Multidimensional Hamiltonian chaos

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In Hamiltonian systems which are close to nondegenerate integrable systems, Arnol'd diffusion does not arise for two degrees of freedom, and for a large number of degrees of freedom it is, in general, exponentially small. If the unperturbed system is degenerate, then diffusion may appear even for two degrees of freedom, leading to a stochastic spider web. It is shown in the paper that in this case the introduction of additional degrees of freedom may lead to a sharp increase of the diffusion rate and of the measure of the chaotic component of phase space, owing to a destruction of the stochastic spider web. These results were obtained for the problem in $2\frac{1}{2}$ degrees of freedom of the motion of a charged particle in a magnetic field and in a wave packet propagating at an angle relative to the magnetic field.

1. INTRODUCTION

The concept of multi-dimensionality has a completely clear definition in the theory of dynamical systems. On the one hand, it is related to the minimal number of degrees of freedom for which a chaotic dynamical system is possible, in principle. On the other hand it is related to the topological properties of the phase space (see, e.g., Ref. 1). Motion in one degree of freedom ($N = 1$) is integrable. Therefore chaos arises for $N > 1$. We consider a perturbation of an integrable system with the Hamiltonian

$$H = H_0 + \varepsilon V,$$

where V is a generic perturbing potential and ε is a dimensionless small parameter ($\varepsilon \ll 1$). In accord with the Kolmogorov-Arnol'd-Moser (KAM) theory, the majority of invariant tori are slightly deformed but retain their principal property, namely to be invariant (see, e.g., Ref. 2). The KAM theory does not address the problem of what happens with the tori which are not preserved and destroyed by the perturbation. However, the theory implies that the measure of the destroyed tori tends to zero for $\varepsilon \to 0$. It is now known that the destruction of the tori is related to the appearance of a region of chaotic dynamics—stochastic leaves and stochastic spider webs (see the review in Ref. 3). Therefore for small ε the measure of the chaotic regions is small and this situation corresponds to weak chaos.

A more delicate problem is that of the topological organization of the chaos zones in phase space. An elementary chaotic region is a stochastic leaf which is formed in the place of a destroyed separatrix (Refs. 1,3). Various stochastic leaves need not, in general, form a single connected net of chaotic dynamics (Fig. 1a). However, sometimes this does happen, and the phase space appears covered by a stochastic web (Fig. 1b).

Individual stochastic leaves can exist within the cells of the spider web, as well as partial spider webs with boundaries which are not connected to the fundamental, global, spider web.

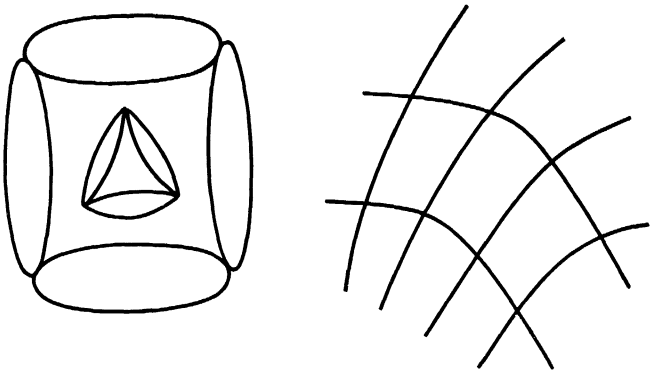
The stochastic spider web plays an important physical role. Unbounded random walks of particles in regular dynamical systems are possible along the channels of the web. The possible existence of the web and the reasons for its appearance were first pointed out by Arnol'd (Ref. 4). He has shown that when the KAM-theory is valid, for $N > 2$ there appears a global spider web. Therefore for $N > 2$ a diffusive motion of a fraction of the particles becomes possible in all of phase space (Arnol'd diffusion). The thickness of the web is exponentially small, of order $\exp(-\text{const}/\sqrt{\varepsilon})$.

One of the reasons for this is related to the topology of phase space, since for $\varepsilon = 0$ and $N \leqslant 2$ the invariant tori of the Hamiltonian H_0 separate the constant-energy hypersurface in phase space (for details see Ref. 3), whereas for $N > 2$ they do not separate it. A second reason is related to the condition that the Hamiltonian H_0 should be nondegenerate, which, for instance in the case of a perturbation potential V which depends periodically in time, has the following form:

$$|\partial^2 H_0/\partial I_j \partial I_k| \neq 0,\ j,\ k = 1, \dots, M,\ N = M + {}^1/_2, \qquad (1.1)$$

where I_j is the set of action variables of the system, $H_0 = H_0(I_1, \dots, I_M)$. In Ref. 5 one can find strict upper bounds on the characteristic diffusion velocity along the channels of the stochastic spider web, and various physical estimates and examples can be found in Refs. 6–8.

A new situation was considered in a series of papers, Refs. 9–12 (see also the review 3). It is related to a group of problems in which the condition (1.1) is not satisfied. In-



a b

FIG. 1. Stochastic leaves do not form (a) or form (b) the connected net of stochastic dynamics.

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stead the determinant vanishes

$$|\partial^2 H_0/\partial I_j\partial I_k|=0, \quad j,\ k=1,\ldots,M. \tag{1.2}$$

Since the frequencies of the nonlinear oscillations are determined by the equations

$$\omega_j=\partial H_0/\partial I_j,\ j=1,\ldots,M,$$

the condition (1.2) is satisfied, in particular, if the unperturbed Hamiltonian H_0 describes linear oscillations. In this case the resonances which are due to the action of the perturbation manifest themselves particularly intensely, and the motion for $\varepsilon=0$ cannot be used as the zeroth approximation. The analysis in the indicated papers showed that, just as in the case when the KAM theory is applicable, weak chaos arises for $N>1$. This means that there exist stochastic leaves and stochastic webs of small measure, a measure which tends to zero as $\varepsilon\to 0$. However, now a global stochastic spider web exists for $N>1$, if the resonance conditions between the perturbation and the unperturbed motion are satisfied.

A stochastic spider web for $N=1\frac{1}{2}$ ($\frac{1}{2}$ degree of freedom corresponds to a perturbation which is periodic in time) has a definite symmetry, and this allows one to make significant progress in the analysis of the stochastic dynamics of the particles along the spiderweb. In Refs. 9 and 11 estimates were obtained for the thickness of the web (see also Ref. 3), under the condition (1.2). The thickness is proportional to $\exp(-\text{const}/\sqrt{\varepsilon})$. This also predetermines a not too fast diffusion along the web. However, the addition of new degrees of freedom may completely change the general picture of weak chaos. This new peculiarity of chaos is the subject matter of the present paper.

This paper considers the concrete physical problem of the motion of a particle in a magnetic field and the field of a wave packet propagating under an angle to the magnetic field. The problem has an immense number of applications in plasma physics (see the review in Ref. 13), and in this case the problem of diffusion is particularly important in this case. The special interest of the problem under investigation notwithstanding, its formal contents and results have a universal meaning for the general problem of the appearance of chaos in multidimensional systems.

The main peculiarity of the system under investigation is the following. For resonant motion of a particle in a magnetic field and the field of a wave packet which is perpendicular to it ($1\frac{1}{2}$ degrees of freedom), there appears a stochastic web, since the condition (1.1) of the KAM theory is not satisfied. The diffusion along the web is relatively slow. For an oblique propagation of the wave packet one additional degree of freedom is added, since a motion longitudinal relative to the magnetic field is included. Even a small perturbation of the particle dynamics transverse to the magnetic field on account of the longitudinal motion leads to a sharp increase of the chaotic region and an enhancement of the diffusion. The new system corresponds to a number of degrees of freedom $N=2\frac{1}{2}$. Very small perturbations due to the longitudinal motion, of the order of $\approx 10^{-3}$–10^{-4}, lead to nonexponentially slow diffusion of the particles, as was the case for Arnol'd diffusion for $N>2$. The cause of this is the existence of a residual web for a part of the degrees of freedom. This shows that real chaos in multidimensional systems can be sufficiently strong if $N>2$ and if there exists partial degeneracy (i.e., with respect to part of the variables) caused by a condition of the type (1.2).

In Sec. 2 the fundamental map for the problem is derived. It is of fourth degree. Section 3 contains a qualitative analysis of some important cases and lists numerical results. Sections 4–7 contain the results of an analytic investigation of different interesting physical cases.

