The CP^{N-1}-model: fractional topological charge and the index theorem for manifolds with boundary

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A new class of self-dual solutions is considered, defined on a manifold with boundary and exhibiting a topological charge $Q = 1/N$. The contribution $\langle\bar\psi\psi\rangle$ of the fluctuations to the fermion condensate is calculated in the supersymmetric version of the CP^{N-1} model. The result is found to be finite, which implies spontaneous breaking of the discrete chiral symmetry. The Atiyah-Patodi-Singer (APS) index theorem is discussed in detail for manifolds with boundary. The necessity of imposing global boundary conditions in this case is explained.

1. INTRODUCTION

The purpose of the present paper is to describe solutions with fractional topological charge Q in two-dimensional CP^{N-1}-theories and to analyze the physical consequences of their existence.

A similar problem for the two-dimensional $O(3)\sigma$-model and for four-dimensional supersymmetric gluodynamics with $SU(3)$ gauge group was considered respectively in Refs. 1 and 2, where it was shown that solutions with half-integer topological charge give a nonvanishing contribution to the fermion condensate $\langle\bar\psi\psi\rangle$ and thus guarantee spontaneous breaking of the discrete chiral symmetry in the theories under discussion. The purpose of the present paper is to go outside the framework of the $SU(2)$ group and to demonstrate for the example of a two-dimensional supersymmetric σ-model based on the $SU(N)$ group (so-called CP^{N-1} theories) the existence of a stable solution with $Q = 1/N$. Furthermore, we will demonstrate that the solution we obtained yields a nonvanishing contribution to the condensate $\langle\bar\psi\psi\rangle$ and that the use of the quasiclassical approximation is fully justified.

It is important to stress the fact that the transition from the group $SU(2)$ to the group $SU(N)$ is not simply an arithmetic problem. The analysis of the $SU(N)$ case carried out in this paper allows one to look at the problem of admissible boundary conditions from more general positions. We shall convince ourselves that it is necessary to impose global boundary conditions on the fermionic fields.[3] This singles out exactly the admissible values of the topological charge $Q \sim 1/N$. Solutions with other fractional values of Q turn out to be unstable. Thus, a consideration of the σ model with group $SU(N)$ allowed one to formulate a general principle for imposing admissible boundary conditions. We hope that the present analysis will help in the future to solve a similar problem for the physically more interesting problem of the four-dimensional $SU(N)$ gauge theory.

Before describing in detail the configurations with fractional values of Q we recall that the integer nature of Q for the instanton (Refs. 4,5) is related to the compactification of the physical space to a sphere, i.e., with an identification of all points which are infinitely remote. A choice of different boundary conditions can lead to fractional topological charges. In particular, in $SU(N)$ gluodynamics defined on the hypertorus $T_1\times T_1\times T_1\times T_1$ the introduction of so-called twisted boundary conditions[6] has allowed one to obtain solutions of the classical equations—torons[7] with $Q = 1/N$ and with the action $S = (8\pi^2/g^2)(1/N)$.

Some words are appropriate here regarding the supersymmetric CP^{N-1} model proper (Refs. 8,9) and on the motivation for considering solutions with fractional topological charge in this case (Ref. 10).

As is known, the model exhibits a naive chiral $U(1)$ symmetry $\psi\to\exp(i\alpha\gamma_5)\psi$, violated by the anomaly

$$\partial_\mu a_\mu = 2NQ, \qquad Q = \frac{1}{4\pi}\int d^2x\,\varepsilon_{\mu\nu}F_{\mu\nu}. \tag{1}$$

However, the discrete symmetry $Z_2\times Z_N$ is preserved. On the other hand, an exact assertion exists (Ref. 9), to the effect that at large N there appears a fermion condensate $\langle\bar\psi\psi\rangle\neq 0$. This in turn signals a violation of the Z_N symmetry mentioned above, and the existence of N vacuum states, labeled by the phase of the condensate $\langle\bar\psi\psi\rangle_k \propto \exp\{2\pi k/N\}$ (Ref. 10). It is natural to expect that a similar behavior is characteristic not only for $N\to\infty$, but for a theory with finite arbitrary N too. A strong argument in favor of this hypothesis is the calculation of the Witten index,[11] which is exactly equal to N and determines the number of vacuum states of the theory. Nevertheless, although one is confident that a condensate $\langle\bar\psi\psi\rangle$ exists, standard instanton calculations yield a zero contribution to $\langle\bar\psi\psi\rangle$ and can only assure a nonvanishing value of the correlator[12]

$$\left\langle \prod_i \bar\Psi\Psi(x_i)\right\rangle \neq 0.$$

The reason for this, as one can easily glean from Eq. (1), is the fact that an instanton is characterized by $2N$ fermionic zero modes. At the same time $\langle\bar\psi\psi\rangle$ can "swallow" only two of them, leading to the trivial vanishing of the instanton contribution to $\langle\bar\psi\psi\rangle$.

It is natural to expect (and this is borne out by experience of working with the group $SU(2)$; Refs. 1,2) that the self-dual solution with $Q = 1/N$ will have only two zero modes and can therefore guarantee a nonvanishing value of $\langle\bar\psi\psi\rangle$. It is exactly this which serves as a motivation for the introduction of torons[1)] into the theory.

