Influence of kinetic effects on the growth of a two-dimensional dendrite

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An analytic study is conducted, within the framework of the exact nonlocal thermal conductivity problem, of the effect of the surface energy anisotropy and the kinetics anisotropy on the selection of the rate and direction of growth of a two-dimensional dendrite. At low supercooling the growth is basically determined by effects related to surface energy, while at high supercooling the growth process may become determined by kinetic effects. If the growth directions determined by surface energy and by kinetics coincide, an increase in any anisotropy leads to an increase in the growth velocity. But when the growth directions determined by these effects do not coincide, the dendrite changes its growth direction as supercooling increases.

1. INTRODUCTION

The surface-energy anisotropy of an interphase boundary substantially affects the growth rate and direction of dendrites growing from the supercooled melt. Taking into account the Gibbs-Thomson effect results in a difference between the temperature on the crystallization front and the melting temperature, by a value proportional to the front curvature. In this case the proportionality coefficient depends on the surface energy of the boundary. It has been shown in a number of computer simulations,[1–4] and analytical studies,[5–10] that for isotropic surface energy there are no solutions of the needle-shaped-dendrite type. Anisotropy, when taken into account, leads to the appearance of a solution, and the growth rate and direction of a steady-state dendrite are uniquely determined and essentially depend on the surface-energy anisotropy.

The influence of kinetic effects on the interphase boundary boils down to the fact that the temperature on the crystallization front differs from the equilibrium temperature and depends on the growth rate. To a first low-velocity approximation the change in temperature is proportional to the normal growth rate. The coefficient of proportionality in this relation, or the kinetic coefficient, is anisotropic in the general case. It has analytically been shown in Ref. 11 that taking account of the anisotropy of the kinetic coefficient results in selecting out the growth rate, even in the absence of the Gibbs-Thomson effect. When both factors are allowed for, the growth rate is mainly determined by surface-energy-connected effects at low supercooling but by kinetic effects at high supercooling. It has been shown in Ref. 12 that the growth rate generally depends on both factors.

An interesting effect was discovered in a numerical solution of the model of the boundary layer.[13] If the directions corresponding to the extrema of the surface energy and of the kinetic coefficient do not coincide, a morphology transition takes place at some supercooling, with a change in the dendrite growth direction. In this case there is an intermediate supercooling region in which no stable solution in the form of a needle dendrite exists in any direction. In this region of supercooling a structure is realized as a result of the splitting of the dendrite tip. The similar structure behavior in viscous-liquid flow was experimentally observed in a model system, the anisotropic Heli-Shaw cell.[13]

This article reports a systematic study of the influence of the surface-energy anisotropy and of the kinetic coefficient on the selection of the velocity and growth direction of two-dimensional dendrites. The exact nonlocal problem of thermal conductivity is considered and the analytic theory developed in Refs. 6, 10, 14, and 15 is used. The results are in qualitative accordance with the results of the numerical solution of the boundary-layer model carried out in Ref. 13.

2. THE GROWTH EQUATIONS

We consider the two-dimensional problem of crystal growth from a supercooled melt. The distribution of the temperature T in a supercooled melt and in a growing crystal is described by the heat conduction equation

$$\partial T/\partial t = D\Delta T. \tag{1}$$

At the crystallization front $y(x,t)$ heat is released and the boundary condition has the form

$$c_p D[\mathbf{n}\nabla T_l - \mathbf{n}\nabla T_c] = -Lv_n. \tag{2}$$

Here c_p and D are the specific heat and the coefficient of temperature diffusivity, which are supposed to be the same in both phases, L is the latent heat of melting, v_n is the growth rate along the normal, $\mathbf{n}$ is a unit vector normal to the interphase boundary, the subsripts l and c refer to the melt and the crystal, respectively. With allowance for the Gibbs-Thomson effect and the kinetic effects at the interphase boundary, the temperature on the crystallization front has the form

$$T[x, y(x, t)] = T_m + (T_m\gamma_E(\theta)/L)\varkappa(x, t) - \tilde\beta(\theta)v_n, \tag{3}$$

where T_m is the melting temperature, $\varkappa$ is the front curvature

$$\varkappa(x, t) = d^2y(x)/dx^2[1+(dy/dx)^2]^{-3/2},$$
$$\gamma_E(\theta) = \gamma(\theta) + d^2\gamma(\theta)/d\theta^2,$$

$\gamma(\theta)$ is the surface energy, θ is the angle between the normal to the front and the y-axis, and $\tilde\beta(\theta)$ is the anisotropic kinetic coefficient ($\gamma_E, \tilde\beta > 0$). Far from the front, the melt is supercooled and has a temperature $T_0 < T_m$.

