \quad m_i \neq m_j, \quad i \neq j.$$

The following hypotheses, for which no proofs have been obtained as yet, seem likely:

1) BS that exist for $\mathbf{K} = 0$ exist for all values of $\mathbf{K}$;

2) in the case $d = 3$, BS exist for all N larger than a certain critical N_c.

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