The plan of the paper is the following. In Sec. 2 the CP^{N-1} theory is formulated and various methods for the description of solutions with fractional Q are discussed. In Sec. 3 an equation is derived for the modes describing quantum fluctuations in the vicinity of the classical solution. Sec-

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tion 4 carries an important conceptual load: it describes the requirements imposed on the modes. These requirements are based on quite general principles, such as the APS index theorem.[3] As shown in Sec. 5, these requirements single out completely definite fractional values $Q \sim 1/N$, guarantee the stability of the corresponding solutions, and yield a nonvanishing contribution to $\langle\bar{\psi}\psi\rangle$.

2. THE CP^{N-1} MODEL. TORONS

In order to describe the toron solution we discuss the duality equations and the Lagrangian of the usual (non-supersymmetric) CP^{N-1} model. The modification made necessary by the presence of fermions will be discussed in the following sections. In terms of a complex N-component unit spinor n_α ($\alpha = 1,...,N$), transforming according to the fundamental representation of $SU(N)$, the action, topological charge, and duality equations have the following form (Refs. 8,9,13):

$$S=\frac{N}{2f}\int d^2x\,|D_\mu n_\alpha|^2, \quad D_\mu=\partial_\mu+iA_\mu,$$

$$A_\mu=-i\bar{n}_\alpha\partial_\mu n_\alpha, \quad \mu=1,2;$$

$$Q=\frac{1}{4\pi}\int d^2x\varepsilon_{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{2}$$

$$\bar{n}_\alpha n_\alpha=1, \quad \alpha=1,\dots,N;$$

$$D_\mu(n_{cl})_\alpha=i\varepsilon_{\mu\nu}(D_\nu n_{cl})_\alpha.$$

Here A_μ, $\mu = 1,2$ is an auxiliary gauge field. It is easy to see that in addition to the $SU(N)$ symmetry, the Lagrangian is also invariant with respect to local $U(1)$ gauge transformations:

$$n_\alpha'=e^{i\theta}n_\alpha; \quad A_\mu'=A_\mu+\partial_\mu\theta. \tag{3}$$

We place the classical solution in an $SU(2)$ subgroup labeled by the indices $\alpha = 1,2$. Then it is easy to show that the self-duality equations (2) are automatically satisfied for an arbitrary analytic function $p_{\alpha(z)}$:

$$(n_\alpha)_{cl}=p_\alpha(z)/|p_\alpha|, \quad |p|^2=\bar{p}_\alpha p_\alpha, \quad \alpha=1,2, \quad z=x_1+ix_2. \tag{4}$$

From the definition (2) of Q it is clear that the topological charge is determined by the phase acquired by the spinor n_α after traversing a circle of large radius:

$$Q=\frac{1}{4\pi}\int d^2x\varepsilon_{\mu\nu}F_{\mu\nu}=\frac{1}{2\pi}\oint_{|x|\to\infty}A_\mu\,dx_\mu=\frac{1}{2\pi}\oint\partial\varphi,$$
$$n(x\to\infty)=n_0e^{i\varphi}, \quad n_0=\text{const}, \quad A_\mu(x\to\infty)=\partial_\mu\varphi. \tag{5}$$

The standard instanton with $Q = 1$ corresponds to the function $p_\alpha(z)$:

$$p_\alpha=u_\alpha\rho+v_\alpha(z-a), \quad u_\alpha\bar{v}_\alpha=0,$$
$$|p|=(|\rho|^2+|z-a|^2)^{1/2}, \quad \bar{u}_\alpha u_\alpha=\bar{v}_\alpha v_\alpha=1, \tag{6}$$

expressed in terms of the unit vectors u_α, v_α associated with the $SU(2)$ subgroup we have selected. Indeed, since $p_\alpha(z)$ depends only on z, the self-duality equations are automatically satisfied. Moreover, after traversing a circle of large

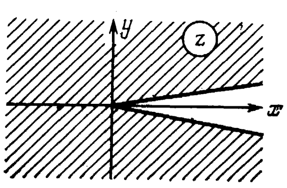

FIG. 1.

radius, the spinor n_α acquires a phase 2π which according to Eq. (5) corresponds to $Q = 1$.

We are now ready, following the logic of Refs. 1 and 2, to describe a self-dual solution defined on N Riemann surfaces and exhibiting $Q = 1/N$. In the same manner as for the $O(3)\sigma$-model (which is equivalent to a CP^{N-1} theory with $N = 2$),[2)] the solution is defined by taking the limit $\Delta\to 0$, corresponding to a regularization of the fixed points of the orbifold (see below):

$$p_\alpha=u_\alpha\Delta^{1/N}+v_\alpha(z-a)^{1/N}, \quad \Delta\to 0, \quad S_{\kappa\pi}=\frac{N\pi}{f}Q=\pi/f,$$
$$|p|=(|\Delta|^{2/N}+|z-a|^{2/N})^{1/2}, \quad Q=1/N. \tag{7}$$

A characteristic feature of the solution (7), just as in the cases discussed in Refs. 1,2, is the existence of a cut. We are thus led to the problem of describing a system defined on a manifold with boundary (see Fig. 1).

If from the outset $\Delta = 0$ one sets then $(n_\alpha)_{cl} = v_\alpha\exp(i\theta/N)$, which according to Eq. (3) corresponds to a gauge rotation of the pure vacuum solution. At a first glance this means that such a configuration cannot cause any physical effects. The experience with the $O(3)\sigma$-model[1] and $SU(2)$ gauge theory[2] as well as the analysis we have carried out shows that this is not so. We convince ourselves that in the supersymmetric CP^{N-1} model the solution (7) for $\Delta\to 0$ insures a nonvanishing value of the chiral condensate.