According to the boundary condition (2), the latent heat of the phase transition is released at the crystallization front $y(x,t)$, i.e., the front is a line along which the sources of heat release are concentrated. Given the movement of the front, the thermal field can be found by using the Green's function of the equation of heat conduction (1). The temperature distribution on the crystallization front, i.e., the

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left-hand side of Eq. (3), will thus be determined only by the function $y(x,t)$ itself. As a result we obtain from Eqs. (1)–(3) a single integro-differential equation which describes the dynamics of the crystallization front[16]:

$$\Delta+\frac{d(\theta)\varkappa(x,t)}{\rho}-\beta(\theta)v_n=\frac{p}{2\pi}\int_0^\infty\frac{d\tau}{\tau}\int_{-\infty}^{\infty}dx'\,\dot{y}(x,t-\tau)\times\exp\left\{-\frac{p}{2\tau}\left[(x-x')^2+[y(x,t)-y(x',t-\tau)]^2\right]\right\}. \tag{4}$$

We used the following dimensionless parameters to consider in what follows the steady-state growth of a near-parabolic dendrite with the parabola parameter ρ and the velocity v, and also to analyze the stability of the steady-state growth in Eq. (4). All lengths are measured in units of ρ, the time in units of ρ/v, $p = v\rho/2D$ is the Péclet number, and $\Delta = (T_m - T_0)c_pL^{-1}$ is the dimensionless supercooling. The capillary length is

$$d=\gamma_E(\theta)T_mc_pL^{-2},\quad \beta(\theta)=\tilde{\beta}(\theta)c_pL^{-1}.$$

Neglecting the surface energy and kinetic effects ($d=0, \beta=0$), Ivantsov[17] obtained a solution of Eq. (4) in the form of a parabola which moves with constant velocity

$$y=t-x^2/2,$$

while the Péclet number is determined by the equation

$$\Delta(p)=2p^{1/2}\exp(p)\int_{p^{1/2}}^{\infty}\exp(-x^2)\,dx. \tag{5}$$

Taking account of a finite surface energy and kinetic effects considerably changes the problem since it leads to the appearance of a term with a higher derivative in Eq. (4). In this case, apart from regular corrections to the shape of the crystallization front, singular corrections arise and in the general case their amplitude increases as we move away from the dendrite tip. The velocity and direction of the dendrite growth are determined from the condition that these corrections be finite.

In the presence of a fourfold symmetry axis, we assume for the functions $d(\theta)$ and $\beta(\theta)$ the simplest model expressions:

$$d(\theta)=d_0(1-\alpha_d\cos4(\theta-\theta_d))=d_0A_d(\theta),$$
$$\beta(\theta)=\beta_0(1-\alpha_\beta\cos4(\theta-\theta_\beta)=\beta_0A_\beta(\theta),$$
$$\operatorname{tg}\theta=dy/dx. \tag{6}$$

Here α_d and α_β are the anisotropy parameters (α_d, $\alpha_\beta \ll 1$), and the angles θ_d and θ_β characterize the deviation of the growth direction from the directions where the functions $d(\theta)$ and $\beta(\theta)$ are minimal. With allowance for the finite values of d and β, the stationary shape of the crystallization front differs from parabolic

$$y=t-x^2/2+\zeta(x).$$

When deriving the equation for ζ, the smallness of the anisotropy parameters α plays a double role. First, it determines the smallness of the correction ζ itself, which allows us to linearize the integral term in Eq. (4). Secondly, the size of the singular region, as will be seen below, is also small at small α. In this case, the derivatives of the correction in the singular region are not small, and because of this the left-hand side of Eq. (4) stays nonlinear even after all the simplifications.[6] Thus, linearizing the integral term in Eg. (4), in the limit of small Péclet numbers, we obtain the following equation for $\zeta(x)$:

$$A_d(\theta)\sigma_d[\zeta''-1][1+(\zeta'-x)^2]^{-3/2}-A_\beta(\theta)\sigma_\beta[1+(\zeta'-x)^2]^{-1/2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dx'(x+x')[\zeta(x)-\zeta(x')]}{(x-x')[1+(x+x')^2/4]}=0, \tag{7}$$

where

$$\sigma_d=d_0/p\rho,\quad \sigma_\beta=\beta_0v/p. \tag{8}$$

For small parameters σ_d and σ_β, the regular correction $\zeta\propto\sigma$ is found from the solution of Eq. (7) neglecting the derivatives which have these small parameters as coefficients. However, the terms with the derivatives represent a singular perturbation, and a solution of Eq. (7) exists only for a definite relation between the parameters σ_d and σ_β. It is this relation that determines the growth rate. In view of the absence of an exact solution of Eq. (7), we use an approximate approach known from the theory of the quantum-mechanical above-barrier reflection. The approach consists in analyzing the equation near singular points in the complex x plane. Indeed, it is seen from Eq. (7) that for small σ and α the effect of the derivatives becomes substantial only in a small region around the singular points $x=\pm i$. In this region Eq. (7) transforms into an inhomogeneous second-order differential equation. This differential equation and the integral equation obtained neglecting the derivatives have a common range of applicability, and the condition that their solutions match in this range determines the growth rate.

The procedure for deriving the differential equation near the singular point $x=i$ from Eq. (7) is described in detail in Refs. 6, 10, and 14. We obtain finally

$$\frac{d^2F}{dz^2}-\frac{2^{1/2}\lambda\tau^{7/4}}{A_d(\tau)}F=-\left[1+2\mu\tau\frac{A_\beta(\tau)}{A_d(\tau)}\right],$$
$$\zeta=\alpha_dF,\quad \tau=z+dF/dz \tag{9}$$
$$x=i(1-\alpha_d^{1/2}z),\quad A_d(\tau)=1-2\exp(4i\theta_d)/\tau^2,$$
$$A_\beta(\tau)=1-2\nu\exp(4i\theta_\beta)/\tau^2,$$

where

$$\lambda=\alpha_d^{7/4}/\sigma_d=\alpha_d^{7/4}p\rho/d_0,$$
$$\mu=\sigma_\beta\alpha_d^{1/2}/\sigma_d=p(\Delta)\alpha_d^{1/2}(2D\beta_0/d_0),\quad \nu=\alpha_\beta/\alpha_d. \tag{10}$$

In Refs. 6, 7, 10, and 14 the problem was considered without taking into account kinetic effects ($\mu=0$), with full linearization of Eq. (7) with respect to $\zeta(x)$ in Ref. 7. In our preceding articles[10,14] we also used a linearized version for obtaining an equation of type (9), which in fact comes down to the substitution $\tau=z$. This approximation is valid only for the asymptote $|z|\gg1$. In Ref. 6, the nonlinear equation (9) was obtained at $\mu=0$. The structure of this nonlinear equation, which is exact at small anisotropy α, is such that the scaling relation $\sigma_i=\lambda\alpha_d^{7/4}$, the same as in the linearized version, is valid for it. Only the numerical value of λ changes. For this reason, the results of the linear analysis are qualitatively true apart from a numerical factor in the scaling relation. Besides, the numerical factor depends on the choice of the model for the anisotropy function.

