Quantum decay of Coulomb-blockade states in low-capacitance tunnel junctions and systems of such junctions

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The properties of a quasistationary state of a Coulomb blockade in a tunnel junction with a low conductance $\alpha \equiv G\pi\hbar/2e^2 \ll 1$ can be described adequately at low temperatures in terms of macroscopic quantum tunneling of a charge. Such a description is dual to the description of junctions with $\alpha \gg 1$ in terms of macroscopic quantum tunneling of a Josephson phase difference. The probability for the macroscopic quantum tunneling of a charge is calculated both for an isolated junction and for a system of several junctions.

INTRODUCTION

Several recent experiments with single tunnel junctions of low capacitance C (Ref. 1) and in systems of tunnel junctions[2–5] have revealed a Coulomb blockade of tunneling,[6–8] which is manifested as a sharp decrease in the conductance of the junction at low voltages

$$|V| < V_t, \tag{1}$$

where V_t is a threshold voltage (for transitions between normal metals, this threshold is $V_t \simeq e/2C$). At relatively high temperatures $T \gtrsim T_0$, tunneling in the Coulomb-blockade regime occurs by virtue of thermal-activation processes, and the probability Γ for this tunneling falls off exponentially with the temperature.[8] As T decreases to values $T \lesssim T_0$, quantum-decay processes become the primary tunneling mechanism. The probability for these processes is determined primarily not by the temperature but by the dissipation level. Measures of the latter are the dimensionless parameters

$$\alpha_T \equiv G_T R_Q \text{ and } \quad \alpha_S \equiv G_S R_Q \text{ where } R_Q \equiv \pi\hbar/2e^2, \tag{2}$$

G_T is the tunneling conductance of the junction, and G_S is the conductance of the resistance shunting the junction (including the resistance of the external circuit).

In Refs. 9–13, Γ was actually calculated for an arbitrary temperature in only the two limiting cases of small Josephson coupling energy[9,10] $E_J \equiv \hbar I_c/2e$ (I_c is the critical current) and small tunneling dissipation[11–13] $\alpha_T \ll \min(1, \alpha_S)$. Because of possible applications of one-electron tunneling in the development of logic and memory devices,[14] the question of quantum limitations on the lifetime $1/\Gamma$ of the quasistationary state in such devices is important. In the present paper we will calculate the probability for the quantum tunneling of an electron, Γ_e (or of a Cooper pair, Γ_{2e}), per unit time for the following cases: (1) a Josephson junction ($\alpha_T = 0$) with either a small or large (in comparison with unity) ratio E_J/E_Q, where $E_Q \equiv e^2/2C$; (2) a tunnel junction between normal metals which is strongly ($\alpha_T \ll \alpha_S$) or weakly ($\alpha_T \gg \alpha_S$) shunted by a resistance; (3) a system of two unshunted normal tunnel junctions, i.e., a one-electron tunnel transistor; and (4) a chain of N such junctions.

1. JOSEPHSON JUNCTION

1. The dynamics of Josephson junctions with a high conductance $\max(\alpha_T, \alpha_S) \gg 1$ at currents $I < I_c$ can be described adequately in terms of macroscopic quantum tunneling of a Josephson phase difference φ, which is a well-defined quantum-mechanical variable.[15–17] In the opposite limit $\max(\alpha_T, \alpha_S) \ll 1$, the dual of φ, the charge q, is well defined. The density matrix is localized in q space. The tunneling of a charge is an incoherent process, and a state with fixed q is a stationary state of the Coulomb blockade.[6,10] Accordingly, the dynamics of junctions of low conductance at voltages $V < V_t$ should be studied in terms of macroscopic quantum tunneling of a charge.[18]

To describe the effective action in the q representation, we note that a Josephson junction which is connected to a current source I and which is shunted by a resistance $R = G_S^{-1}$ can be described by an equivalent circuit (Fig. 1) with a voltage source $V = I/G_S$ and a low inductance L, which determines the cutoff frequency $\omega_c = 1/G_S L$ of the ohmic spectrum.[9,10] In the absence of quasiparticle tunneling ($\alpha_T = 0$), the Hamiltonian of a system of this sort is

$$H = H_0 + H_L + H_\xi, \tag{3a}$$
$$H_0 = (q - 2en)^2/2C + E_J \cos\varphi, \tag{3b}$$
$$H_L = \Phi^2/2L - q(V - V_\xi). \tag{3c}$$

Here $q = -i\hbar\partial/\partial\Phi$ is the operator representing a charge which has passed through the external circuit (R, L, V), $n = -i\partial/\partial\varphi$ is the operator representing the number of Cooper pairs that have tunneled across the junction, the operator V_ξ represents the voltage across the shunt, and H_ξ is the Hamiltonian of the shunt. Adopting the usual model of the shunt, a set of harmonic oscillators with continuously distributed frequencies,[15] we can write

$$H_\xi = \frac{1}{2}\sum_\alpha \left(\frac{p_\alpha^2}{m_\alpha} + m_\alpha\omega_\alpha^2 x_\alpha^2 + \frac{c_\alpha^2}{m_\alpha\omega_\alpha^2} q^2\right), \qquad V_\xi = \sum_\alpha c_\alpha x_\alpha, \tag{4a}$$

where the coefficients c_α are defined by

$$J(\omega) \equiv \frac{\pi}{2}\sum_\alpha \frac{c_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega - \omega_\alpha) = \frac{\omega}{G_S}. \tag{4b}$$

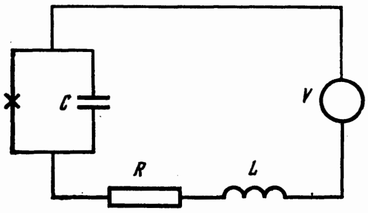

FIG. 1. Circuit connecting the tunnel junction to the external source.