Below we set $a = 0, \Delta = 1$ without loss of generality. We thus place the toron at the origin and measure all quantities on the scale of the parameter Δ. The dependence on Δ can be reestablished by dimensionalizing the final expressions.

Some words are in order about the interpretation of the toron solution (see footnote[1)]). For the sake of concreteness we consider the case $N = 2$. We map the manifold with cut (Fig. 1) conformally onto a disk of radius R (Fig. 2):

$$\omega=R(1+i\tilde{z})/(1-i\tilde{z}), \quad z=\tilde{z}^2. \tag{8}$$

In terms of the variable ω the physical space corresponds to the interior of the disk of radius R. Defining the theory in the disk and taking the limit $R\to\infty$ at the end of the computations, we arrive at the original formulation of the theory on two-dimensional Euclidean space. In this interpretation the

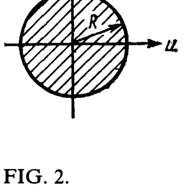

FIG. 2.

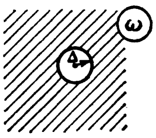

FIG. 3.

toron solution is "smeared" out over all of space, and in this respect it is strongly reminiscent of the solution of Ref. 7.

One can proceed differently and map conformally onto the exterior of a circle of radius $\Delta\to 0$ (Fig. 3). In this case the toron solution can be interpreted as a point effect (for $\Delta\to 0$). Since the Lagrangian (2) is invariant under conformal mappings, the magnitude of the toron action $S_{\rm cl} = \pi/f$ does not depend on the presence of the dimensional parameters Δ and R.

Thus, the description of the self-dual solution with fractional topological charge on a manifold with boundary does not present particular difficulties. Problems arise elsewhere: is this solution stable with respect to quantum fluctuations? Is the toron contribution to $\langle\bar\psi\psi\rangle$ non-zero after taking the limit $\Delta\to 0$ (or $R\to\infty$)? Anticipating the events we note that although on the manifolds with boundary under consideration one can construct solutions with arbitrary topological charge, only certain values of $Q\sim 1/N$ correspond to stable solutions.

There exists an alternative point of view on the toron solution. It is related to a compactification of the complex $\tilde z$ plane into a sphere and identification of the appropriate points (for details see the Appendix of Ref. 1). The manifold constructed this way is an orbifold. We shall not dwell here in detail on this question, since it is more convenient technically to work with the initial manifold of Fig. 1.

Thus, in order to answer the questions posed above it is necessary to calculate the quantum fluctuations superposed on the classical solution and to determine the toron measure. The sections that follow are dedicated to the consideration of these problems.

To summarize one may assert the following. The standard instanton solution is defined on a compact manifold without boundary (a sphere). The toron solution can be defined only on a manifold with boundary. In this case all the characteristic sizes z which determine the physical quantities (action, etc.) are characterized by an external parameter $\Delta\to 0$. Thus one interprets the toron solution as a point defect for $\Delta = 0$, regularized in such a manner that the self-duality equations should be satisfied also for nonzero values of Δ. As we shall see below, such a regularization of a point defect insures a nonvanishing value for $\langle\bar\psi\psi\rangle$ for $\Delta\to 0$. In addition, in asymptotically free theories (and the CP^{N-1} models belong to this category) when $z\sim\Delta\to 0, g(z\sim\Delta\to 0)\to 0$, the standard quasiclassical calculation, based on the toron solution, is fully controllable.

3. THE EIGENVALUE PROBLEM FOR FLUCTUATIONS IN THE FIELD OF A TORON

As is usual for quasiclassical calculation, it is necessary to expand the field n_α in the vicinity of the classical solution $(N_\alpha)_{cl}$. Retaining only the quadratic terms we are led to the expression (Refs. 12,14–16):

$$S=S_{\rm cl}+\int d^2x\,\overline{\delta n_\alpha}M_{\alpha\beta}\delta n_\beta,\qquad \partial=\frac{1}{2}(\partial_1-i\partial_2),$$
$$-M_{\alpha\beta}=|p|\partial\,\frac{1}{|p|^2}\left(\delta_{\alpha\beta}-\frac{p_\alpha\bar p_\beta}{|p|^2}\right)\bar\partial|p|,\qquad \overline{\delta n_\alpha}p_\alpha=0. \tag{9}$$

Here δn_α is a fluctuation describing a small deviation from the classical solution, and $p_\alpha(z)$ is the toron solution, defined by the expression (7). The supplementary condition $\delta n_\alpha\bar p_\alpha$ written out in Eq. (9) is a consequence of the constraint $n_\alpha\bar n_\alpha = |(n_\alpha)_{cl}+\delta n_\alpha|^2 = 1$.