At large values of the argument $z(|z| \gg \max\{1, \nu^{1/2}\})$ but still near the singularity ($|z| \ll \alpha_d^{-1/2}$), we have a particular solution of Eq. (9)

$$F \propto (1+2\mu z) z^{-3/2},$$

which is obtained by neglecting the derivatives. This solution matches the solution of the original integral equation (7). However, this particular solution is realized only for a definite connection between the parameters λ, μ, and ν in Eq. (9). Indeed, at large $|z|$ the general solution of the linearized homogeneous equation (9)

$$F_{1,2} \propto \exp(\pm {}^4/_7 \cdot 2^{1/4} \lambda^{1/2} z^{7/4}) \tag{11}$$

shows the fastest growth along the rays $\arg(z) = 0, \pm 4\pi/7$. The particular solution, which decreases as we recede away from the singularity, can be obtained only under the condition that this exponential growth will be suppressed. For its suppression on the three rays, we have only two integration constants at our disposal. Therefore, the third condition connects the parameters, i.e., λ appears a perfectly defined function of μ and ν. This function can be found by numerical integration of Eq. (9) under the condition that the solution be bounded at large values of $|z|$ along the rays $\arg(z) = 0$, $\pm 4\pi/7$. Besides, various asymptotes of the function $\lambda(\mu,\nu)$ can be discerned analytically.

In final analysis, knowledge of the function $\lambda(\mu,\nu)$ determines the dependence of the growth rate on the supercooling and other parameters of the problem. It is worth noting that the parameter μ in Eq. (10) does not depend on the growth rate, and at given parameters of the system it depends only on the supercooling through the function $p(\Delta)$ of Eq. (5). On the other hand, the parameter

$$\mu^2/\lambda = \beta_0 (2D\beta_0/d_0) \alpha_d^{-3/4} v \tag{12}$$

is the growth rate v, accurate to the fixed parameters of the system. Therefore, by determining the dependence of μ^2/λ on μ we obtain in fact the dependence of the growth rate v on $p(\Delta)$, where Δ is the supercooling.

Let us note an important detail mentioned in Ref. 10. Although the integral term in Eq. (7) is obtained in the limit of small Péclet numbers, $p \ll 1$, Eq. (9) for the singular part of perturbation has a wider range of applicability in p. The point is that the size of the singular region is small to within the small parameter $\alpha = \max\{\alpha_d, \alpha_\beta\}$, and the range of applicability of Eq. (9) is determined by the condition $p\alpha^{1/2} \ll 1$, which at $\alpha \ll 1$ allows us to extend the results into the region of large Péclet numbers.[10]

Besides obtaining the stationary solutions, it is necessary to investigate their stability. In this case we seek the correction to the stationary shape in the form

$$\zeta_1(x,t) \propto \xi_1(x) \exp(\Omega t),$$

and $\operatorname{Re}\Omega < 0$ corresponds to stable solutions. An analytical theory of the stability of needle-dendrite solutions is developed in Ref. 15. Taking account of the nonlinearity of Eqs. (7) and (9), we obtain near the singular point an equation for the eigenvalue spectrum

$$\frac{d^3\varphi}{dz^3} - \left[F \frac{d}{d\tau}\left[\frac{2^{1/2}\lambda\tau^{3/2}}{A_d(\tau)}\right] - 2\mu \frac{d}{d\tau}\left[\tau \frac{A_\beta(\tau)}{A_d(\tau)}\right]\right]\frac{d^2\varphi}{dz^2} - \frac{2^{1/2}\lambda\tau^{3/2}}{A_d(\tau)}\left[\frac{d\varphi}{dz} + 2\omega\varphi\right] = 0, \tag{13}$$

where $\omega = \Omega\alpha_d^{1/2}$, and the function φ is the integral of ζ_1

$$\varphi = \int_{-\infty}^{x} \zeta_1(x')\,dx'.$$

Parameters μ and $\lambda(\mu)$, as well as the fucntions $F(z)$ and $\tau(z)$, are determined from the stationary solution. At large values of $|z|$, the functions $\varphi_{1,2}$ behave in the same way as Eq. (11), and

$$\varphi_3 \propto \exp(-2\omega z).$$

From the condition (11) for the suppression of exponential growth along the rays $\arg(z) = 0, \pm 4\pi/7$ we can find two integration constants and the spectrum of ω (in view of linearity and homogeneity of the equation, the third constant is left arbitrary.[15]

3. SELECTION OF THE VELOCITY AND DIRECTION OF GROWTH

We proceed now to report the results of the calculation of the rate and direction of the dendrite growth, and of its stability.