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2. The effective action for a system with Hamiltonian (3) is calculated on trajectories specified by the continuous coordinate $q(\tau)$ and the discrete coordinate $n(\tau)$ (n is an integer). We first consider the case of fairly large values of the Josephson coupling energy, $E_J \gg \alpha_S^{1/2} E_Q$, in which the time scale of the variation in $q(\tau)$ is significantly greater than the average time between successive tunneling events. Assuming that the change in $q(\tau)$ occurs adiabatically slowly, we can replace the Hamiltonian H_0 by its expectation value in terms of the fast variables $n(\tau)$, which is equal to the ground-state energy $E_0(q)$ at sufficiently low temperatures. The expression for the action then takes the form[15] (here and below, $\hbar = 1$, $k_B = 1$)

$$S=\int_0^\beta\left[\frac{1}{2}L\dot{q}^2+E_0(q)-qV\right]d\tau$$
$$+\frac{1}{4\pi G_S}\int_{-\infty}^{\infty}d\tau\int_0^\beta d\tau'\left[\frac{q(\tau)-q(\tau')}{\tau-\tau'}\right]^2,\quad \beta=\frac{1}{T}. \quad (5)$$

a) In the limit of strong Josephson coupling, $E_J \gg E_Q$, the functional dependence

$$E_0(q)=E_0-\Delta\cos(\pi q/e), \quad (6)$$

where

$$E_0=(2E_JE_Q)^{1/2},\ \Delta=16(E_JE_Q/\pi)^{1/2}(E_J/2E_Q)^{1/4}\times\exp[-(8E_J/E_Q)^{1/2}],$$

has the same form (to within the constant part of E_0) as the Josephson potential energy $E_J \cos\varphi$. The substitutions

$$q\to\varphi/2e,\quad 2e\to 2\pi/2e,\quad V\to I,$$
$$\Delta\to E_J,\quad L\to C,\quad G_S\to 1/G_S \quad (7)$$

thus convert action (5) into the action of a shunted Josephson junction in the φ representation,[15] with the parameters

$$E_J'=\Delta,\quad E_Q'=\frac{\pi R_Q}{4L},\quad G_S'=\frac{1}{G_SR_Q^2},$$
$$\left(\alpha_S'=\frac{1}{\alpha_S}\right),\quad I'=\frac{V}{R_Q}. \quad (8)$$

The evolution of a junction of this sort in the φ representation will be exactly the same as that of the original junction in the q representation. The cutoff frequency should be chosen high enough that the strong-viscosity condition $L \ll G_S^{-2} e/\pi(V_t^2 - V^2)^{1/2}$, $V_t = \pi\Delta/e$ holds and such that the properties of a system with the action (5) do not depend strongly on L. The duality between effective actions which we have established is a generalization of the well-known duality of mobilities[9,17] and diagonal density matrix elements[10] in the φ and q representations.

The tunneling probability $\Gamma_{2e} = B\exp(-A)$ for the action (5), (6) is given by the following expressions[19] in the strong-viscosity approximation:

$$A=\begin{cases}\dfrac{1}{\alpha_S}\ln\dfrac{V_t^2}{V^2+(\pi T/\alpha_S e)^2}+\dfrac{2}{\alpha_S}-\dfrac{2eV}{\pi T}\operatorname{arctg}\dfrac{\pi T}{\alpha_S eV}, & T<T_0, \quad (9a)\\ \dfrac{2e}{\pi T}[(V_t^2-V^2)^{1/2}-V\arccos(V/V_t)], & T\geq T_0, \quad (9b)\end{cases}$$

$$B=2^{1/2}\left(\frac{e}{\pi L}\right)^2G_S^{-1/2}\left[V^2+\left(\frac{\pi T}{\alpha_S e}\right)^2\right]^{-1/2},\quad T<T_0, \quad (10)$$

where $T_0 = \pi^{-1}\alpha_S e(V_t^2 - V^2)^{1/2}$. In the particular case $T = 0$ we have

$$\Gamma_{2e}=2^{1/2}\left(\frac{e}{\pi L}\right)^2G_S^{-1/2}\frac{1}{V_t}\left(\frac{V}{V_t}\right)^{2/\alpha_S-1}. \quad (11)$$

By analogy with Ref. 20, we can introduce a voltage which is renormalized as a result of the quantum fluctuations near the equilibrium position:

$$V_t^*=V_t\{1+\alpha_S\ln[\pi G_S^2L(V_t^2-V_0^2)^{1/2}/e]\}.$$

This renormalized voltage is the voltage which would actually be measured in an experiment (e.g., which would be determined from the V dependence of $\langle q\rangle$ at voltages $V \approx V_0 < V_t^*$). Making using of the results of Ref. 20 for $V_t - V_t^* \ll V_t - V_0$, $T = 0$ we can write the expression (11) in the form

$$\Gamma_{2e}=(2G_S)^{1/2}\frac{V_t^{*2}-V_0^2}{V_t^*}\left(\frac{V}{V_t^*}\right)^{2/\alpha_S-1}. \quad (12)$$

The generalization to the case of arbitrary T also follows from the results of Ref. 20.