Further, following Ref. 16, we first consider fluctuations

$$\delta n_\alpha=\omega_\alpha G,\quad \alpha=3,4,\dots,N,\quad \omega_\alpha\bar u_\alpha=0,\quad \omega_\alpha\bar v_\alpha=0, \tag{10}$$

which do not belong to the subspace of the $SU(2)$ subgroup spanned by the unit vectors u_α and v_α, Eq. (7). Here ω_α is a constant unit vector, orthogonal to both vectors u_α and v_α. We separately consider the fluctuations

$$\delta n_\alpha=t_\alpha F,\quad \alpha=1,2,\quad t_\alpha=(v_\alpha\bar\Delta^{1/N}-u_\alpha\bar z^{1/N})/|p|,$$
$$t_\alpha\bar t_\alpha=1,\quad t_\alpha\bar p_\alpha=0,\quad t_\alpha\bar\omega_\alpha=0 \tag{11}$$

belonging to the $SU(2)$ subgroup which contains the classical solution (7) and which are orthogonal to the latter.

Thus, the additional requirements $\delta n_\alpha(n_\alpha)_{\rm cl}=0$ which is due to the constraint is satisfied and the problem reduces to the determination of the scalar functions G, Eq. (10), and F, Eq. (11). Crucial for the subsequent analysis is the following change of variables:

$$\eta=\frac{|z|^{2/N}-1}{|z|^{2/N}+1},\quad \varphi=\operatorname{arctg}\frac{x_2}{x_1},\quad -1\leqslant\eta\leqslant 1,\quad 0\leqslant\varphi<2\pi. \tag{12}$$

The physical meaning of η and φ is obvious: they are the corresponding coordinates on the sphere obtained by compactification of the complex z plane with boundary (Fig. 1). In terms of the variables η and φ the problem of diagonalizing the quadratic form (9) we are interested in reduces to the well-known equations for the d-functions. Indeed, substituting the relations (10)–(12) into Eq. (9) we are led to the following expression for the quadratic form:

$$S=S_{\rm cl}+\int_{-1}^{1}d\eta\int_0^{2\pi}d\varphi\bar G\Big\{-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}-\frac{1}{1-\eta^2}\Big[N^2\frac{\partial^2}{\partial\varphi^2}$$
$$-iN(1+\eta)\frac{\partial}{\partial\varphi}-\frac{1}{2}(1+\eta)\Big]-\frac{3}{4}\Big\}G$$
$$+\int_{-1}^{1}d\eta\int_0^{2\pi}d\varphi\bar F\Big\{-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}-\frac{1}{1-\eta^2}\Big[N^2\frac{\partial^2}{\partial\varphi^2}$$
$$-2iN(1+\eta)\frac{\partial}{\partial\varphi}-2(1+\eta)\Big]-2\Big\}F. \tag{13}$$

We note that the standard requirement of single-valuedness for the functions $F,G\propto\exp(il\varphi), l=0,1,\dots$ is not mandatory since our initial manifold (Fig. 1) had a boundary for $\varphi=0$ and 2π. Thus the points $\varphi=0$ and $\varphi=2\pi$ are not identified. However, as can be seen from the expression (7), the toron solution is defined on a Riemann surface of N sheets with the appropriate gluing of the Nth sheet to the first and the corresponding identification. This guarantees exactly a φ-dependence of the functions F, G of the form

$$F,\ G\propto\exp(im\varphi/N),\quad m=0,\ 1,\ldots \tag{14}$$

As will be seen below the behavior (14) is responsible for the existence of regular solutions. One could proceed otherwise, namely require the regularity of the functions F and G. This would inevitably lead to Eq. (14). Thus, taking into account Eqs. (13) and (14) we are led to the following eigenvalue problem:

$$\Big\{-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}+\frac{1}{1-\eta^2}\Big[\Big(m^2-m+\frac{1}{2}\Big)-\eta\Big(m-\frac{1}{2}\Big)\Big]-\frac{3}{4}\Big\}G=\lambda_G G, \tag{15a}$$

$$\Big\{-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}+\frac{1}{1-\eta^2}\times[(m^2-2m+2)-\eta(2m-2)]-2\Big\}F=\lambda_F F. \tag{15b}$$

The regular solutions of Eqs. (15) are well known and are described, e.g., in Ref. 17. The result has the form

$$G\sim\exp\Big(i\frac{m}{N}\varphi\Big)d^{j+1/2}_{m-1/2,-1/2}(\eta),\quad \lambda_G=j(j+2),\quad m,j=0,1,\ldots, \tag{16a}$$

$$F\sim\exp\Big(i\frac{m}{N}\varphi\Big)d^{j+1}_{1-m,1}(\eta),\quad \lambda_F=j(j+3),\quad m,j=0,1,\ldots \tag{16b}$$

We note that the eigenvalues (16) agree exactly with the values obtained in instanton calculations (Ref. 16). However the degree of degeneracy g_j of each of the modes is different in the two cases. If this were not so, on account of the agreement of the eigenvalues λ_j the toron and the instanton determinants

$$\propto\exp(d^2x\delta\bar n_\alpha M_{\alpha\beta}\delta n_\beta)\propto\prod_j(\lambda_j)^{-1/2g_j}$$

would coincide exactly, which is obviously not the case. A similar situation has arisen in the $O(3)\sigma$-model (Ref. 1), where important consequences of this difference were pointed out. Additional requirements which will be discussed in the following section, single out from the whole set of solutions (16) the admissible ones and determine the degree of degeneracy of each of the modes. As usual, in computing the functional integral, it is necessary to normalize to the vacuum solution, which for definiteness we align with the vector v_α. In this case the modes satisfy the standard equation for the Legendre polynomials:

$$\Big[-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}+\frac{m^2}{1-\eta^2}\Big]\delta n_\alpha=\lambda\delta n_\alpha, \tag{17}$$

$$\delta n_\alpha\sim(u_\alpha,\omega_\alpha)\exp\Big(i\frac{m}{N}\varphi\Big)P_{jm}(\eta),\quad \bar u_\alpha v_\alpha=0,\quad \bar\omega_\alpha v_\alpha=0$$

with the eigenvalues $\lambda=j(j+1)$.