At arbitrary angles θ_d and θ_β the coefficients A_d and A_β from Eq. (9) are complex and this equation has no solutions for real λ and μ (Ref. 10). Therefore, the possible values of the angles are θ_d, $\theta_\beta = 0$, $\pi/4$. This means that the crystal can grow in the direction of either the minimum $d(\theta)$ or the minimum $\beta(\theta)$, and the directions of these minima should either coincide or make an angle $\pi/4$. The latter condition is fulfilled in our case by virtue of the symmetry C_{4v} (and not merely C_4).

1. If we neglect the kinetic effects ($\beta_0 = 0, \mu = 0$), Eq. (9) has a solution at $\theta_d = 0$ and $\lambda(\mu = 0) = \lambda_0$. Then from Eqs. (10) and (12) we find

$$v = 2Dp^2\alpha_d^{7/4}/\lambda_0 d_0. \tag{14}$$

The value of the parameter $\lambda_0 \approx 0.42$ was obtained numerically. Other possible discrete values of λ, at which Eq. (9) has solutions, correspond to instability of the dendrite tip-splitting type tip. This unstability results from the presence of eigenvalues $\omega > 0$ in Eq. (13) (see Ref. 15). Note that in the presence of kinetic effects the asymptote (14) takes place for small values of the parameter μ (i.e., in the limit of small supercooling). The asymptote (14) is realized for $\nu \lesssim 1$ at $\mu \ll 1$ and for $\nu \gg 1$ at $\mu \ll 1/\nu$. The growth regime (14) corresponds to conditions when the effects connected with the surface energy play the main role in the selection of the growth rate and direction.

2. In the limit of negligible surface energy ($d_0 \to 0$), the parameters μ and λ are large. In this case it is necessary to keep in Eq. (9) only the terms which contain these large parameters. The first-order equation thus obtained can be reduced by the substitutions $z \to \nu^{1/2} z$ and $F \to F\nu$ to an equation containing only the single parameter

$$\gamma = \lambda\nu^{5/4}/\mu. \tag{15}$$

Equation (9) has then a solution when $\theta_\beta = 0$, and the numerical parameter γ runs through a discrete spectrum of values. Only the minimal value γ_0 corresponds to the stable solution. From Eqs. (10) and (12) we find for the growth rate

$$v = p\alpha_\beta^{5/4}/\gamma_0\beta_0. \tag{16}$$

This limit was considered within the framework of the linear version in Ref. 11. The growth regime (16) is realized at large values of μ when the kinetic effects play a major role in selecting the growth rate and direction. The criterion of applicability of Eq. (16) depends on the anisotropy ratio ν. At $\nu \gg 1$ the asymptote (16) holds if $\mu \gg \nu^{-1/2}$, but at $\nu \ll 1$ Eq. (16) is true if $\mu \gg \nu^{-3/2}$.

3. Let us consider the limit of isotropic kinetics, when the parameter $\nu \to 0$. The dendrite grows in the direction of minimum $d(\theta)$, i.e., $\theta_d = 0$. A plot of μ^2/λ versus μ obtained by numerical solution of Eq. (9) at $\nu = 0$ is shown in Fig. 1. At $\mu \ll 1$ the parameter $\lambda \approx \lambda_0$, and in accordance with Sec. 3.1 the asymptote (14) is realized. However at $\mu \gtrsim 1$ this law is violated and at $\mu \gg 1$ another asymptote is realized, in which case the small values $|z| \ll 1$ in Eq. (9) become important. In this limit, by the substitutions $z \to z\mu^{-1/3}$ and $F \to F\mu^{-2/3}$, we will obtain for λ

$$\lambda \propto \mu^{1\,1/6}. \tag{17}$$

From Eqs. (10), (12), and (17) we find

$$v \propto (2D\beta_0/d_0)^{-5/6} p^{1/6} \alpha_d^{5/6}/\beta_0. \tag{18}$$

A feature of this asymptote is that the growth rate depends on the kinetic coefficient β_0 but not on its anisotropy, i.e., on the parameter α_β. At small but finite values of $\nu \ll 1$, the asymptote (18) is intermediate and valid at

$$1 \ll \mu \ll \nu^{-3/2}, \tag{19}$$

and at $\mu \gg \nu^{-3/2}$ the growth rate, in accordance with Sec. 3.2, is described by Eq. (16).