b) In the opposite limit of a weak Josephson coupling, $E_J \ll E_Q$, the functional dependence $E_0(q)$ is given by

$$E_0(q)=4\pi^{-2}E_Q\arcsin^2[D^{-1/2}\sin(\pi q/2e)], \quad (13)$$

where

$$D=1+2\pi^2\varkappa^3\quad\text{for}\quad \min_n\{(q-e)/2e-n\}\ll 1,$$
$$\varkappa^3\equiv E_J^2/128E_Q^2.$$

If the voltage V is not too large, and the relation $1 - V/V_t \gg \varkappa$ holds, with $V_t = e/C$, we should break up the potential $E_0(q)$ into two parts: $E_0(q) = E^{(0)}(q) + E^{(1)}(q)$. Far from the edges of the Brillouin zone we have

$$E_0(q)\approx E^{(0)}(q)=\min_n[(q-2en)^2/2C]$$
$$\text{for}\quad |(q-e)/2e-n|\gg E_J/E_Q. \quad (14)$$

The probability $\Gamma_{2e}^{(0)} = B_0\exp(-A_0)$, i.e., the probability for a tunneling to the nearest minimum of the potential $E^{(0)}(q) - qV$ was calculated in Refs. 21 and 22. In the case $V \ll V_t$, we find, at $T = 0$,

$$A_0=\frac{2}{\alpha_S}\left(\ln\frac{2V_t}{\pi V}+\bar{c}-1\right), \quad (15)$$

where $\bar{c} \approx 0.577$ is Euler's constant. For voltages $\varkappa \ll 1 - V/V_t \ll 1$ and $T \ll T_0 = G_S/2\pi C$ we have, according to Ref. 20,

$$A_0=\frac{\pi^2}{4\alpha_S}p\left\{1-p\left[\frac{1}{2}+\frac{1}{3}\left(\frac{T}{T_0}\right)^2\right]\right\}\left(1-\frac{V}{V_t}\right)^2, \quad (16)$$

$$B_0=b\alpha_S^{1/2}\left[1+\frac{p}{12\pi(1-p)}\left(\frac{T}{T_0}\right)^2\right]E_Q, \quad (17a)$$

where b is the logarithmic factor

$$b=4\pi^{-3/2}[\ln(p\xi C/G_S^2L)+\bar{c}](-\ln p\xi-\bar{c})^{-1/2}, \quad (17b)$$

and p is the root of the equation

$$p(1-\bar{c}-\ln p\xi)=1,\quad \xi=\frac{\pi}{2}\left(1-\frac{V}{V_t}\right). \quad (17c)$$

Expressions (15)–(17) hold in the case of strong viscosity, $L \ll G_S^{-2} C(1 - V/V_t)$.

The remainder of the potential, $E^{(1)}(q)$, can be treated as a small perturbation near the points $q_n = 2e(n + 1/2)$; the contribution of this perturbation to the argument of the exponential function is given by

$$A_1 = \int_0^\beta E^{(1)}(q^e(\tau))\,d\tau, \tag{18}$$

where $q^e(\tau)$ is the extreme trajectory in the potential[21,22] $E^{(0)}(q) - qV$. When (16) is valid, and with $L \lesssim G_S^{-2} C E_J/E_Q$, we find

$$A_1 = -\frac{\pi^2}{32\alpha_S}\left(\frac{E_J}{E_Q}\right)^2. \tag{19}$$

The probability for a macroscopic quantum tunneling, $\Gamma_{2e} = B\exp(-A)$, $A = A_0 + A_1$, $B \sim B_0$, is given by (16), (17), and (19) under these conditions. Numerical results for arbitrary T and V (subject to the condition $1 - V/V_t \gg \varkappa$) can be found from the general relations of Refs. 21 and 22.

At even higher voltages, $0 < 1 - V/V_t' \ll \varkappa$, $V_t' = V_t(1 - 3\varkappa)$, for which the minimum q_m of the potential $E_0(q) - qV$ lies near the inflection point of the function $E_0(q)$, $q_m \approx e(1 - 3\varkappa)$, we can approximate the potential near q_m by a cubic parabola and use the results of Ref. 16. In the strong-viscosity approximation, $L \ll G_S^{-2} C \varkappa^{1/2}(1 - V/V_t')^{-1/2}$ at low temperatures

$$T < T_0 = (3/\varkappa)^{1/4}(1 - V/V_t')^{1/4} G_S/2\pi C,$$

we have

$$A = \frac{2^{-1/4}\pi^2}{3\alpha_S}\left(\frac{E_J}{E_Q}\right)^{3/2}\left[1 - \frac{1}{3}\left(\frac{T}{T_0}\right)^2\right]\left(1 - \frac{V}{V_t'}\right), \tag{20}$$

$$B = \frac{2^{1/4}}{6}\frac{eC}{L^2}\left(\frac{E_J}{E_Q}\right)^{3/2} G_S^{-1/2}. \tag{21}$$

When the renormalization of V_t' is taken into account, the expression for B is similar to (12) and is given in Ref. 18.