2. DERIVATION OF THE MAP

The initial equations of motion in the field of a wave packet propagating under an angle relative to a constant magnetic field have the form

$$\ddot{\mathbf{r}}=\frac{e}{m_0}\mathbf{E}(\mathbf{r},t)+\frac{e}{m_0c}[\dot{\mathbf{r}},\mathbf{B}_0], \tag{2.1}$$

where $\mathbf{B}_0$ is along the z axis. The electric field $\mathbf{E}(\mathbf{r},t)$ is in the xz plane and is chosen in the following form

$$\mathbf{E}(\mathbf{r},t)=-\mathbf{E}_0\sum_{n=-\infty}^{\infty}\sin(\mathbf{kr}-n\Delta\omega t)$$
$$=-\mathbf{E}_0T\sin(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{2.2}$$

where, as was done in Ref. 9, it was assumed that the wave packet is homogeneous and of sufficiently large spectral width. The time interval $T=2\pi/\Delta\omega$ is determined by the frequency interval between the harmonics of the packet. On account of the assumed potential character of the electric field, the wave vector $\mathbf{k}$, as well as the amplitude vector $\mathbf{E}_0$ has only two components k_x and k_z related by

$$k_z/k_x=E_{0z}/E_{0x}=\beta=\text{const}. \tag{2.3}$$

We write out Eq. (2.1) in components, taking into account the representation (2.2) for the electric field:

$$\begin{aligned}\ddot{x}&=-\frac{e}{m_0}TE_{0x}\sin(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT)+\omega_0\dot{y},\\ \ddot{y}&=-\omega_0\dot{x},\\ \ddot{z}&=-\frac{e}{m_0}TE_{0z}\sin(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT),\end{aligned} \tag{2.4}$$

where $\omega_0=eB_0/mc$ is the cyclotron frequency. The second equation in the system (2.4) can be integrated once, yielding

$$\dot{y}+\omega_0x=\text{const}. \tag{2.5}$$

On account of this constant of the motion, the equations of motion (2.4) of the particle reduce to a system of two equations:

$$\begin{aligned}\ddot{x}+\omega_0{}^2x&=-\frac{e}{m_0}TE_{0x}\sin(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT),\\ \ddot{z}&=-\frac{e}{m_0}TE_{0z}\sin(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT),\end{aligned} \tag{2.6}$$

where the constant in Eq. (2.5) may be set equal to zero without loss of generality. In the case of oblique propagation of the wave packet ($k_x, k_z\neq 0$), it follows from the equations of motion (2.6) that the longitudinal and transverse degrees of freedom are coupled. The Hamiltonian of the system (2.6) has the following form:

$$H=\frac{1}{2m_0}(p_x^2+p_z^2)$$
$$+\frac{m_0}{2}\omega_0^2x^2-e\varphi_0T\cos(k_xx+k_zz)\sum_{n=-\infty}^{\infty}\delta(t-nT), \qquad (2.7)$$

where $\varphi_0=E_{0x}/k_x=E_{0z}/k_z$ is the amplitude of the potential of the electric field, and p_x and p_z are the respective components of the particle momentum:

$$p_x=m_0\dot{x}, \quad p_z=m_0\dot{z}. \qquad (2.8)$$

In place of the system of differential equations (2.6) one can write a finite-difference system. Between two successive actions of the delta-functions the trajectory of the particle satisfies the equations

$$\ddot{x}+\omega_0^2x=0, \ \ddot{z}=0. \qquad (2.9)$$

As they pass through the delta function at the time $t_n=nT$ the solutions (2.9) must satisfy the boundary conditions

$$x(t_n+0)=x(t_n-0), \quad z(t_n+0)=z(t_n-0),$$
$$\dot{x}(t_n+0)=\dot{x}(t_n-0)-\frac{e}{m_0}TE_{0x}\sin[k_xx(t_n)+k_zz(t_n)],$$
$$\dot{z}(t_n+0)=\dot{z}(t_n-0)-\frac{e}{m_0}TE_{0z}\sin[k_xx(t_n)+k_zz(t_n)]. \qquad (2.10)$$

With the help of these equations we obtain from (2.6)

$$\dot{x}_{n+1}=-\omega_0x_n\sin\omega_0T+\left[\dot{x}_n-\frac{eE_{0x}}{m_0}T\sin(k_xx_n+k_zz_n)\right]\cos\omega_0T,$$
$$x_{n+1}=x_n\cos\omega_0T+\left[\dot{x}_n-\frac{eE_{0x}}{m_0}T\sin(k_xx_n+k_zz_n)\right]\sin\omega_0T, \qquad (2.11)$$
$$\dot{z}_{n+1}=\dot{z}_n-\frac{eE_{0z}}{m_0}T\sin(k_xx_n+k_zz_n), \quad z_{n+1}=z_n+T\dot{z}_{n+1},$$

where we have denoted:

$$\dot{x}_n=\dot{x}(nT-0), \quad x_n=x(nT-0),$$
$$\dot{z}_n=\dot{z}(nT-0), \quad z_n=z(nT-0).$$

Going over to more convenient dimensionless variables

$$u_n=k_x\dot{x}_n/\omega_0, \ v_n=-k_xx_n,$$
$$w_n=k_z\dot{z}_n/\omega_0, \quad Z_n=k_zz_n, \qquad (2.12)$$

we can rewrite (2.11) in the following form

$$u_{n+1}=v_n\sin\alpha+[u_n+K\sin(v_n-Z_n)]\cos\alpha,$$
$$v_{n+1}=v_n\cos\alpha-[u_n+K\sin(v_n-Z_n)]\sin\alpha,$$
$$w_{n+1}=w_n+K\beta^2\sin(v_n-Z_n), \quad Z_{n+1}=Z_n+\alpha w_{n+1}, \qquad (2.13)$$

with the notation

$$\alpha=\omega_0T, \quad K=\frac{eE_{0x}k_x}{m_0\omega_0}T. \qquad (2.14)$$

We now consider some extreme situations. If the wave packet propagates along the magnetic field ($E_{0x}=k_x=0$), the electric field of the packet does not influence the Larmor rotation of the particles in a plane perpendicular to the magnetic field, i.e., the longitudinal and transverse degrees of freedom decouple. The system (2.13) reduces to two independent maps. The first of these:

$$u_{n+1}=v_n\sin\alpha+u_n\cos\alpha, \quad v_{n+1}=v_n\cos\alpha-u_n\sin\alpha,$$

describes a simple rotation of the particle in the magnetic field. The second one has the form

$$w_{n+1}=w_n-K\beta^2\sin Z_n, \quad Z_{n+1}=Z_n+\alpha w_{n+1} \qquad (2.15)$$

and describes only the longitudinal motion along the magnetic field. After the substitution $\alpha w=I$ it reduces to the standard Chirikov map[6] with the nonlinearity parameter

$$K_0=\alpha\beta^2K=\frac{eE_{0z}k_z}{m_0}T^2.$$

In the other extreme case, when the wave packet propagates strictly perpendicular to the magnetic field ($E_{0z}=k_z=0$), the longitudinal motion is free ($\beta=0$) and the transverse one is described by the map with a twist (Ref. 9):

$$u_{n+1}=v_n\sin\alpha+(u_n+K\sin v_n)\cos\alpha,$$
$$v_{n+1}=v_n\cos\alpha-(u_n+K\sin v_n)\sin\alpha. \qquad (2.16)$$

This map is the generator of the stochastic web with a symmetry of order q, if the resonance condition $\alpha=\alpha_q$ is satisfied, where

$$\alpha_q=2\pi/q \qquad (2.17)$$

and q is an integer. The condition (2.17) means that over a full rotation of the particle in the magnetic field it experiences exactly q kicks from the wave field. If (2.17) is satisfied in the case of strictly transverse propagation of the wave packet, the phase plane (u,v) of the map (2.16) is covered by the stochastic web for arbitrarily small K (Refs. 9,10). We derive below the conditions for the conservation of the stochastic web for nonorthogonal propagation of the wave packet, and investigate the metamorphoses of the phase portrait as the parameters of the system are changed.

In conclusion of this section we represent the Hamiltonian (2.7) making use of the dimensionless variables $(U,v;Z,w)$ defined in Eq. (2.12). We have

$$\mathcal{H}=\mathcal{H}(u,v;Z,w)$$
$$=\frac{v^2+u^2}{2}+\frac{1}{2\beta^2}w^2-K\cos(v-Z)\sum_{n=-\infty}^{\infty}\delta(\tau-n\alpha), \qquad (2.18)$$

where we have introduced the dimensionless time

$$\tau=\omega_0t, \qquad (2.19)$$

and the Hamiltonian equations of motion

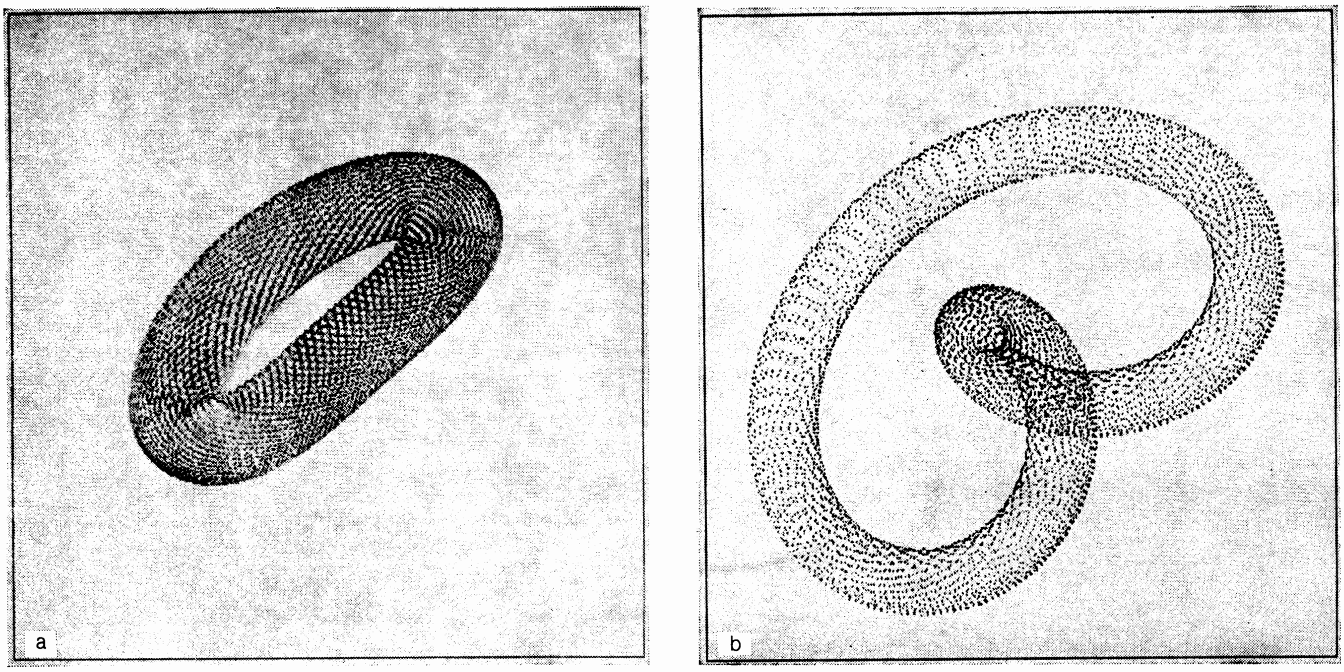
$$\frac{dv}{d\tau}=-\frac{\partial\mathcal{H}}{\partial u}, \quad \frac{du}{d\tau}=\frac{\partial\mathcal{H}}{\partial v},$$
$$\frac{dw}{d\tau}=-\beta^2\frac{\partial\mathcal{H}}{\partial Z}, \quad \frac{dZ}{d\tau}=\beta^2\frac{\partial\mathcal{H}}{\partial w}. \qquad (2.20)$$