Before preoccupying ourselves fully with the selection of admissible modes from among those enumerated in Eqs. (16) and (17) we briefly recall some results referring to the fermion determinant.

In the quasiclassical approximation the correction to the action is defined by the expression (Ref. 12):

$$\Delta S_f=\int d^2x\bar\psi_\alpha L_{\alpha\beta}\psi_\beta,\quad \psi_\alpha(\bar n_\alpha)_{cl}=0,$$

$$L_{\alpha\beta}=i\Big(\delta_{\alpha\beta}-\frac{p_\alpha\bar p_\beta}{|p|^2}\Big)\begin{pmatrix}0 & |p|\partial\frac{1}{|p|}\\ \frac{1}{|p|}\bar\partial|p| & 0\end{pmatrix}. \tag{18}$$

Here

$$\psi_\alpha=\begin{pmatrix}\psi_\alpha^1\\ \psi_\alpha^2\end{pmatrix}$$

is a two-component spinor each component of which transforms according to the fundamental representation of $SU(N)$. In the sequel we shall need the following important property of the spinor ψ_α: the eigenfunctions of its upper component ψ^1_α satisfy exactly the corresponding bosonic equations (9) and thus the eigenvalues of the operator L, Eq. (18), are equal to $\pm\lambda^{1/2}_{F,G}$. Strictly speaking this assertion refers only to non-zero modes. However, since the null eigenfunctions are orthogonal to all the others, it is clear that they too coincide with the bosonic modes. Indeed, as in the boson case, separating the necessary structure which ensures orthogonally to the classical solution

$$\psi_\alpha=\omega_\alpha\begin{pmatrix}G_1\\ iG_2\end{pmatrix}+t_\alpha\begin{pmatrix}F_1\\ iF_2\end{pmatrix}, \tag{19}$$

we are led to the following equations for the fermionic modes:

$$\begin{cases}-|p|\partial\frac{1}{|p|}G_2=\lambda G_1,\\ \frac{1}{|p|}\bar\partial|p|G_1=\lambda G_2,\end{cases}\qquad\begin{cases}-|p|^2\partial\frac{1}{|p|^2}F_2=\lambda F_1,\\ \frac{1}{|p|^2}\bar\partial|p|^2F_1=\lambda F_2.\end{cases} \tag{20}$$

Combining Eqs. (20) it is easy to see that the upper components G_1 and F_1 satisfy exactly the equations defined by the bosonic form (9) with eigenvalues $\lambda^2=\lambda_{F,G}$ and the lower components G_2 and F_2 can be reconstructed uniquely from the upper components by means of Eq. (20). It is clear that the twofold degeneracy of the fermionic modes (the presence of solutions with [...1...]) is a consequence of the chiral symmetry, and the eigenvalues coincide because of the supersymmetry of the model.

The cancellation of the contributions of the non-zero modes of the bosons and fermions in the functional integral now looks perfectly obvious. In spite of this cancellation it is necessary to know (essentially owing to the existence of zero modes) which among the modes enumerated in Eqs. (16), (17), (19) satisfy all the requirements to be discussed below. The next section is dedicated to developing the appropriate criteria.

4. ON THE REQUIREMENTS IMPOSED ON THE MODELS

The selection criteria for modes are simplest to formulate for the example of the CP^1 theory, which is equivalent to the $O(3)\sigma$-model. In this case the Lagrangian can be written in a form which contains only physical degrees of freedom. In terms of a single complex field $\varphi(x_1,x_2)$ without any constraint, the Lagrangian of the $O(3)\sigma$-model has the form

$$S=\frac{1}{f}\int d^2x\frac{|\partial_\mu\varphi|^2}{(1+\bar{\varphi}\varphi)^2}. \qquad (21)$$

This can be shown to be equivalent to the original formulation (2) by means of the relation

$$\varphi=n_2/n_1. \qquad (22)$$

Here n_1 (n_2) is the magnitude of the projection of the spinor n_α onto the directions u_α (v_α). In addition we restrict our attention to the analysis of zero modes only, on account of the cancellation, noted above, of the contributions of the non-zero modes in supersymmetric theories. Recall (Ref. 1), Eq. (7) that in terms of Eq. (22) the toron solution corresponds to the function $\varphi(z)=(\Delta/z)^{1/2}$, and the equation for the zero mode is simply the Cauchy-Riemann equation:

$$\bar{\partial}(\delta\varphi_0)=0. \qquad (23)$$

An important condition reducing the arbitrariness in the selection of zero modes is related to the finiteness requirement, which in the case of the $O(3)\sigma$-model has the following form

$$|\delta\varphi_0|^2/(1+\bar{\varphi}_{cl}\varphi_{cl})^2_{z\to 0}=\text{const}.$$

This condition, in addition to the single-valuedness requirement on the physical plane (or, what amounts to the same, on one Riemann sheet) leads to the existence of only one nontrivial complex zero mode $\delta\varphi_0\propto 1/z$ (Ref. 1).