4. Let us consider now another limit, when the surface energy anisotropy is smaller than the anisotropy of the kinetic coefficient, $\nu \gg 1$. Then, in conformity with Secs. 3.1 and 3.2, the asymptote (14) is realized at $\mu \ll \nu^{-1}$ and the asymptote (16) at $\mu \gg \nu^{-1/2}$. In the intermediate region

$$\nu^{-1} \ll \mu \ll \nu^{-1/2} \tag{20}$$

another type of asymptote exists. The dendrite grows along the direction of the minimum $\beta(\theta)$, i.e., $\theta_\beta = 0$, in which case the values $|z| \sim \mu\nu \gg 1$ are important in Eq. (9). Using the substitution $z \to z\mu\nu$, $F \to F\mu^2\nu^2$ we find for λ

$$\lambda = \gamma_1(\mu\nu)^{-7/2}, \tag{21}$$

where γ_1 is a numerical factor. From Eqs. (10), (12), and (21) we have for the growth rate

$$v = (2D\beta_0/d_0)^{9/2} p^{1\,1/2} \alpha_\beta^{7/2}/\gamma_1\beta_0. \tag{22}$$

The expression (22) for the growth rate does not depend on the surface-energy anisotropy α_d. For isotropic surface energy, the region of existence of the asymptote (14) vanishes, and the transition from Eq. (22) to Eq. (16) occurs at

$$p\alpha_\beta^{1/2}(2D\beta_0/d_0) \sim 1. \tag{23}$$

It is important to point out that for isotropic surface energy the solution (22) exists at any small supercooling. It was pointed out in Ref. 12 that in this case the authors failed to find a solution analytically at small supercooling and such a solution was found only numerically at finite values of supercooling.

A plot of μ^2/λ as a function of μ obtained by numerical solution of Eq. (9) in case of isotropic surface energy is presented in Fig. 2, with α_β replacing α_d in expressions (10) and (12) for μ and μ^2/λ.

When α_d and α_β are of the same order, i.e., at $\nu \sim 1$, the regions of existence of the intermediate asymptotes (18) and (22) vanish. There are only two limiting expressions (14) and (16) in this case, and the transition from (14) to (16) occurs at $\mu \sim 1$. The dependence of μ^2/λ on μ at $\theta_d = \theta_\beta = 0$ and $\nu = 1$ is represented in Figs. 1 and 2 by curve 1. It is seen from the comparison of this curve with curves 2 corresponding to isotropic kinetics (Fig. 1) and to isotropic surface energy (Fig. 2) that when the directions of the minima of $d(\theta)$ and $\beta(\theta)$ coincide, an increase in either anisotropy (α or α_β) leads to an increase in the growth rate.

5. Let us consider now the most interesting case, when the directions of the minima $d(\theta)$ and $\beta(\theta)$ do not coincide and the angle between them is equal to $\pi/4$. The dependence of μ^2/λ on μ for this case at $\nu = 1$ is presented in Fig. 3. Curve 1 corresponds to growth along the direction of the minimum $d(\theta)$, i.e., $\theta_d = 0$, $\theta_\beta = \pi/4$, and curve 2 corresponds to growth along the direction of the minimum $\beta(\theta)$, i.e., $\theta_d = \pi/4$, $\theta_\beta = 0$.

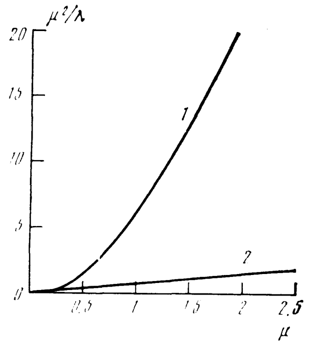

FIG. 1. Dependences of the reduced growth rate μ^2/λ on the reduced supercooling μ at $\theta_d = \theta_\beta = 0$. Curve 1 corresponds to $\alpha_d = \alpha_\beta$, curve 2 to $\alpha_\beta = 0$.