3. We turn now to the case of small E_J, in which the adiabatic approximation cannot be used, and the n dependence of the Hamiltonian must be taken into account explicitly. In the q, n representation, Hamiltonian (3) becomes

$$H_0 = \frac{(q-2en)^2}{2C} + \frac{E_J}{2}(|n+1\rangle\langle n| + |n\rangle\langle n+1|). \tag{22}$$

In this form, the Hamiltonian is convenient for the use of perturbation theory in E_J (the terms H_L and H_ξ remain unchanged). In second order in E_J, there can be tunneling of only a single Cooper pair, $n \to n \pm 1$, so it is sufficient to consider two states, e.g., those with $n = 0$ and $n = 1$. In this case, the Hamiltonian (3), (22) reduces to that examined in Ref. 23 and can be rewritten in the form

$$H = \frac{E_J}{2}\sigma_x + \sigma_z e(V - V_\xi') + H_\xi', \tag{23}$$

where V_ξ' and H_ξ' are given by (4) with

$$J(\omega) = \frac{\omega/G_S}{1 + (\omega C/G_S)^2} \quad \text{as} \quad L \to 0 \tag{24}$$

[in expression (4a) for H_ξ', the last term should be discarded]. The role of a cutoff frequency is played here by the quantity $\omega_c = G_S/C$, and the small inductance L does not appear in the result. The quantity Γ_{2e}, which is the probability for a transition from the state $|\uparrow\rangle$ ($n = 0$) to the state $|\downarrow\rangle$ ($n = 1$) per unit time, was calculated in Ref. 24 for the case of small V, for which the Lorentz cutoff in (24) can be replaced by an exponential cutoff. Going through the corresponding calculations for a Lorentz cutoff, we find, at absolute zero,

$$\Gamma_{2e} = \frac{\pi}{2}\frac{E_J^2 C}{G_S}\frac{1}{\Gamma(2/\alpha_S)}\left(\frac{\pi V}{\alpha_S V_t}\right)^{2/\alpha_S - 1} \qquad V \ll \alpha_s V_t, \tag{25}$$

$$\Gamma_{2e} = \frac{\pi^{1/2}}{4}\frac{E_J^2 C}{G_S}\left(\frac{\alpha_S}{\lambda}\right)^{1/2}\exp\left[-\frac{\pi^2}{4\lambda\alpha_S}\left(1 - \frac{V}{V_t}\right)^2\right],$$

$$0 < 1 - \frac{V}{V_t} \ll 1, \tag{26}$$

where $V_t = e/C$, and $\lambda \approx \ln \alpha_S^{-1/2}$. The tunneling probability (25), (26) determines the current-voltage characteristic of the junction, $I_T = 2e\Gamma_{2e}(V)$, which is the same as that which was calculated in Ref. 9 by means of the density matrix in the φ representation.

In concluding this section of the paper, we note that the functional dependence $\Gamma_{2e} \sim V^{2/\alpha_S}$ at low voltages [see (11), (15), (25)], with $\alpha_S \ll 1$, appears to be of universal applicability for an arbitrary value of the ratio E_J/E_Q.

2. TUNNEL JUNCTION BETWEEN NORMAL METALS

In describing Josephson junctions, we took into consideration only the tunneling of Cooper pairs, ignoring the quasiparticle (one-electron) tunneling. We now consider the tunneling of individual electrons at a junction between normal metals (the results will also apply to one-electron tunneling in a Josephson junction with $E_J \ll E_Q$). The Hamiltonian for such a junctioin differs from (3) in the term H_0, which in this case becomes

$$H_0 = (q - en)^2/2C + H_T, \tag{27}$$

where H_T is the tunneling Hamiltonian [in which we also include the Hamiltonians of the electrodes of the junction; see (37)].

1. We first consider the case of a weakly shunted junction, $\alpha_S \ll \alpha_T \ll 1$, in which we can use the adiabatic approximation (Secs. 1 and 2). The value of $E_0(q)$ averaged over the fast coordinate $n(\tau)$ for Hamiltonian (27) is determined by the partition function, an expression for which was derived in Ref. 25 (see also Ref. 26). In particular, in the region $|q - e/2| \ll e/2$, at absolute zero, we have

$$E_0(q) = \min_{n=0,1}(E^{(n)}(q)) + \frac{\alpha_T}{\pi^2}|W|\left(1 + \ln\frac{\omega_c}{|W|}\right)$$

$$\text{for} \quad \alpha_T\omega_c \lesssim |W| \ll \omega_c, \tag{28}$$

where

$$E^{(n)}(q) = (q - en)^2/2C, \quad W(q) = E^{(1)}(q) - E^{(0)}(q), \tag{29}$$

and ω_c is the cutoff frequency of the tunneling dissipation, which is usually assumed to be equal to E_Q (Ref. 25).

The potential $U(q) = E_0(q) - qV$, which appears in expression (5) for the action, has a minimum (Fig. 2) at the

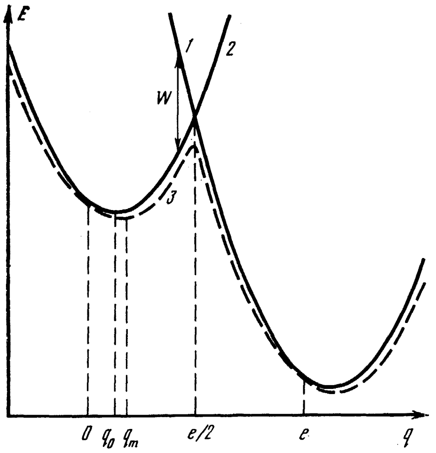

FIG. 2. The Coulomb energy $E^{(n)}(q)-qV$ of quantum states with $n=0$ (curve 1) and $n=1$ (2) and the effective renormalized potential $U(q)$ (3) versus the magnitude of the charge.

point $q_m=CV'$, determined by

$$V'=V+\frac{2\alpha_T}{\pi^2}V_t\ln\frac{\omega_c}{W(q_m)},\qquad(30)$$

$$\frac{2\alpha_T}{\pi^2}\ln\frac{\pi^2}{\alpha_T}\leqslant 1-\frac{V}{V_t}\ll 1,\qquad(31)$$

where $V_t=e/2C$. We introduce the new coordinate $q_1=q-q_m$, and we break up the potential into two parts, $U=U_0+U_1$:

$$U_0(q_1)=\frac{q_1^2}{2C}-2(1-\chi)V_t(q_1-\bar{q})\theta(q_1-\bar{q}),\qquad(32)$$

$$U_1(q_1)=2\alpha_T V_t\left\{|q_1-\bar{q}|\ln\left(\frac{\bar{q}}{|q_1-\bar{q}|}+1\right)-\bar{q}\right\},\qquad(33)$$

where $\chi=2\pi^{-2}\alpha_T\ln(\omega_c/W(q_m))$, $\bar{q}=e/2-q_m$.