In order to answer the question how the conditions listed above look in the formulation (2) which contains a local gauge invariance and redundant degrees of freedom, we recall that the number of nontrivial zero modes is determined by the difference between the number of zero modes in the field of the toron (16b) and in the field of the vacuum (17). In the vacuum field the solution (17) $\delta n_\alpha\propto u_\alpha P_{00}$ with $\lambda=0$ corresponds to the values $j=0$, $m=0$. Thus the degree of degeneracy is in this case equal to $g_{vac}=2$ (Ref. 3). In the toron field three values $m=0,1,2$ correspond to the zero mode $\lambda_F=0$ given by Eq. (16b) with $j=0$. However, only two of these functions are orthogonal on the physical sheet (on the complete Riemann surface which contains two sheets, all three components are, of course, orthogonal):

$$\delta n_\alpha\propto t_\alpha F,\quad F\propto d_{11}^1\sim(1-\eta), \qquad (24a)$$

$$t_\alpha=(v_\alpha\bar{\Delta}^{1/2}-u_\alpha\bar{z}^{1/2})/|p|,\quad F\propto d_{-11}^1e^{i\varphi}\propto(1+\eta)e^{i\varphi}. \qquad (24b)$$

Thus, from the complete set of solutions (16) we select a system which is orthonormal on one physical sheet. As explained in Ref. 1 for the example of the $O(3)\sigma$-model, this leads exactly to a reduction of the degeneracy g_j compared to the instanton case, such that the measure automatically retains its renormalization-group invariant form.

Another requirement consists in the absence of singularities in the eigenfunctions in the whole range of their definition. This condition singles out from the whole set of solutions the regular ones, (16), which are proportional to d-functions.

Thus in a toron field there are four zero modes (24); in free space there are only two. Consequently there exist only 4–2 = 2 nontrivial zero modes in agreement with the existence of the two parameters (a) which describe the position of the toron (7) (we recall that the parameter $\Delta\to 0$ in Eq. (7) is a regulator and not a collective variable).

We now convince ourselves that both of the formulations described lead to the same result for the modes. For this we determine the relation between the fluctuations $\delta\varphi_0$ in the formulation (21) and the fluctuations δn_α in terms of Eq. (2). Using the relation (22) we have

$$\delta\varphi=[\delta n_2(n_1)_{cl}-\delta n_1(n_2)_{cl}]/(n_1)_{cl}^2. \qquad (25)$$

To the solution (24a) corresponds the zero mode $\delta\varphi_0=1$ and to (24b), $\delta\varphi_0\propto 1/z$. As far as the mode $\delta\varphi_0=1$ is concerned, it is trivial and related to the possibility of variation of the boundary conditions (a similar mode exists also in the absence of the toron, i.e., in vacuum). The other nontrivial mode $\delta\varphi_0\propto 1/z$ had already been obtained independently.

The lesson one can learn from the preceding analysis is the following. The mode $\delta\varphi_0\propto 1/z$ in the formulation (21) is single-valued. The same mode, but in the formulation (2) with local gauge invariance is not of this nature (t_α contains the factor $\bar{z}^{1/2}$). However, the invariant quantity $\overline{\delta n_\alpha}\delta n_\alpha$ is a single-valued function.

Thus, the requirement that has to be imposed on the eigenfunctions for theories with redundant degrees of freedom [of the type of gauge theories (2)] consists in the single-valuedness of invariant quantities. Just such quantities have a physical meaning.

The same requirement can be understood from a totally different point of view, namely that of the APS index theorem[3] for manifolds with boundary. As is well known (Ref. 3), in order to calculate the number of fermionic zero modes in this case, it is necessary to impose global boundary conditions so that the operator (18) would be self-adjoint on the manifold with boundary. Since the operator L in Eq. (18) is of first order, self-adjointness means simply the possibility of integrating by parts and throwing away the total derivative. As is easy to see,[3] this total derivative coincides with the integral $\int dy$ over the boundary Y of the scalar product of some eigenfunctions of the operator (18). Thus, the global boundary conditions have the form (Ref. 3)

$$\psi_\alpha=\begin{pmatrix}\psi_\alpha^1\\ \psi_\alpha^2\end{pmatrix},\qquad \int_Y dy\bar{\psi}_\alpha^1\psi_\alpha^2+\text{h.c.}=0. \qquad (26)$$

In particular, for the manifold of Fig. 1 with the boundary defined by the condition $\varphi=2\pi,\varphi=0$ we have

$$\int_0^\infty dr(\bar{\psi}_\alpha^1\psi_\alpha^2)|_{\varphi=0}-\int_0^\infty dr(\bar{\psi}_\alpha^1\psi_\alpha^2)|_{\varphi=2\pi}+\text{h.c.}=0.$$

In more detail the application of the APS theorem[3] to our conditions is discussed in Ref. 19. We note here that if the manifold were without boundary and the eigenfunctions were single-valued, then the relation (26) would be automatically fulfilled. In the case under consideration the condition (26) is very strong: it requires that any invariant quantities should coincide on the two edges of the cut, i.e., for $\varphi=2\pi$ and $\varphi=0$. But this coincides with the condition formulated above for the bosonic degrees of freedom. The fact that the bosonic and fermionic degrees of freedom are related has already been mentioned at the end of the preceding section. Here we emphasize the fact that the single-valuedness requirement of the invariant quantities (rather than the functions themselves) appears from a completely different side: the requirement that the operator should be self-adjoint and the condition (26) related to it.