One can easily verify that the system (2.20) leads to the map (2.13).

3. RAPID DIFFUSION (QUALITATIVE ANALYSIS AND SOME RESULTS)

In this section we describe a preliminary qualitative analysis which shows how the addition of one degree of freedom can lead to an acceleration of the diffusion of particles throughout phase space.

The map (2.13) has, in particular, trajectories which



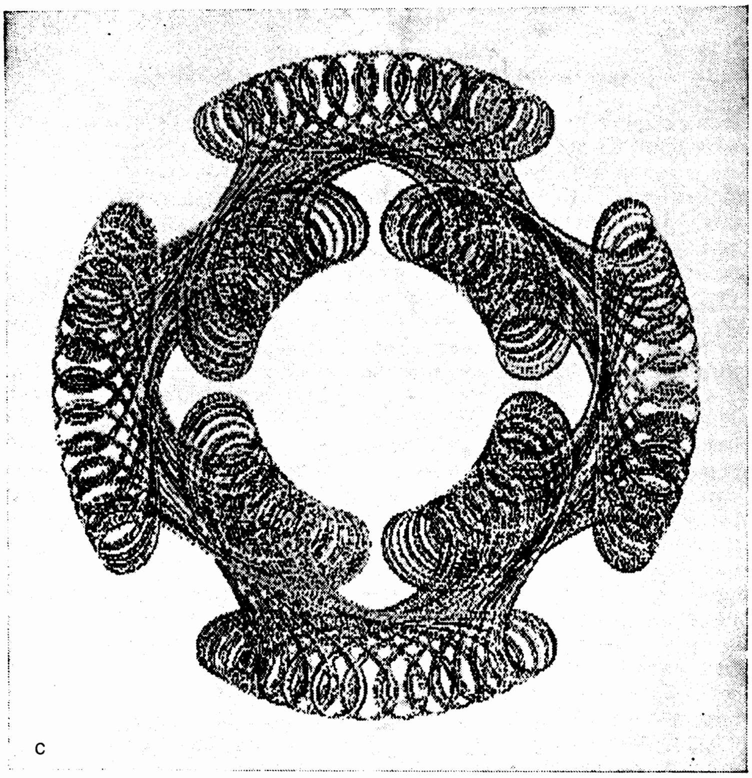


FIG. 2. Various examples of trajectories which are windings of the invariant submanifolds [the plane (u,v)]: a,b—$K = 0.05$, $\beta = 0.0001$, $w_0 = 0.47$, the size of the square is $(0.05\pi)^2$; c—$K = 0.001$, $\beta^2 = 1.0$, $w_0 = 1.0$, the size of the square is $(0.2\pi)^2$.

are windings of the invariant submanifolds. Examples are given in Fig. 2 and will not be considered further. We discuss in more detail the case which is close to a two-dimensional web. For instance, let there exist a resonance of fourth order, i.e., in Eq. (2.17) $q = 4$, and $\alpha = \pi/2$. We shall consider the parameter K in Eq. (2.13) to be small. Then for $w_0 = 0$, $Z_0 = 0$ and $\beta = 0$ there occurs a two-dimensional version of Eq. (2.16), exhibiting a two-dimensional web with the symmetry of a square lattice (Refs. 9,10,3), Fig. 3a. We now take into account the perturbation in (2.13) for small $\beta \ll 1$. It follows from the third equation in (2.13) that w starts to vary slowly and that on the average $|w|$ increases over a sufficiently long time. At the same time there appear slow variations of the magnitude of Z. Thus one may consider the first pair of equations (2.13) with a slowly varying parameter Z. These changes lead to a slow drift of the orbits of the two-dimensional mapping (2.16). The simplest form of the reasoning is based on the fact that, for example, while unwinding slowly, the orbit intersects the stochastic web. Its further path proceeds in the form of random wanderings inside the web, until the same slow drift leads the orbit inside the cell. Thus the effective width of the web becomes considerably larger than $\beta = 0$.

Another path of reasoning turns out in the sequel to be

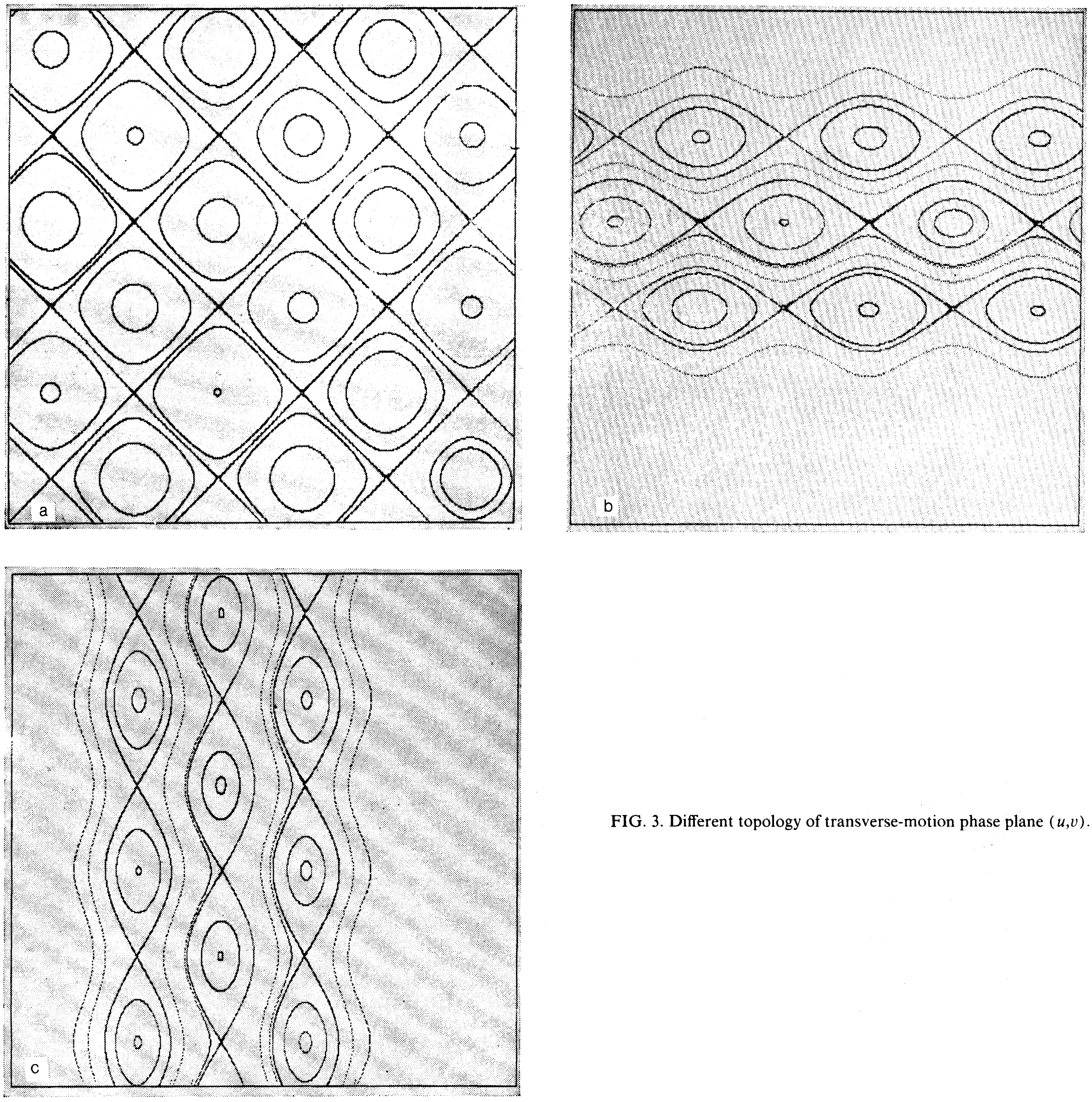


FIG. 3. Different topology of transverse-motion phase plane (u,v).

more effective. We consider, for instance, a resonance of fourth order, i.e., we put in the system (2.13) $\alpha = 2\pi/4 = \pi/2$. For small values of β and K the change in the longitudinal momentum w is small. Integer values $w = s$ signify in dimensional units that the condition for longitudinal cyclotron resonance $k_z \dot{z} = s\omega_0$ are satisfied.