The tunneling probability $\Gamma_e^{(0)}$ in potential U_0 is given by (16) and (17) for $T=0$, after the following substitutions are made:

$$V\to V',\ b\to(1-\chi)b,\ \xi\to\xi/(1-\chi),\ \alpha_S\to 4\alpha_S,\ E_Q\to E_Q/4\qquad(34)$$

(the reason for the last two of these substitutions is that we are discussing one-electron tunneling, rather than the tunneling of Cooper pairs).

We treat potential U_1 as a perturbation. The increment A_1 in the argument of the exponential function is then given by the value of action (18) [after the replacement $E^{(1)}\to U_1$] on the extreme trajectory[21]

$$q_1^e(\tau)=e\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{\omega(1+\tau_S|\omega|)}[\sin\omega(\tau_0-\tau)+\sin\omega(\tau_0+\tau)],\qquad(35)$$

in the potential U_0, where $\tau_S=C/G_S$, and the time τ_0 is determined by the relation $q_1^e(\tau_0)=\bar{q}$. From (18) and (35) we find

$$A_1\approx-(\alpha_T/\pi^2\alpha_S)(1-V'/V_t)\ln^{-2}(1-V'/V_t).\qquad(36)$$

The quantity Γ_e is determined by (16), (17), (34), and (36), which hold in the adiabatic approximation, $\alpha_S/(1-V'/V_t)\ll\alpha_T\ll 1$. The effect of tunneling dissipation is seen primarily in the renormalization of the voltage V in (30). The change in the argument of the exponential function, $A_0(V')-A_0(V)$, as a result of this renormalization is greater by a factor $\sim\ln^2(1-V'/V_t)\gg 1$ than the quantity A_1, which arises because of the deviation of the potential from a parabolic shape. We would also like to stress that the conductance of the shunt, $\alpha_S\ll\alpha_T\ll 1$, has a governing effect on Γ_e under the conditions α_S [see (16), (17), (34)]. This shunt constitutes a "bottleneck" for the tunneling process. Despite the frequent virtual processes of a tunneling of electrons across the junction, $n=0\rightleftarrows n=1$ (the frequency of the attempts is $\sim\tau_T^{-1}$, where $\tau_T=C/G_T$), the system is usually in an excited state ($n=1$) for a short time interval $\sim W^{-1}$. If tunneling is to actually occur, the system must spend a sufficiently long time $\tau^*\approx\tau_S(1-V/V_t)\gg W^{-1}$ in this state [see (28)]; over this time, a charge $(e/2)(1-V/V_t)$ will pass through the shunt, and the state with $n=1$ will become favored from the energy standpoint. An estimate of the action for this process,

$$S\approx\tau^*W\approx(1-V/V_t)^2/\alpha_S,$$

agrees within a factor ~ 1 with the value which we found for A.

2. We turn now to highly shunted junctions, with $\alpha_T\ll\alpha_S\ll 1$. We write the tunneling Hamiltonian H_T in (27) in terms of Bose operators,[13,27] in the form

$$H_T=\sum_\alpha\sum_{k=\pm1}\left\{g_\alpha(a_{k,\alpha}^+ +a_{-k,\alpha})\sum_n|n+k\rangle\langle n|+\omega_\alpha a_{k,\alpha}^+a_{k,\alpha}\right\},$$

$$\sum_\alpha g_\alpha^2\delta(\omega-\omega_\alpha)=\alpha_T\omega/\pi^2.\qquad(37)$$

This form of the Hamiltonian is convenient for the use of perturbation theory in α_T [the operators $\exp(ik\varphi/2)$ (Ref. 13) are written in the charge representation: $\exp(ik\varphi/2)|n\rangle=|n+k\rangle$, where $k=\pm1$]. Introducing the two states $|\uparrow\rangle$ ($n=0$) and $|\downarrow\rangle$ ($n=1$), and proceeding as in the derivation of (23), we put the Hamiltonian (3), (27) in the form

$$H=\sum_\alpha\Big\{g_\alpha[(a_{-1,\alpha}^+ +a_{1,\alpha})\sigma_+ +(a_{-1,\alpha}+a_{1,\alpha}^+)\sigma_-]+\omega_\alpha\sum_{k=\pm1}a_{k,\alpha}^+a_{k,\alpha}\Big\}$$
$$+\sigma_z e(V-V_\xi')/2+H_\xi'.\qquad(38)$$