As was already said, the toron solution can be described on a disk (Fig. 2) or the exterior of a circle (Fig. 3). For the appropriate analysis, see Ref. 19. We note here that the number of nontrivial fermion zero modes equals two, the same as for the bosonic modes. This assertion is valid both for the disk and for the manifold of Fig. 1. Indeed, the spinor (19) with vanishing lower component ($F_2 = G_2 = 0$) and nonvanishing upper component equal to $F_1 = F(\lambda = 0)$, $G_1 = G(\lambda = 0)$, Eq. (16), automatically satisfies the equations (20) with eigenvalue $\lambda = 0$.

Thus, the number of nontrivial bosonic and fermionic zero modes coincides. This assertion has a fairly general foundation (Ref. 18).

As an application of the requirements considered above we describe the zero modes for the group $SU(3)$ with $Q = 1/3$. As we shall see, for other fractional values of q for the group $SU(3)$ it is impossible to satisfy the conditions discussed above. We interpret this fact as an instability of the corresponding configurations. Thus, the form of the group $SU(3)$ uniquely fixes the admissible value $Q = 1/3$. The result has a trivial generalization to arbitrary N.

We start the analysis in the vacuum (17). The number of zero modes corresponding to the value $j = 0$ equals $2(N-1)_{N=3} = 4$ (see footnote 3). We now consider modes in the toron field belonging to the distinguished $SU(2)$ subgroup [the F functions (16b)]. In this case it is easy to see that the requirements of orthogonality and single-valuedness of the scalar products on the physical sheet are satisfied only by one complex (two real) modes among the three ($m = 0,1,2$). The situation with the states along the unit vector ω_3 [the G functions (16a)] is less trivial. The value $\lambda = 0$ corresponds to $j = 0$, $m = 0,1$ in (16a). It is easy to show that one can construct two combinations from these solutions, satisfying all the imposed requirements:

$$\delta n_3{}^{\rm I} \sim \omega_3 d^{(1/2)}_{-1/2\,-1/2} \sim \omega_3 (1-\eta)^{1/2}, \tag{27}$$

$$\delta n_3{}^{\rm II} \sim \omega_3 \left[d^{(1/2)}_{-1/2\,-1/2} + \frac{8}{3^{3/2}} e^{i2\pi/3} d^{(1/2)}_{1/2\,-1/2} e^{i\varphi/3} \right]$$

$$\sim \omega_3 \left[(1-\eta)^{1/2} + \frac{8}{3^{3/2}} (1+\eta)^{1/2} e^{i(2\pi+\varphi)/3} \right].$$

The modes (27) are mutually orthogonal:

$$\int_{-1}^{1} d\eta \int_0^{2\pi} d\varphi \overline{\delta n_3{}^{\rm I}} \delta n_3{}^{\rm II} = 0 \tag{28}$$

and single-valued on the physical sheet:

$$|\delta n_3{}^{\rm I}|^2 \sim (1-\eta),$$

$$|\delta n_3{}^{\rm II}|^2 \sim (1-\eta) + \left(\frac{8}{3^{3/2}}\right)^2 (1+\eta) + \frac{16}{3^{3/2}} \cos\left(\frac{2\pi+\varphi}{3}\right) (1-\eta^2)^{1/2}. \tag{29}$$

Indeed, the only nontrivial function of the angle φ occurs in Eq. (29) in the form $\cos[2\pi + \varphi)/3]$, having the same limit for $\varphi \sim \varepsilon \to 0$ and $\varphi \sim 2\pi - \varepsilon$. Thus, there are four zero modes (27), along the unit vector ω_3. Together with the two zero modes from the group $SU(2)$ this ensures the existence of six zero modes in the toron field. Only $6 - 4 = 2$ of them are nontrivial, which is due to the existence of four vacuum modes (see above). This number (two) agrees, of course, with the existence of two parameters a which characterize the position of the toron (7). The number of nontrivial fermionic zero modes is also equal to two, and for the reason mentioned above their explicit expressions coincide exactly with the form of the bosonic zero modes.

We note that the existence of exactly two nontrivial fermionic zero modes is a consequence of the index theorem for manifolds with boundary (Refs. 3,19):

$$n_+ - n_- = 2[NQ]. \tag{30}$$

Here the square brackets [ ] denote the integer part of a number. Only the specific realizations of this theorem are of interest.

In concluding the present section we briefly describe the situation which arises when the number N increases. In going from $N = 3$ to $N = 4$ two new features appear: first, the zero mode acquires a phase $\exp(i\varphi/4)$ [Eq. (16a)], in place of $\exp(i\varphi/3)$, for the $SU(3)$ case; second, new modes appear together with the appearance of a new unit vector ω_4. These two facts have the consequence that there are two functions (depending nontrivially on the angle φ) which are proportional to $\cos[(2\pi + \varphi)/4]$ and $\cos[(4\pi + \varphi)/4]$, out of which one can construct a linear combination proportional to $\cos[(2\pi + \varphi)/4] + \cos[(4\pi + \varphi)/4]$ which is single-valued on the edges of the cut (for $\varphi \to 0$ and $\varphi \to 2\pi$). The situation is generic: with the growth of N and the decrease of the phase angle $\exp(i\varphi/N)$ there appears an additional term proportional to $\cos[(2\pi k + \varphi)/N]$, related to the appearance of the unit vector ω_N. This guarantees the existence of two nontrivial zero modes (bosonic and fermionic) in accord with the existence of two parameters a, describing the position of the toron. These modes satisfy all the necessary requirements.