Owing to the smallness of the changes in w the resonance lasts a long time. Upon a fourfold iteration of the map (2.13) with $w = s = \text{const}$, the quantity Z returns to its initial value (modulo 2π), and for (u,v) there appears a mapping depending on Z as a parameter. The phase portrait of this map for $s = 1$ for Z in the interval $(0,\pi/2)$ changes approximately from that depicted in Fig. 3c to the form in Fig. 3b. For $Z = \pi/4$ it passes through the structure in Fig. 3a. The slow motion of the stochastic leaf in the vicinity of all separatrices effectively magnifies its thickness, which reaches values of the order of unity (!), in spite of the fact that the perturbation is of order $\beta^2 \sim 10^{-4}$–10^{-6} (Fig. 4a). At the same time long motions of the particle in the plane (u,v) are possible (Fig. 4b,c). Such "jumps" may have a characteristic length of up to 10^3 cells or more. They are called Lévy jumps (for Lévy random walks, see the review, Ref. 14). Random walks accompanied by Lévy jumps were discovered in systems with dynamic chaos in Refs. 15,16. It is because of this character of the stochastic dynamics that strong intermittency occurs. Partially, the acceleration of diffusion is accompanied by Lévy jumps.

If the initial value of w_0 is situated in some neighborhood of the singular values $w_0 = 0, \pm 1, \ldots$, but not too close to them, then for small β there occur slow quasi-regular changes of the quantity Z, with nonzero average. This may lead to a drift of the whole picture represented in Fig. 3a, i.e.,

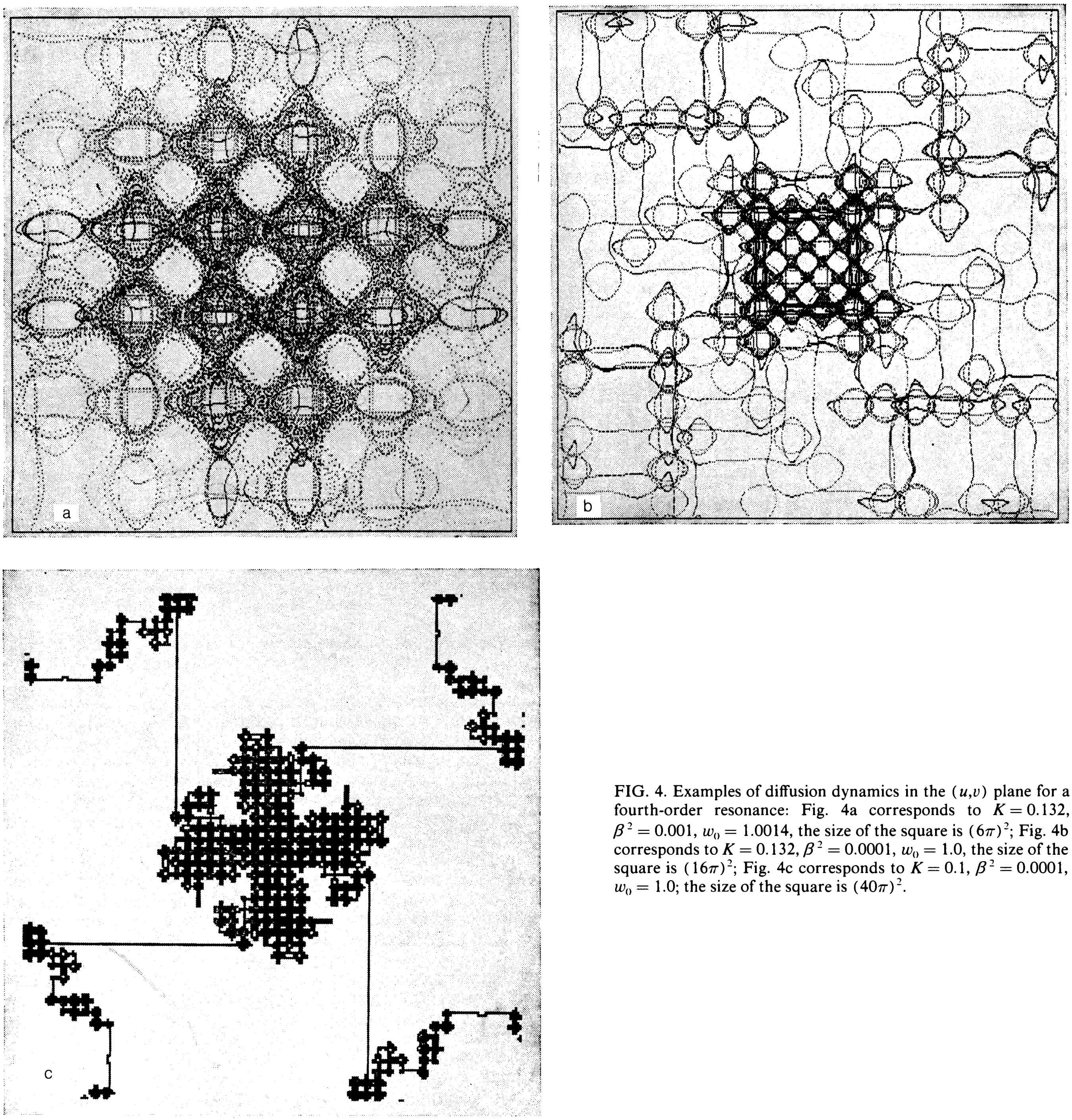


FIG. 4. Examples of diffusion dynamics in the (u,v) plane for a fourth-order resonance: Fig. 4a corresponds to $K = 0.132$, $\beta^2 = 0.001$, $w_0 = 1.0014$, the size of the square is $(6\pi)^2$; Fig. 4b corresponds to $K = 0.132$, $\beta^2 = 0.0001$, $w_0 = 1.0$, the size of the square is $(16\pi)^2$; Fig. 4c corresponds to $K = 0.1$, $\beta^2 = 0.0001$, $w_0 = 1.0$; the size of the square is $(40\pi)^2$.

of the original web. As a result of this the web as a whole is destroyed, and only separate small parts conserve the structure of an arbitrarily oriented square lattice (Fig. 5). As before, the diffusion remains rapid.

We present a few more examples for other resonances. For $q = 3$ fast diffusion is also possible, with preservation of the web in the four-dimensional phase space. An example is shown in Fig. 6. It is analogous to the picture in Fig. 4 for $q = 4$. A more complicated picture appears for $q = 5$. Over a long time ($\sim 1/\beta$) the motion occurs in the potential well of one cell of the web, until the drifting particle intersects the web. After that it will undergo a diffusive motion along the web with a weak manifestation of a symmetry of order 5 (Fig. 7) until again it leaves the net, etc. This is reminiscent of the picture represented in Fig. 4b.

Thus, the existence of a residual spider web due to the symmetry of the problem for $N = 1\frac{1}{2}$ leads to a sharp increase in the diffusion of particles if one more degree of freedom is added, i.e., the propagation of the wave packet is weakly perpendicular to the magnetic field.

The chaos which appears is structured. Significant portions of the trajectory have a weakly broken symmetry they had in the absence of the additional degree of freedom. Another possible variant is that a web with a symmetry of arbitrary order q is preserved also for $N = 2\frac{1}{2}$. Strong intermittency is a consequence of such random walks.

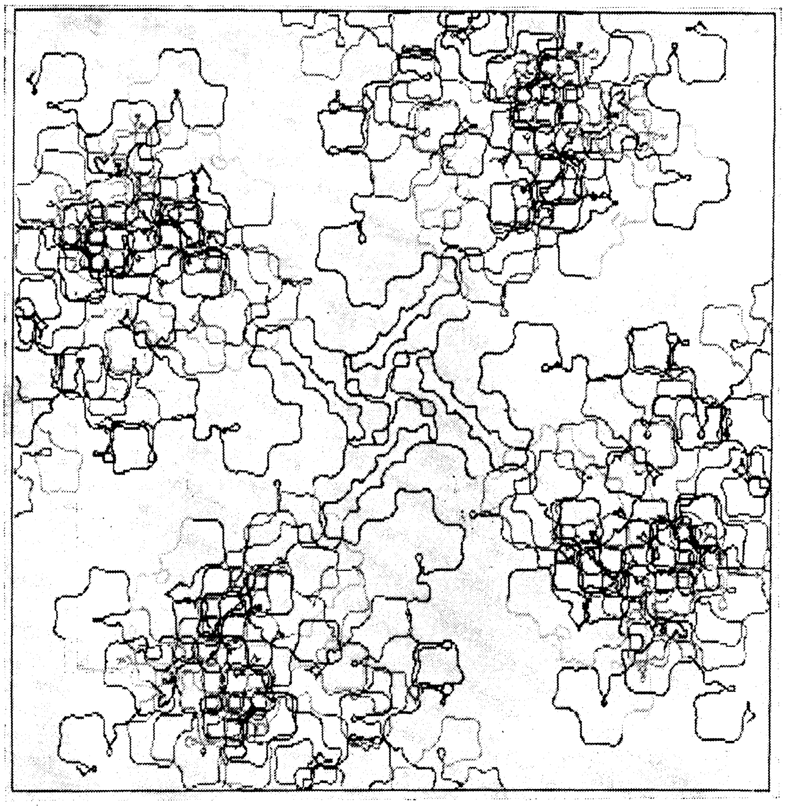


FIG. 5. Fast diffusion in the (u,v) plane for a fourth-order resonance: $K = 0.1, \beta^2 = 0.0001, w_0 = 1.01$, the size of the square is $(32\pi)^2$.