Using the Hamiltonian (38), and working in second-order perturbation theory in g_α, we can derive a general expression for Γ_e (by analogy with the approach in Ref. 24). In particular, at $T=0$ we have

$$\Gamma_e=\frac{8\alpha_T\alpha_S E_Q}{\pi^2\Gamma(2+1/2\alpha_S)}\left(\frac{\pi V}{4\alpha_S V_t}\right)^{1/2\alpha_S+1},\quad V\ll\alpha_S V_t,\qquad(39)$$

$$\Gamma_e=32\pi^{-9/2}\alpha_T E_Q\frac{\alpha^{*3/2}}{(1-V/V_t)^2}\exp\left[-\frac{\pi^2}{16\alpha^*}\left(1-\frac{V}{V_t}\right)^2\right],$$

$$4\alpha^{*1/2}\ll 1-V/V_t\ll 1,\qquad(40)$$

where $\alpha^*\approx-0.5\alpha_S\ln\alpha_S$. The tunneling probability (39), (40) agrees with the current-voltage characteristic $I_T=e\Gamma_e(V)$ of the junction which was calculated by a different method in Refs. 11 and 13. The behavior of Γ_e as a function of V and α_S is of the same nature as in the case $\alpha_S\ll\alpha_T\ll 1$ [see (16) and (17) with (34)].

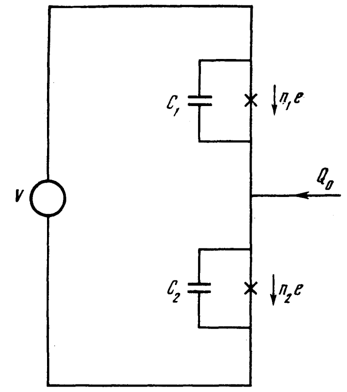

FIG. 3. Equivalent circuit of a one-electron tunnel transistor.

3. ONE-ELECTRON TUNNEL TRANSISTOR

Let us examine the one-electron tunnel transistor,[14] which is a series connection of two unshunted tunnel junctions connected to a dc voltage source V (Fig. 3). The change in the charge on the central electrode, Q_0, which can be controlled at the scale of e, causes a change in the electron tunneling conditions, so it becomes possible to control the tunneling current I_T (see the experiments of Ref. 3). In particular, at voltages below the threshold, $V < V_t(Q_0)$, a Coulomb blockade of tunneling comes into play. Even in the Coulomb-blockade region, however, a weak tunneling current flows through the system, because of (1) thermally activated spontaneous tunneling which is important at relatively high temperatures[14] and (2) quantum tunneling, which dominates at low temperatures, and which will be discussed below.

We write the Hamiltonian of the transistor as the sum

$$\hat{H}=E(n_1, n_2)+H_T^{(1)}+H_T^{(2)} \tag{41}$$

of the Coulomb energy[14]

$$E(n_1, n_2)=\frac{1}{2C_\Sigma}[(n_1-n_2)e+Q_0]^2-eV\left(n_1\frac{C_2}{C_\Sigma}+n_2\frac{C_1}{C_\Sigma}\right), \quad C_\Sigma=C_1+C_2 \tag{42}$$

(n_ie is the charge which has passed through the ith junction, n_i is an integer, and $i = 1,2$) and the tunneling Hamiltonians $H_T^{(1)}$, $H_T^{(2)}$, of the junctions with tunneling conductances $\alpha_T^{(1)}, \alpha_T^{(2)} \ll 1$. For Hamiltonians $H_T^{(1)}$, $H_T^{(2)}$, we will use the representation (37), which is convenient for a perturbation-theory approach. We assume $T = 0$, and we assume that the system is in one of the quasistationary states $|n_1^0,n_2^0\rangle$. In this case, the tunneling of an arbitrary number of electrons through any of the junctions increases $E(n_1,n_2)$. The state $|n_1^0,n_2^0\rangle$, with zero excitation quanta in the modes (k,α_i), $k = \pm 1$, $i = 1,2$ in (37), is an eigenstate of Hamiltonian (41) with $g_\alpha = 0$. In first order, the perturbation [the term with g_α in (37)] causes only transitions to the states $|n_1^0 \pm 1,n_2^0\rangle$ and $|n_1^0,n_2^0 \pm 1\rangle$ which are allowed from the energy standpoint; here

$$\begin{aligned} E_1&=E(n_1^0+1, n_2^0)-E(n_1^0, n_2^0)>0,\\ E_2&=E(n_1^0, n_2^0+1)-E(n_1^0, n_2^0)>0. \end{aligned} \tag{43}$$

Favored from the energy standpoint (at $V > 0$) is a second-order transition to the state $|n_1^0 + 1,n_2^0 + 1\rangle$, accompanied by the creation of two excitations, in the modes $(1,\alpha_1)$ and $(1,\alpha_2)$ with the frequencies ω_1 and ω_2:

$$\omega_1+\omega_2=E(n_1^0, n_2^0)-E(n_1^0+1, n_2^0+1)=eV. \tag{44}$$

The matrix element for a transition of this sort through an intermediate state $|n_1^0 + 1,n_2^0\rangle$ or $|n_1^0,n_2^0 + 1\rangle$ is

$$M=g_\alpha^{(1)}g_\alpha^{(2)}\left\{\frac{1}{E_1+\omega_1}+\frac{1}{E_2+\omega_2}\right\}. \tag{45}$$

The final-state density $\rho_f = \rho_1(\omega_1)\rho_2(\omega_2)$ is determined by the number of modes $(1,\alpha_i)$ in a unit energy interval $\rho_i(\omega)$; as can be seen from (37), we have

$$\rho_i(\omega)g_i^2(\omega)=\alpha_T^{(i)}\omega/\pi^2, \quad i=1,2, \tag{46}$$

where $g_i(\omega) = g_\alpha^{(i)}$ at $\omega = \omega_\alpha^{(i)}$.