Summarizing, it should be noted that from completely different points of view (the index theorem, Ref. 3, and the analysis of the CP^{N-1}-model in terms of the unconstrained field φ) we have developed selection criteria for admissible modes. It turned out that the appropriate conditions for the group $SU(N)$ are satisfied only by configurations with $Q \sim 1/N$. In addition, the hermiticity of the Hamiltonian, the equality of gauge invariant quantities across the cut, the existence of an othonormal set of eigenfunctions, all these questions are interrelated in the formulation of the theory on a manifold with boundary. In addition to the quantities mentioned, the distinguished character of configurations with $Q \sim 1/N$ manifests itself in their stability.[19]

Having shown that the modes determined above satisfy all the requirements, one can easily calculate the toron measure and the condensate $\langle\bar{\psi}\psi\rangle$ in the supersymmetric CP^{N-1} model. As we explained, the contribution of the nonzero modes cancels exactly and therefore only the zero mode analysis carried out above is required.

We recall (Refs. 12,14–16) that in the calculation of the functional integral Z in the quasiclassical approximation, each nontrivial bosonic zero mode leads to a factor $M_0 dx_0$. Here M_0 is the ultraviolet cutoff, dx_0 is the integration measure with respect to a collective variable corresponding to this zero mode. Each fermionic zero mode is accompanied by a factor $d\varepsilon/M_0^{1/2}$, where $d\varepsilon$ is the measure associated to integration with respect to a collective Grassmann variable.

Taking into account what was said, the toron measure takes the form

$$Z_{\text{toron}} \sim M_0{}^2 d^2a\,(d^2\varepsilon/M_0)\exp[-\pi/f(M_0)] = md^2ad^2\varepsilon,$$
$$m = M_0\exp(-\pi/f). \qquad (31)$$

Here the factor $M_0^2 d^2 a$ is due to the two bosons, and the factor d_ε^2/M_0 is due to the two fermionic zero modes, mentioned above. The factor $\exp(-\pi/f)$ is the contribution of the classical action of the toron (7) and the parameter α describes the position of the toron. Just as for the instanton calculations (Ref. 12), the expression (31) has an exactly renormalization-group-invariant form. This phenomenon is easily explained: as the action decreases the number of zero modes decreases, which regenerates exactly the renormalization-group invariant result.

Now everything is prepared for the calculation of the chiral condensate $\langle\bar\psi\psi\rangle$. Substituting the zero modes for the fields ψ and taking into account that the Brezin integral over the collective fermion variables in (31) is done according to the rule $\varepsilon\bar\varepsilon d\varepsilon d\bar\varepsilon = 1$, we convince ourselves that

$$\langle\bar\psi\psi\rangle \propto m\int d^2a\,\bar\psi_0(x-a)\,\psi_0(x-a) = m. \qquad (32)$$

During the last stage we used the value of the normalization integral for the zero mode, and subsequently removed the regularization by setting $\Delta = 0$.

As is known (Ref. 9), the relation (32) signifies spontaneous breaking of the discrete chiral symmetry. We note that an instanton may guarantee a nonvanishing value only for the correlator $\langle\Pi\bar\psi\psi(x_i)\rangle$ (Ref. 12) in accord with the fact that the solution with $Q = 1$ changes the chiral charge ΔQ_5 (Eq. 1), by two units and has two nontrivial zero modes. Thus the corresponding vacuum-to-vacuum transition is accompanied by the creation of a pair $\bar\psi\psi$, which is explicitly demonstrated by the calculation (32).

5. CONCLUSION

The main purpose of this paper was to describe a self-dual solution with $Q = 1/N$ on a manifold with boundary and the analysis of the physical consequences related to this solution, on the example of the supersymmetric CP^{N-1}-model. Arguments were advanced in favor of an interpretation of the toron as a point defect, regularized in such a manner that the self-duality equations are preserved and a nonzero value is guaranteed for $\langle\bar\psi\psi\rangle$ after removing the regularization ($\Delta\to 0$).

Other topics considered in the paper are related to the development of a selection criterion for admissible modes in the field of a toron. We have formulated the self-adjointness requirement for the Hamiltonian on a manifold with boundary. This ensured single-valuedness of the gauge-invariant quantities on the two sides of the cut. In turn, this means the validity of global boundary conditions for the fermionic modes (Ref. 3) and the stability of the classical configuration for the bosonic ones. This is the principle which selects the admissible values $Q\sim 1/N$.

We conjecture that the formulated principle is general enough and will, in particular, determine the admissible values of the fractional topological charge in a supersymmetric Yang-Mills theory with arbitrary gauge group.[20]

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[1)] In the present paper we retain the term "toron" introduced in Ref. 7 and utilized in Refs. 1 and 2. By this we underline the fact that the solution minimizes the action and has topological charge $Q = 1/N$, i.e., exhibits all the properties characteristic for a toron in the sense of Ref. 7.

[2)] The equivalence with a $O(3)\sigma$-model is verified by using the relations: $n^a = \bar n\sigma^a n$, where σ^a are the ordinary Pauli matrices, and n^α is a real unit vector, characterizing the dynamics of the $O(3)\sigma$-model.

[3)] We count the number of real zero modes. The solution (17) is defined up to a complex phase, which leads to the appearance of the additional factor 2.

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