In the process of random walk for small values of β, i.e., weakly nonperpendicular propagation of the wave packet, the variations of w remain bounded. Therefore the phase space is stratified (foliated). In various leaves bounded by the plane (u,v), $Z\in(-\pi,\pi)$ and a small region of values of w, the properties of the dynamics may differ strongly, as can be seen, for example, in Fig. 4a,b.

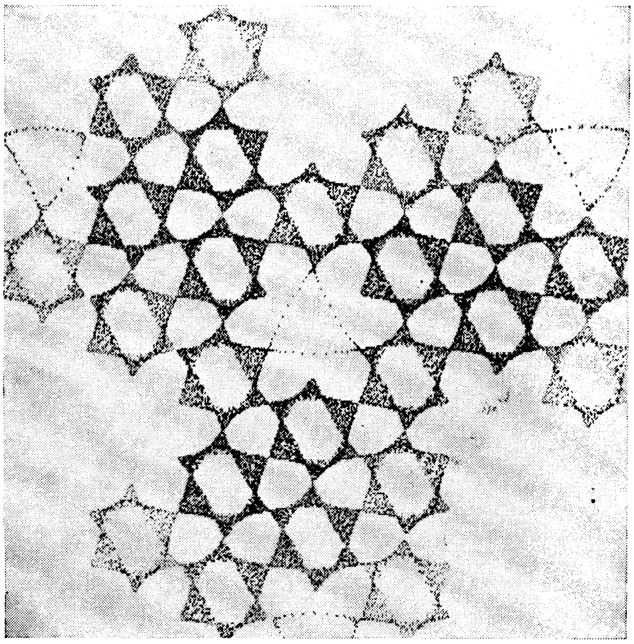


FIG. 6. Fast diffusion in the (u,v) plane for a third-order resonance: $K = 0.4, \beta^2 = 0.00001, w_0 = 3.0$; the size of the square is $(8\pi)^2$.

We consider in the sequel the theory of these and some other cases of dynamics.

4. THE RESONANCE HAMILTONIAN

If the resonance condition (2.17) is satisfied:

$$\Delta\omega = q\omega_0, \quad q=1, 2, \dots, \tag{4.1}$$

the Hamiltonian (2.7) can be rewritten in a more convenient form. For this purpose we make a canonical transformation from the variables (x,p_x) to the action–angle variables (J,θ) for the transverse degree of freedom:

$$x=\rho\sin\theta, \quad p_x=m_0\omega_0\rho\cos\theta, \quad J=m_0\omega_0\rho^2/2, \tag{4.2}$$

where ρ is the Larmor radius. With the help of the generating function

$$F=(\theta-\omega_0 t)I \tag{4.3}$$

a transformation is made to the new variables $J = I$, $\varphi = \theta - \omega_0 t$ in a coordinate system which rotates with the cyclotron frequency ω_0. In terms of these variables the Hamiltonian has the form

$$\bar{H}=H+\frac{\partial F}{\partial t}=\frac{1}{2m_0}p_z^2$$

$$-e\varphi_0 T\cos[k_x\rho\sin(\varphi+\omega_0 t)+k_z z]\sum_{n=-\infty}^{\infty}\delta(t-nT). \tag{4.4}$$

Following Ref. 17 we transform the delta-function series

$$\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{j=1}^{q}\sum_{m=-\infty}^{\infty}\delta(t-(mq+j)T). \tag{4.5}$$

Substituting this expression into Eq. (4.4) and making use of

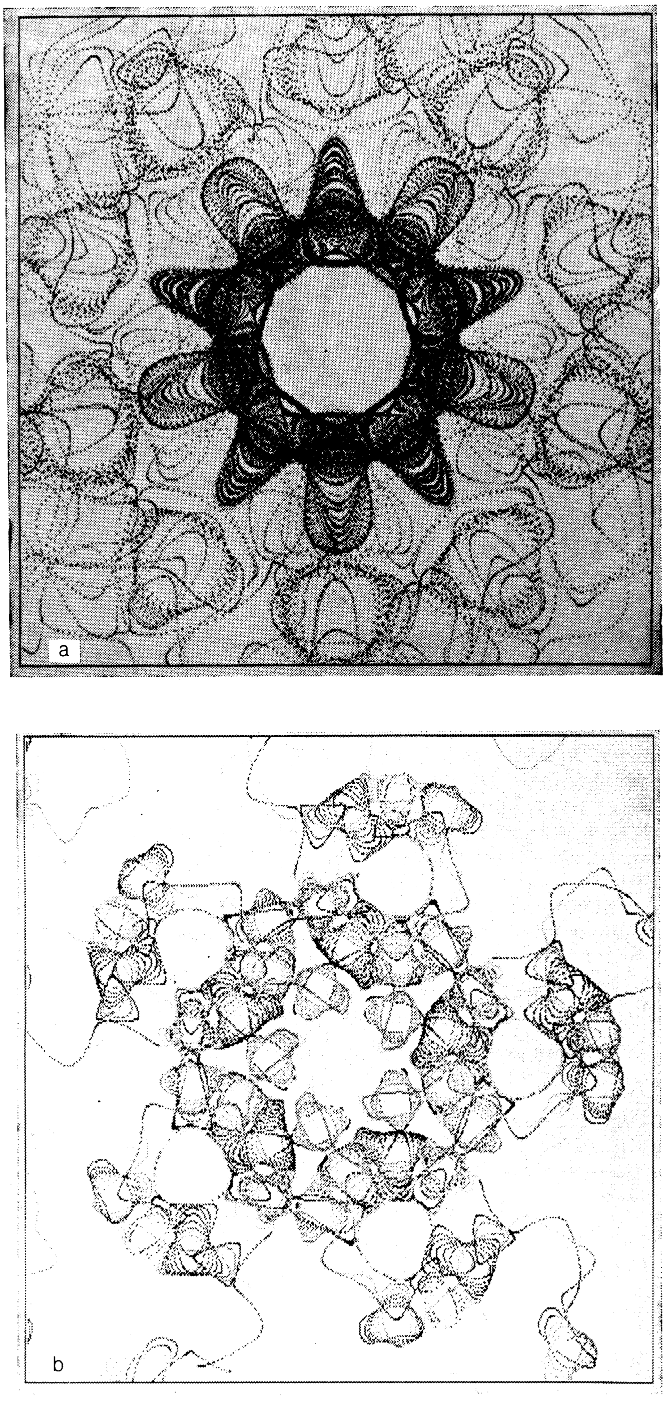


FIG. 7. Diffusion dynamics in the (u,v) plane for a fifth-order resonance: a) $K=0.132$, $\beta^2=0.00005$, $w_0=0$, size of the square $(8\pi)^2$; b) $K=0.132$, $\beta^2=0.00001$, $w_0=1.0$, size of the square $(16\pi)^2$.

the representation

$$\sum_{m=-\infty}^{\infty}\delta(t-(mq+j)T)=\frac{1}{qT}\sum_{m=-\infty}^{\infty}\exp\left(2\pi i m\frac{t/T-j}{q}\right), \quad (4.6)$$

we find the resonance Hamiltonian

$$\tilde{H}=\frac{1}{2m_0}p_z^2-\frac{e\varphi_0}{q}\sum_{j=1}^{q}\cos\left(k_x x\cos\frac{2\pi j}{q}+k_x\frac{p_x}{m_0\omega_0}\sin\frac{2\pi j}{q}+k_z z\right)$$
$$\times\left[1+2\sum_{m=1}^{\infty}\cos\left(m\omega_0 t-\frac{2\pi}{q}mj\right)\right]. \quad (4.7)$$

In the sequel we shall have use for the following representation of this Hamiltonian

$$\tilde{H}=H_q+V_q,$$

$$H_q=\frac{1}{2m_0}p_z^2-\frac{e\varphi_0}{q}\sum_{j=1}^{q}\cos\left(v\cos\frac{2\pi j}{q}-u\sin\frac{2\pi j}{q}-k_z z\right),$$

$$V_q=-2\frac{e\varphi_0}{q}\sum_{j=1}^{q}\cos\left(v\cos\frac{2\pi j}{q}-u\sin\frac{2\pi j}{q}-k_z z\right)$$
$$\times\sum_{m=1}^{\infty}\cos\left(m\omega_0 t-\frac{2\pi}{q}mj\right). \quad (4.8)$$

where (u,v) are the coordinates (2.12) in a plane perpendicular to the magnetic field. The expression for H_q will be called the mean Hamiltonian. It represents the expression for $\tilde{H}$, averaged over a period of the Larmor rotation. The mean Hamiltonian H_q describes the motion of a particle which differs little from the actual motion under the condition

$$\omega_0\gg\max\{k_z v_z,\ (e\varphi_0 k_x^2/m_0)^{1/2}\}. \quad (4.9)$$

In dimensionless units the condition (4.9) is equivalent to the following:

$$w\ll 1,\ \Omega_\perp/\omega_0\ll 1,$$

where we have introduced the transverse bounce-frequency

$$\Omega_\perp=(e\varphi_0 k_x^2/m_0)^{1/2}, \quad (4.9')$$

corresponding to the frequency of small oscillations of a particle in the field of a plane wave with amplitude φ_0.