Using the general expression for the probability for a second-order transition per unit time,[28]

$$\Gamma_e=2\pi|M|^2\rho_f, \tag{47}$$

along with (44)–(46), we find

$$\Gamma_e=2\pi^{-3}\alpha_T^{(1)}\alpha_T^{(2)}\int_0^{eV}d\omega\,\omega(eV-\omega)\left\{\frac{1}{E_1+\omega}+\frac{1}{E_2+eV-\omega}\right\}^2. \tag{48}$$

The integral in (48) can be evaluated easily; the result is

$$\Gamma_e=2\pi^{-3}\alpha_T^{(1)}\alpha_T^{(2)} \times\left\{\left(1+\frac{2}{eV}\frac{E_1E_2}{E_1+E_2+eV}\right)\left[\sum_{i=1,2}\ln\left(1+\frac{eV}{E_i}\right)\right]-2\right\}eV, \tag{49}$$

which determines the tunneling current $I_T = e\Gamma_e$ in the Coulomb-blockade region at absolute zero. At low voltages $eV \ll \min(E_1,E_2)$ we have

$$\Gamma_e=\frac{1}{3\pi^3}\alpha_T^{(1)}\alpha_T^{(2)}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)^2(eV)^3. \tag{50}$$

Comparisons of these expressions with Ref. 14 shows that in the Coulomb-blockade region the current decreases exponentially with the temperature[14] only at sufficiently high temperatures

$$T \gtrsim T_0=E_i/\ln(\pi^2/\alpha_T^{(i)}),$$

where $E_i = \min(E_1,E_2)$, and it approaches the finite value in (49) for $T \ll T_0$. The probability for spontaneous quantum tunneling of electrons given by (49), in contrast with the probability for tunneling in an isolated junction, is not exponentially small. The reason lies in the purely discrete mechanism for the transport of charge through a one-electron transistor (see the discussion below).

4. CHAIN OF TUNNEL JUNCTIONS

We now consider a chain of N series-connected tunnel junctions,[29] with the equivalent circuit in Fig. 4. The Hamiltonian of the chain is

$$H=E(n_1, n_2, \dots, n_N)+\sum_{i=1}^{N}H_T^{(i)}, \tag{51}$$

where $H_T^{(i)}$ are the tunneling Hamiltonians of the junctions, (37), and $E(n_1,n_2,...,n_N)$ is the Coulomb energy, which can

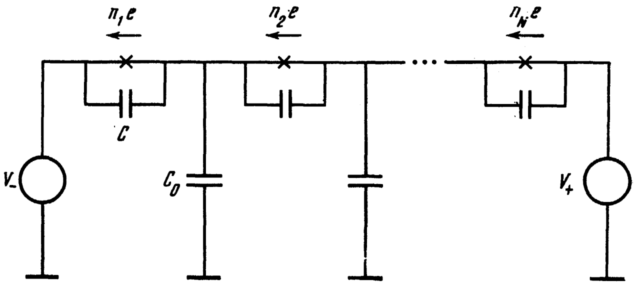

FIG. 4. Equivalent circuit of a chain of tunnel junctions. C—Self-capacitance of junction; C_0—capacitance between the electrodes of the junction and the substrate.

be expressed[29] as a function of the charges $n_i e$ which have tunneled through the junctions with $i = 1,2,...,N$.

From this point on, the arguments are analogous to those for a transistor. At voltages $|V| < V_t(U)$, where $V = V_+ - V_-$, $U = (V_+ + V_-)/2$, a Coulomb blockade occurs in the chain.[29] Consequently, the system is in one of the quasistationary states

$$s_0 \equiv |n_1^0, n_2^0, \ldots, n_N^0\rangle,$$

in which the tunneling of any charge $n_i e$ through any $N-1$ junctions leads to an increase in the Coulomb energy. For $V > 0$, a transition of order N to the state

$$s_N \equiv |n_1^0+1, n_2^0+1, \ldots, n_N^0+1\rangle$$

is allowed from the energy standpoint. Such a transition corresponds to the passage of an electron through the entire chain and is accompanied by the creation of N excitation quanta in the $(1,\alpha_i)$ modes [see (37)], with the frequencies ω_i satisfying

$$\sum_{i=1}^{N} \omega_i = eV. \qquad (52)$$

The matrix element M of the Nth order transition, which appears in the general expression (47) for Γ_e, is

$$M(s_0 \to s_N) = \sum_{\{s_1,s_2,\ldots,s_{N-1}\}} \left(\prod_{j=1}^{N-1} \frac{V(s_{j+1}, s_j)}{\varepsilon(s_j) - \varepsilon(s_0)} \right) V(s_1, s_0). \qquad (53)$$

Here $V(s_{j+1}, s_j)$ is a matrix element of the perturbation operator which is the sum of all the terms in (51) which contain g_α. The summation is over all possible sets of intermediate states $\{s_1,s_2,...,s_{N-1}\}$. Each such set is specified by listing the sequence of indices of the junctions, $\{i_1,i_2,...,i_N\}$, $1 \leqslant i_j \leqslant N$, $i_j \neq i_l$ for $j \neq l$, through which the electron tunneling occurs (there are $N!$ different sequences) and by listing the quantum frequencies $\omega_1,\omega_2,...,\omega_N$ in (52). For each sequence $\{i_1,i_2,...,i_N\}$ the energies of the intermediate states are

$$\varepsilon(s_j) = E(s_j) + \Omega(s_j), \quad \Omega(s_j) = \sum_{k=1}^{j} \omega_{i_k}, \qquad (54)$$

where $E(s_j)$ is the Coulomb energy of state s_j, which is found from s_0 by means of the substitutions $n_l^0 \to n_l^0 + 1$ for $l = i_1,i_2,...,i_j$. We fix the values $\omega_1,\omega_2,...,\omega_N$, and we denote by S the following sum over all possible sequences $\{i_1,i_2,...,i_N\}$:

$$S(\omega_1, \omega_2, \ldots, \omega_N) = \sum_{\{i_1,i_2,\ldots,i_N\}} \prod_{j=1}^{N-1} \frac{1}{\varepsilon(s_j) - \varepsilon(s_0)}. \qquad (55)$$