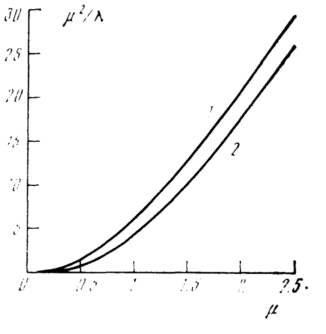

FIG. 2. The same as in Fig. 1 but curve 2 corresponds to $\alpha_d = 0$.

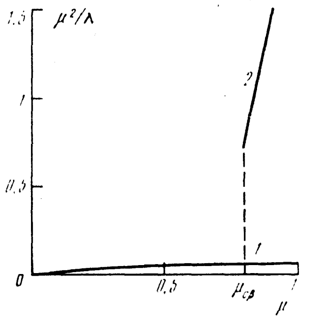

FIG. 3. Dependences of the reduced velocity μ^2/λ on the reduced supercooling μ at $\alpha_d = \alpha_\beta$. Curve 1 corresponds to $\theta_d = 0$, $\theta_\beta = \pi/4$; curve 2 to $\theta_d = \pi/4$, $\theta_\beta = 0$ (a solution exists only at $\mu > \mu_{c\beta}$).

At small values $\mu \ll 1$, curve 1 is described by the asymptote (14). As μ rises (by increasing the supercooling), the stationary solution corresponding to the dendrite growing along the direction of the minimum $d(\theta)$, i.e., along the direction predetermined by surface energy, has another asymptote

$$\lambda \approx (5.5\mu)^2(1+\gamma/\mu), \quad \mu \gg 1, \quad \nu = 1, \tag{24}$$

where the numerical parameter γ runs through a discrete spectrum of values. This result can be obtained analytically, but the calculations are too cumbersome. From Eqs. (10), (12), and (24), we find the growth rate in the form

$$v \approx 0.03\alpha_d^{3/4}(d_0/2D\beta_0^2)(1-\gamma/\mu). \tag{25}$$

Only the solution corresponding to the maximum rate or the minimum value of γ is stable. Other solutions corresponding to larger values of γ are unstable and parametrically located near the principal stable solution. This can cause the solution that is stable to small fluctuations to be destroyed by fluctuations of finite amplitude. As is seen from Fig. 3, the asymptotes (24) and (25) set in early enough, i.e., already at $\mu \sim 1$. On the other hand, stationary solutions corresponding to growth along the direction of the minimum $\beta(\theta)$ exist only in the region $\mu > \mu_{c\beta}$ (curve 2 in Fig. 3), and at $\mu \gg 1$ they are described by the asymptote (16). Thus, at small supercooling a needle-shaped dendrite grows along the direction of the minimum $d(\theta)$, but at large supercooling the dendrite grows along the direction of the minimum $\beta(\theta)$. The critical $\mu_{c\beta}$ depends on the parameter ν, with $\mu_{c\beta} \sim 1$ at $\nu \sim 1$. At $\nu \ll 1$ the region of the transition from a dendrite growing along the minimum $d(\theta)$ to a dendrite growing along the minimum $\beta(\theta)$ shifts to larger μ, so that $\mu_{c\beta} \sim \nu^{-3/2}$, but at $\nu \gg 1$ the critical value is small, $\mu_{c\beta} \sim \nu^{-1}$.

These results on the changes in the direction of the dendrite growth as the supercooling increases are in qualitative agreement with those in Ref. 13. In that article the morphologic transitions from a dendrite growing along the minimum $d(\theta)$ to a dense branched structure and then to a dendrite growing along the minimum $\beta(\theta)$ were revealed both experimentally in anisotropic Heli-Shaw cell and theoretically by numerical analysis of a model of the boundary layer. The observed[13] supersaturation region where the dense branched structure develops can be connected with the fact that, when a dendrite growing along the direction of the minimum $d(\theta)$, is destroyed by fluctuations before a dendrite can grow in the kinetic direction, i.e., along the minimum $\beta(\theta)$. Besides the above experiments on the model sytem (anisotropic Heli-Shaw cell), we note that in experiments on three-dimensional growth from supersaturated solutions[18] and, from melts,[19,20] and in experiments on electrochemical deposition,[21] changes in the direction of the dendrite growth and in the slope of the growth-rate plot vs supersaturation were observed as the supersaturation increased.

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