The character of the mean motion is determined to a large degree by the number q of resonances. The values $q=1,2$ and $q=4$ belong to the category of trivial resonances, when the mean motion is exactly integrable. For example, for $q=4$ we obtain from Eq. (4.8)

$$H_4=\frac{1}{2m_0}p_z^2-\frac{e\varphi_0}{4}\sum_{j=1}^{4}\cos\left(v\cos\frac{\pi j}{2}-u\sin\frac{\pi j}{2}-k_z z\right)$$
$$=\frac{1}{2m_0}p_z^2-\frac{e\varphi_0}{2}\cos k_z z(\cos u+\cos v). \quad (4.10)$$

In dimensionless units $(u,v;Z,w)$ this Hamiltonian may be rewritten in a form analogous to Eq. (2.18):

$$\mathscr{H}_4=\frac{1}{2\beta^2}w^2-\frac{K}{\pi}\cos Z(\cos u+\cos v) \quad (4.10')$$

where $\mathscr{H}_4$ and H_4 differ from each other only by a constant factor. Hence the first pair of the equations of motion (2.20) takes on the form

$$\frac{dv}{d\tau}=-\frac{K}{\pi}\cos Z\sin u,\quad \frac{du}{d\tau}=+\frac{K}{\pi}\cos Z\sin v.$$

It follows from these equations that the quantity

$$\cos u+\cos v=\text{const}\equiv C_0$$

is an invariant of the motion. Therefore the second pair of the equations (2.20):

$$\frac{dw}{d\tau}=-\beta^2\frac{K}{\pi}C_0\sin Z,\quad \frac{dZ}{d\tau}=w,$$

reduces to the equation of the pendulum and is directly integrable. The terms V_4 which have been discarded destroy the separatrices of the motion in the (Z,w) plane and in the (u,v) plane. As a result a web is formed in four-dimensional space, exhibiting the symmetry of a square lattice and a periodic structure in the (Z,w) plane typical of a pendulum. Thus, the problem of stochastic spider web in the case (4.9) is easily solved.

In all other cases ($q \neq 1,2,4$) the averaged Hamiltonian corresponds to a nonintegrable Hamiltonian system, the dynamics of which is determined by the relation between the frequencies that characterize the longitudinal and transverse degrees of freedom.

If the following condition is satisfied in place of (4.9)

$$\frac{v_z}{(e\varphi_0/m_0)^{1/2}} \lesssim 1, \quad \frac{k_x}{k_z\omega_0}\left[\frac{e\varphi_0 k_x^2}{m_0}\right]^{1/2} \ll 1, \tag{4.11}$$

or in a different form

$$w\omega_0/\beta\Omega_\perp \lesssim 1, \quad \Omega_\perp/\beta\omega_0 \ll 1, \tag{4.11'}$$

then the problem allows for a relatively complete analytic investigation for arbitrary values of q. In this case the averaged transverse motion is slow compared to the longitudinal motion, and there exists an additional approximate first integral. In order to find it we write the expression (4.8) for H_q in the form of the Hamiltonian of a pendulum:

$$H_q = \frac{1}{2m_0} p_z^2 - A(u,v)\cos(k_z z - \Phi(u,v)), \tag{4.12}$$

with a slowly varying amplitude

$$A(u,v) = \frac{e\varphi_0}{q}\left\{\left[\sum_{j=1}^{q}\cos\left(v\cos\frac{2\pi}{q}j - u\sin\frac{2\pi}{q}j\right)\right]^2 + \left[\sum_{j=1}^{q}\sin\left(v\cos\frac{2\pi}{q}j - u\sin\frac{2\pi}{q}j\right)\right]^2\right\}^{1/2} \tag{4.13}$$

and phase

$$\Phi(u,v) = \operatorname{arctg}\sum_{j=1}^{q}\sin\left(v\cos\frac{2\pi}{q}j - u\sin\frac{2\pi}{q}j\right) \Big/ \sum_{j=1}^{q}\cos\left(v\cos\frac{2\pi}{q}j - u\sin\frac{2\pi}{q}j\right). \tag{4.14}$$

The slowness of the transverse motion leads to the appearance of the adiabatic invariant

$$J(H_q, A(u,v)) = \frac{1}{2\pi}\oint p_z\, dz = \frac{1}{2\pi}\oint [2m_0(H_q + A(u,v)\cos z)]^{1/2} dz \approx \text{const}. \tag{4.15}$$

Since $H_q = \text{const}$ on the trajectory, Eq. (4.15) leads to an approximate constant of the motion $A(u,v) = \text{const}$.

If the initial conditions (z,p_z) are taken not on the separatrix (4.12) of the pendulum, then the relation $A(u,v) = \text{const}$ means that the phase point does not get close to the separatrix, and the adiabatic condition is not violated.

From Eq. (4.13) we obtain that the additional constant of the motion, under the conditions (4.9) and (4.11), has

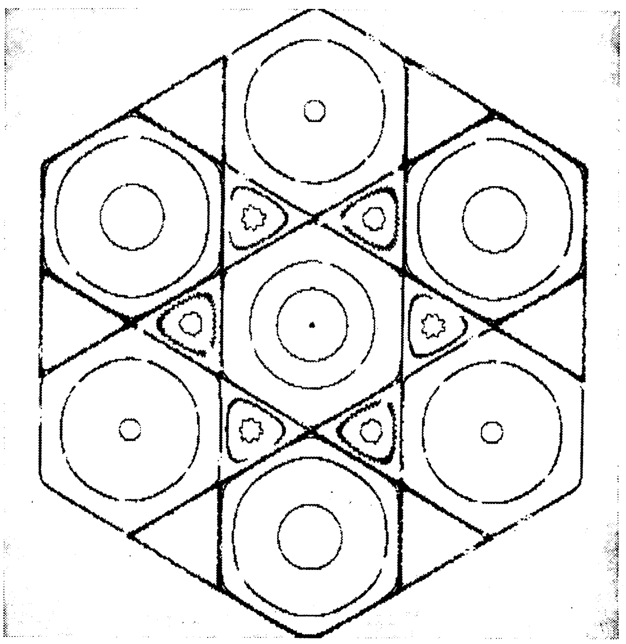


FIG. 8. Adiabatic motion in the (u,v) plane for a third-order resonance: $K = 0.001$, $\beta^2 = 1.0$, $w_0 = 0$, size of the square $(4\pi)^2$.

the form

$$\sum_{j=1}^{q}\sum_{i\neq j}^{q}\cos\left[v\left(\cos\frac{2\pi}{q}i - \cos\frac{2\pi}{q}j\right) - u\left(\sin\frac{2\pi}{q}i - \sin\frac{2\pi}{q}j\right)\right] = \text{const}. \tag{4.16}$$

In particular, for $q = 3$ we obtain from Eq. (4.16) we have

$$\cos\sqrt{3}u + \cos\left(\frac{\sqrt{3}}{2}u + \frac{3}{2}v\right) + \cos\left(\frac{\sqrt{3}}{2}u - \frac{3}{2}v\right) = \text{const}. \tag{4.17}$$

Figure 8 shows the trajectories (u_n, v_n) for several initial conditions with $q = 3$, when the conditions (4.9) and (4.11) are satisifed. The stochastic web can be seen near the separatrix level lines of the approximate constant of the motion (4.17).

An analytic investigation is possible also in the other extreme case when the second condition (4.11) is replaced by its opposite:

$$\frac{v_z}{(e\varphi_0/m_0)^{1/2}} \lesssim 1, \quad \frac{k_x}{k_z\omega_0}\left[\frac{e\varphi_0 k_x^2}{m_0}\right]^{1/2} \gg 1.$$

Here the averaged transverse motion is rapid compared to the longitudinal motion. The system has an adiabatic invariant. However, during the drift of the slow variables (z,p_z) the phase point may hit the separatrix of the fast motion in the (u,v) plane, and the adiabaticity conditions are violated. This leads to a destruction of the adiabatic invariance and a chaotization of the motion. The phenomena which occur here are discussed in detail in Sec. 6.