Now writing an integral over frequencies, and using (46), (52)–(55), we find from the general expression (47)

$$\Gamma_e = 2\pi \left(\prod_{i=1}^{N} \frac{\alpha_T^{(i)}}{\pi^2} \right) \int_0^\infty S^2(\omega_1, \omega_2, \ldots, \omega_N)\, \delta \times \left(eV - \sum_{i=1}^{N} \omega_i \right) \prod_{i=1}^{N} \omega_i d\omega_i. \qquad (56)$$

In the particular case of low voltages, $V \ll V_t$, with $E(s_j) - E(s_0) \gg \Omega(s_j)$, we can ignore the energy of the quanta in evaluating (54) and (55), and we can take S^2 through the integral sign in (56):

$$\Gamma_e = 2\pi \left(\prod_{i=1}^{N} \frac{\alpha_T^{(i)}}{\pi^2} \right) \frac{S^2}{(2N-1)!} (eV)^{2N-1}. \qquad (57)$$

In the simple limiting case in which the mutual capacitance of the electrodes in the chain, C, is zero, and the expression for the Coulomb energy is

$$E(n_1, n_2, \ldots, n_N) = \frac{e^2}{2C_0} \sum_{i=1}^{N-1} (n_{i+1} - n_i)^2 + n_1 eV_- - n_N eV_+, \quad (58)$$

we can derive the following estimate for (55):

$$S = (C_0/e^2)^{N-1} \exp(N + 1.5 \ln N + O(1)) \quad \text{for} \quad V \ll V_t = e/C_0, \qquad (59)$$

where C_0 is the self-capacitance of the electrodes. The tunneling current $I_T = e\Gamma_e$ through a chain of N junctions at low voltages $V \ll V_t$, at $T = 0$ and under the condition $\alpha_T \ll 1$, is proportional to the quantity

$$(V/V_t)^{2N-1} \prod_i \alpha_T^{(i)},$$

which is an exponentially decreasing function of N.

5. CONCLUSION

We have been discussing the macroscopic quantum tunneling of a charge in isolated tunnel junctions with a small conductance, $\max(\alpha_T, \alpha_S) \ll 1$, and in systems of such junctions. Because of the incoherent nature of the tunneling, the quantity Γ which we have calculated determines not only the current-voltage characteristic of the junction,

$$I_T = ne\Gamma_{ne}(V), \quad n = 1, 2,$$

but also the frequency spectrum of the fluctuations in the tunneling current,

$$\langle I_T^2 \rangle_\omega = \frac{ne}{2\pi} I_T,$$

in the region $V < V_t$.

The results which we have found reveal an interesting general aspect of the behavior of the tunneling probability Γ as a function of the dissipation level α in various systems. In an unshunted isolated junction ($N = 1$) which is connected directly to a voltage source, the tunneling of an electron occurs immediately to an energetically favored state, and the quantity

$$\Gamma_e \propto |M^{(1)}_{s_0\to s_1}|^2 \propto \alpha_T$$

is proportional to the square matrix element of the tunneling Hamiltonian, In a transistor ($N=2$) and in a chain of junctions ($N=3,4,...$), the tunneling occurs through a sequence of $N-1$ intermediate states, and the tunneling probability

$$\Gamma_e \propto |M^{(N)}_{s_0\to s_N}|^2 \propto \alpha_T^N$$

is determined by Nth order matrix element (53). As the number of intermediate states is increased, the quantity Γ_e approaches zero progressively more rapidly (in proportion to α_T^N) with decreasing α_T. We wish to stress that the possibility of singling out a finite number $N-1$ of intermediate states arises because of the discrete nature of the tunneling dissipation, which describes transport of a charge which is a multiple of e. The circuit in Fig. 1 may be thought of, at least qualitatively, as the limiting case of an infinite chain (a resistor) to which one more tunnel junction is connected in series. A transition from the initial state to the final state occurs through a continuum of intermediate states (which correspond to different values of the continuous coordinate q). The tunneling probability $\Gamma \propto \exp(-\text{const}/\alpha_S)$ falls off exponentially with decreasing dissipation.

The results derived above are important for possible applications of small tunnel junctions for developing one-electron digital circuits, which have recently been discussed.[14] If such a circuit is to operate correctly, the Coulomb-blockade states will have to be highly stable with respect to thermal and quantum fluctuations of the charge at the junction. This statement means that a high probability for a quantum decay of Coulomb-blockade states in a one-electron transistor (a probability which falls off only as α_T^2 with decreasing conductance of the junctions, α_T) would apparently make such a transistor useless as a basic element in digital devices. It seems very likely that this difficulty can be overcome by replacing the transistor by a chain of a large number of junctions. At the typical experimental values at the moment, $\alpha_T = 10^{-3}$ and $C_0 = 10^{-15}$ Φ, for example, increasing the number of junctions to $N=5$ makes the probability for a quantum decay in the chain, (57), quite small: $\Gamma_e \sim 10^{-16}$ s^{-1} (for $V=0.1V_t$, and $C=0$).

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