5. THE AVERAGED MOTION FOR CYCLOTRON RESONANCE

In this section we discuss in more detail the properties of the resonant Hamiltonian (4.8) for the condition of a longitudinal cyclotron resonance

$$k_z v_z = s\omega_0, \tag{5.1}$$

where s is an integer. For simplicity we restrict our attention to the case $q=4$, when the expression (4.8) for the resonant Hamiltonian can be represented in the form

$$H = \frac{1}{2m_0} p_z^2 - \frac{e\varphi_0}{2} \cos k_z z \times \Big\{ \cos u + \cos v + 2 \sum_{m=1}^{\infty} [\cos v + (-1)^m \cos u] \times \cos(2m\omega_0 t) \Big\} - e\varphi_0 \sin k_z z \sum_{m=1}^{\infty} \{(-1)^m \sin u \sin[(2m-1)\omega_0 t] + \sin v \cos[(2m-1)\omega_0 t]\}. \tag{5.2}$$

We assume that the motion of the particles along the magnetic field is close to the resonant motion (5.1) and that the cyclotron rotation is high-frequency:

$$\max\{|k_z v_z - s\omega_0|, (e\varphi_0 k_x^2/m_0)^{1/2}\} \ll \omega_0. \tag{5.3}$$

When these conditions are satisfied one can separate in the Hamiltonian (5.2) resonant terms, i.e., one can average under the conditions of cyclotron resonance. If the number s in Eq. (5.1) is even, $s=2m$, the corresponding resonant Hamiltonian has the form

$$\langle H \rangle = \frac{1}{2m_0} p_z^2 - \frac{e\varphi_0}{2} \cos(k_z z - 2m\omega_0 t) [\cos v + (-1)^m \cos u]. \tag{5.4}$$

In particular, for $m=0$ one obtains the expression (4.10). The mean motion described by the Hamiltonian (5.4) can be analyzed similarly to (4.1). It is regular, since the problem admits an additional first integral

$$\cos v + (-1)^m \cos u = \text{const}. \tag{5.5}$$

Taking into account the high-frequency terms omitted during the averaging leads to the result that the system of separatrices of the average motion in the (u,v)-plane is destroyed and chaos is generated in a small neighborhood of them. It follows from Eq. (5.5) that just as in the case of normal propagation of the wave packet, the system of separatrices in the (u,v) plane forms a single separatrix net, and the destruction of this net leads to the formation of the stochastic spider web also in the case of oblique propagation of the wave packet.

A different situation may appear if the number s in Eq. (5.1) is odd, i.e., if $s=2m-1$. The resonant part of the Hamiltonian (5.2) then has the form

$$\langle H \rangle = \frac{1}{2m_0} p_z^2 - \frac{e\varphi_0}{2} \{(-1)^m \sin u \cos[k_z z - (2m-1)\omega_0 t] + \sin v \sin[k_z z - (2m-1)\omega_0 t]\}. \tag{5.6}$$

In a coordinate system moving along the direction of the magnetic field with a velocity $v_z = (2m-1)\omega_0/k_z$, the Hamiltonian will correspond to a conservative system:

$$\langle \bar{H} \rangle = \frac{1}{2m_0} \tilde{p}_z^2 - \frac{e\varphi_0}{2} [(-1)^m \sin u \cos k_z \tilde{z} + \sin v \sin k_z \tilde{z}], \tag{5.7}$$

where

$$\tilde{z} = z - (2m-1)\omega_0 t/k_z, \quad \tilde{p}_z = p_z - m_0(2m-1)\omega_0/k_z. \tag{5.8}$$

In general, the system (5.7) is not integrable. It is simple to investigate the case when the conditions (4.11) are satisfied for $\tilde{p}_z$. Then the variables $\tilde{p}_z$ and $\tilde{z}$ are fast variables, and u and v are slow variables. Similar to Sec. 4, the adiabatic invariance of the "action" of the fast motion yields an approximate integral of the slow motion.

$$\cos 2u + \cos 2v = \text{const}.$$

The opposite limiting case when the variables $\tilde{p}_z$ and $\tilde{z}$ are slow and the variables u and v are fast, is considerably more complicated. It is analyzed in the following section.

6. MOTION IN AN ALMOST PERPENDICULAR WAVE PACKET

In this section we consider one of the most interesting cases of three-dimensional motion of particles, when the wave packet propagates almost perpendicularly to the magnetic field, i.e., the parameters $\beta = k_z/k_x$ is very small. Then the second inequality in (4.11) gets reversed, i.e.,

$$\beta\omega_0/\Omega_\perp \ll 1. \tag{6.1}$$

We rewrite the Hamiltonian (5.7) of the averaged motion, $\langle H \rangle$, in a moving coordinate system, in the following dimensionless form, assuming for definiteness that m is odd:

$$F = \frac{1}{2\beta^2} w^2 + \frac{K}{\pi} (\sin u \cos Z - \sin v \sin Z) = \frac{1}{2\beta^2} w^2 + \frac{K}{\pi} V(Z, u, v), \tag{6.2}$$

where $F = (k_x^2/m_0\omega_0^2)\langle H \rangle$, the tilde has been omitted for simplicity, and we have utilized the notation (2.12). The equations of motion have the form (2.20) [see also Eq. (2.18)] in terms of dimensionless time $\tau = \omega_0 t$.

In the system (6.2) the variables (u,v) are the fast ones and (Z,w) are slow. The particle rotates rapidly in the cells of the spider web and is at the same time subject to a slow motion (drift), intersecting the cell. For a rough description of this motion it suffices to average it over the fast rotations. The mean equations have an integral (adiabatic invariant)—the action I of the system in terms of u,v for frozen Z. The averaged equations for Z and w are derived from Eq. (6.2).

According to the second pair of (2.20) we have

$$\frac{d^2 Z}{d\tau^2} = -\beta^2 \frac{K}{\pi} \overline{\frac{\partial V}{\partial Z}} = \beta^2 \frac{K}{\pi} (\overline{\sin u} \sin Z + \overline{\sin v} \cos Z), \tag{6.3}$$

where the bar on top denotes averaging over the oscillation period in the (u,v) plane for a frozen value of Z. Here all the averaged quantities are functions of the action I of the fast system. Relative to the coordinates (u,v) for frozen values of (Z,w) we call the system (6.2) fast, whereas the system described by the averaged equations will be called slow. In order to investigate Eq. (6.3) it is necessary to know the values of $\overline{\sin u}$ and $\overline{\sin v}$ which depend on I,Z as parameters. For $Z=\pi/4$ the system (6.2) has a net of separatrices in the shape of a square lattice (similar to Fig. 3a, without the stochastic web). In the cases when the frozen value $Z \in (0,\pi/4)$

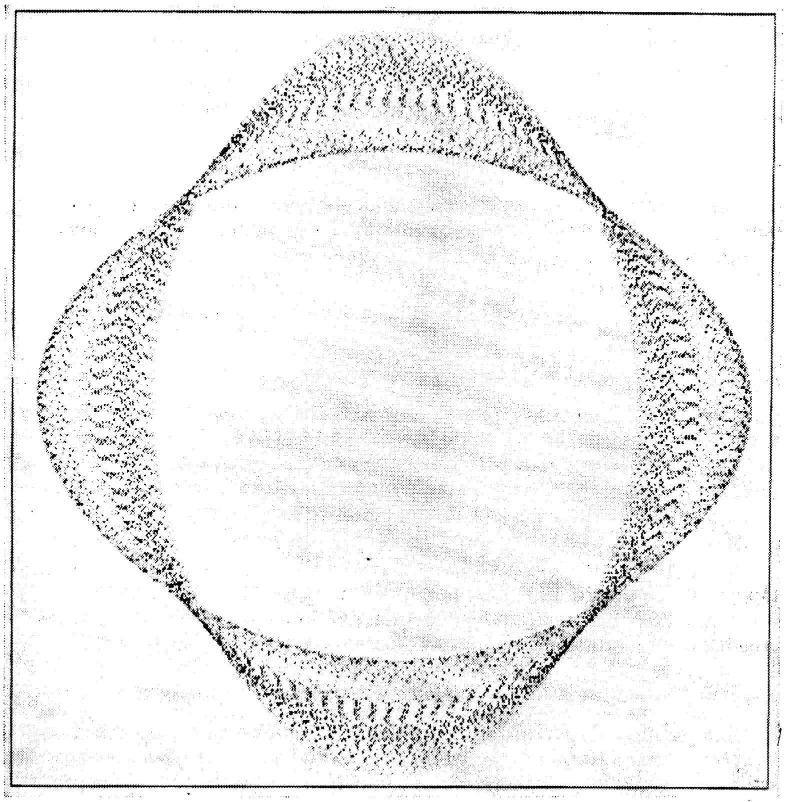


FIG. 9. Stable regular motions in the (u,v) plane for a fourth order resonance: $K = 0.05, \beta^2 = 10^{-6}$, $w_0 = 1.0$; the size of the square is $(0.9\pi)^2$.

the phase portrait of the fast system for (u,v) is similar to the one shown in Fig. 3c, and for $Z \in (\pi/4, \pi/2)$ the portrait is the one in Fig. 3b. Further change in Z leads to a repetition of these pictures with a shift by π in u or v. Therefore the expressions for $\overline{\sin u}$ and $\overline{\sin v}$ need to be calculated taking into account the described types of phase portrait.

We discuss next in more detail the motion in a neighborhood of the equilibria of Eq. (6.3), i.e., the slow system. We have

$$d^2Z/d\tau^2 = \beta^2 K \operatorname{const}(Z - Z_0)^2, \quad (6.4)$$

where Z_0 is the equilibrium position and const is a number of order unity, obtained from (6.3) after expanding in $Z - Z_0$ and calculating the averages of $\overline{\sin u}$ and $\overline{\sin v}$. Simple calculations show that the value $Z_0 = \pi/4$ is an equilibrium of the slow system for any I. For small I this equilibrium is stable if the trajectory of the fast system over which the averaging is effected surrounds the point $u = -\pi/2, v = \pi/2$; the equilibrium is unstable if the trajectory of the fast system surrounds the point $u = \pi/2, v = -\pi/2$. For large values of I, corresponding to trajectories of the fast system, close to the separatrices, the equilibrium $Z_0 = 0$ is, on the contrary, stable for trajectories surrounding $u = \pi/2, v = -\pi/2$ and unstable for trajectories surrounding $u = -\pi/2, v = \pi/2$. For $Z_0 = 5\pi/4$ the stability properties are replaced by their opposites. For motions close to a stable equilibrium the phase trajectory of the fast motion pulsates periodically (Fig. 9), and the motion is regular.

The region of stable regular motions is bounded in this case. Its relative size is of the order of one. Outside this region instability leads to chaotic dynamics (see Fig. 4). The relative size of the unstable region is also of the order one. That is why rapid diffusion occurs in a large (not exponentially small) phase volume. Fig. 10 demonstrates in detail the mechanism of "capture" of the trajectory in the unstable region of a cell of the web and succeeding different methods of exiting from it.

Here we encounter the usual situation when slow variations of an adiabatic invariant of a system lead its passing through a separatrix (Refs. 18–20). The crossing of the separatrix yields a finite jump in the adiabatic invariant. Such crossings occur randomly in time. The characteristic interval Δt_0 between such crossings is determined by the drift time of the particle through the instability region, i.e., through a length of the order of the size of the cell of the spider web:

$$\Delta t_0 \sim 1/(\beta^2 K)^{1/2} \omega_0, \quad (6.5)$$

[see Eq. (6.4)]. This information is, however, insufficient for understanding the general character of the particle motion.

7. LÉVY JUMPS AND INTERMITTENCY

We have considered above the influence of the slow dynamics in the neighborhood of a singularity $Z = \pi/4 + \pi n$ (n an integer). The phase portrait of the fast system (u,v), i.e., the motion transverse to the magnetic field, has a square separatrix lattice. If the values of Z are near $n\pi$, then, as was noted, the phase portrait in the (u,v) plane has the form represented in Fig. 3c. For $Z = n\pi$ it degenerates into a family of straight lines $u = \text{const}$. Along these lines, arbitrarily long "free" paths of the particles are possible. A simple analysis in the linear approximation shows that the trajectory $u = C = \text{const}, 0 < C < \pi/2$ corresponds to an unstable equilibrium position of the slow system for $Z = 0$ and to a stable one for $Z = \pi$. Stable equilibria corresponds to an unbound-

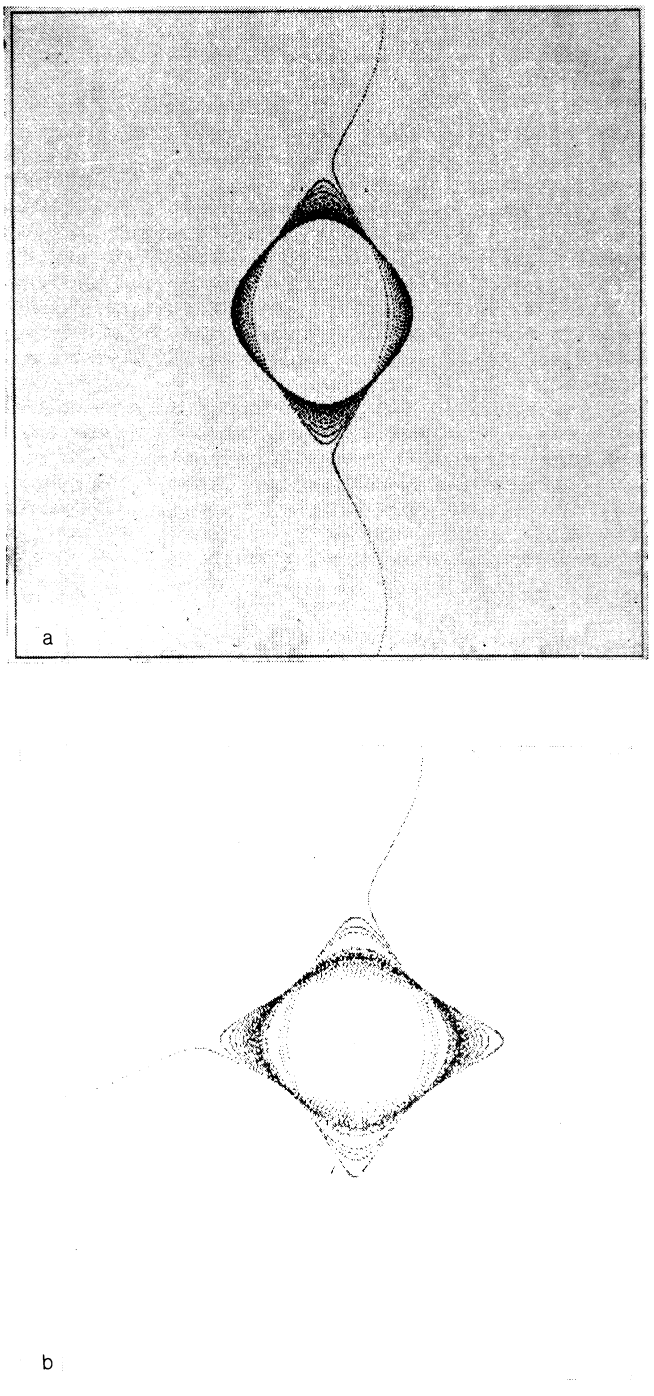


FIG. 10. Captures of the trajectory into a cell of the web and exits from it: $K = 0.05$, $\beta^2 = 10^{-6}$.

ed acceleration of the particles. In the neighborhood of an unstable equilibrium position the acceleration will continue only over a bounded time interval of the same order as (6.5), but taking account of the values of $\sin u$ and $\sin v$ for the appropriate values of Z.

Similar long "free" paths of the particles are possible also in the horizontal direction, if Z is close to the values $\pi/2 + \pi n$. The phase portrait for this case is shown in Fig. 3b.

Long "free" paths correspond to "uncaptured" trajectories of the particles. One can now imagine practically the entire picture as a whole. For $Z = \pi/4 + \pi n$ the whole phase plane is covered by a square web which admits only finite trajectories bounded by one cell. Long trajectories exist only inside the stochastic web, whose thickness is small. In the process of slow changes of Z there occurs an "unlocking" of the cells along the horizontal or the vertical in the (u,v) plane. In this time interval long "free paths" are possible for particles whose orbits intersect the separatrices and enter the region of non-finite motion. Further variation of Z leads to a reclosing of the separatrices and changes the character of the motion.

Thus, the whole motion of the particles with unstable stochastic orbits consists of the following three elements: 1) bounded oscillations with a characteristic amplitude of the order of the size of one cell of the web; 2) "free" flights of various lengths along the directions $u = \text{const}$ or $v = \text{const}$; 3) random walks on the stochastic web, existing even for $\beta = 0$. These three elements create a fast progression of the stochastic particle transport, an example of which is shown in Fig. 4. Compared to the diffusion of particles along the stochastic web ($\beta \neq 0$) the acceleration of the transport process for $\beta \neq 0$ is caused, first, by a step of the random walk which is substantially larger than the cell size, and second, by the existence of anomalously long portions of almost free motion in the phase plane (u,v) (see Fig. 4c).

The described form of the motion can be considered as a process known as a Lévy random walk (Ref. 14). It is important to note that in the case under consideration the random walk arises as a result of the dynamic chaos. However, the kinetic description of the dynamics is the next step, after the fact of stochasticity can be considered as established. It is in the stage of replacing Newtonian dynamics by kinetic theory that the approximate description, close to the kinetic theory of Lévy random processes, makes its appearance. The long stretches of regular motion represent Lévy jumps. Their appearance is typical for systems in which dynamical chaos is related to the existence of a stochastic web.[16]

The presented model of realizations of Lévy random walks exhibits one unique peculiarity: it allows one to clarify in all detail the origin and fundamental properties of the Lévy jumps. Moreover, it is obvious that intermittency of the process of random walks turns out to be associated to Lévy jumps. Indeed, on the one hand, the Lévy jumps, and on the other hand the long captures inside the cells of the spider web, create the quasi-regular "insertions" into the particle trajectories.

8. CONCLUSION

The problem of motion of a particle in a complex field configuration discussed here leads to the possibility of obtaining a very complicated process of formation of chaotic dynamics in the multi-dimensional case, since a system with $2\frac{1}{2}$ degrees of freedom already exhibits the main traits of multidimensional systems. The main peculiarity of the problem under consideration consist in the fact that degeneracy with respect to some of its degrees of freedom exists. Formally, this is reflected by the fact that two of the four equations of the mapping create a stochastic web. The existence of this web leads to a considerable acceleration of the diffusion and to an increase of the measure of the part of phase space in which the dynamics is stochastic. Both these characteristics turn out to be higher than in the case of Arnol'd diffusion.

Another peculiarity of the dynamics is that the chaos has some structure conditioned by the nature of the degeneracy in part of the degrees of freedom. The structured nature of the chaos manifests itself in the existence of stable or metastable regions with symmetry. Other manifestations of this property of multi-dimensional chaos are random walks with Lévy jumps and intermittency.

The paper considered mainly the case of degeneracy with a fourth-order symmetry. However, in other cases with a quasi-symmetry of order five or higher, the properties of acceleration of random walks, growth of the region of stochastic dynamics, and structured nature of the chaos are preserved.

In conclusion the authors express their sincere gratitude to V. I. Arnol'd for interesting discussions of this work.

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