Quasistationary plasma waves of small and finite amplitude

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(Submitted 19 August 1988)
Zh. Eksp. Teor. Fiz. **95**, 1951–1961 (June 1989)

The adiabatic approximation is used to construct periodic solutions to the system of Vlasov–Poisson equations, which describe quasistationary Langmuir waves. A nonlinear dispersion relation is derived in the small-amplitude limit. Finally, the contribution of resonant particles to the energy and momentum of the waves is determined.

1. INTRODUCTION

In the theory of stationary plasma waves two approaches to the solution of the problem are well known. The first of these, based on the linearization of the system of Vlasov–Poisson equations, leads to the Van Kampen modes.[1,2] The second is based on a rigorous solution of the equations and is valid for waves of arbitrary intensity and spatial configuration. These solutions are known as Bernstein-Greene–Kruskal (BGK) waves.[2,3] However, even periodic BGK solutions form a very wide class. Partially for this reason the form of the stationary (in the wave frame) charged particle distribution function is frequently postulated without justification, and the electrostatic potential of a small-amplitude wave is assumed to be sinusoidal.[4–6] Obviously, the most natural choice of the stationary particle distribution can be made by considering waves with slowly varying parameters. Indeed, under actual conditions the waves are frequently excited by weak sources so that the intensity of the oscillations grows very slowly. Then the generation process can be assumed to be adiabatic, and the excited wave can be assumed quasistationary. In this case the form of the distribution function, and with it the profile of the self-consistent potential, are uniquely determined.

In the present article a rigorous solution of the system of Vlasov–Poisson equations is constructed which corresponds to quasistationary plasma waves of finite amplitude. A nonlinear dispersion relation is derived. It is shown that even in the limit of infinitely small amplitudes the periodic wave, generally speaking, is not strictly harmonic. Finally, expressions are found for the momentum and energy of the quasistationary waves, correctly taking into account the contribution of the resonant particles.

2. REFERENCE EQUATIONS AND FORMULATION OF THE PROBLEM

Let us consider the problem of the excitation of a Langmuir wave by a weak source in a collisionless plasma in the one-dimensional case. We will begin with the kinetic Vlasov equation with an external periodic "driving force" and the Poisson equation[1)]

$$\frac{\partial f}{\partial t}+V\frac{\partial f}{\partial x}-\frac{e}{m}\frac{\partial}{\partial x}(\varphi+\varphi_{ext})\frac{\partial f}{\partial V}=0, \tag{1}$$

$$-\frac{\partial^2\varphi}{\partial x^2}=4\pi e n_0\Big(1-\int_{-\infty}^{\infty}dV\,f\Big). \tag{2}$$

Here we assume the external force to arise from a potential, as is frequently the case in practice, for example, in the excitation of a plasma wave by a periodic high-frequency potential (a beat wave).

Let a periodic perturbation with wavelength $\lambda=2\pi/k$ with an initially ($t\to-\infty$) infinitesimally small amplitude ($A\to0$) propagate with variable phase velocity $V_{\rm ph}(t)$. By not excluding variations of the phase velocity beforehand we thereby allow for the possibility of a nonlinear frequency shift.[5–8] We will denote the limiting value of the wave velocity by $A\to0$ ($t\to-\infty$) by $V_{\rm ph0}=\omega_0/k$.

Introducing the dimensionless variables

$$\tau=2^{1/2}\omega_p t,\quad v=\tilde V-u,\quad \xi=\frac{k}{2}\Big(x-\int_{-\infty}^{t}dt'\,V_\Phi(t')\Big),$$
$$\tilde V=\frac{V}{v_0},\quad u=V_\Phi(t)/v_0,\quad u_0=\frac{V_{\Phi0}}{v_0},\quad v_0=2^{1/2}\omega_p/k,$$
$$\tilde f=v_0 f,\quad \Phi=-e\varphi k^2/4m\omega_p^2,\quad \Phi_{ext}=-e\varphi_{ext}k^2/4m\omega_p^2 \tag{3}$$

we transform to the noninertial reference system associated with the wave (the tilde above $\tilde f$ and $\tilde V$ is omitted for brevity)

$$\frac{\partial f}{\partial\tau}+\frac{v}{2}\frac{\partial f}{\partial\xi}-\frac{\partial}{\partial\xi}(\Phi+\Phi_{ext})\frac{\partial f}{\partial v}-\frac{du}{d\tau}\frac{\partial f}{\partial v}=0, \tag{4}$$

$$\frac{\partial^2\Phi}{\partial\xi^2}=1-\int_{-\infty}^{\infty}dv\,f(v,\xi,\tau). \tag{5}$$

In this system all quantities are assumed to depend weakly on time ($\partial/\partial\tau\sim\Phi_{\rm ext}\to0$), i.e., the perturbations are quasistationary. As a particular case, it is possible with the help of Eqs. (4) and (5) to describe the excitation of a stationary wave by a weak external source, acting over a finite time with continuous growth and decay of its intensity. Similarly, it is possible to consider the process of feeding additional energy to an already existing BGK wave.

Thus, the problem reduces to the solution of Eqs. (4) and (5) with periodic boundary conditions and initial conditions ($t\to-\infty$) $|\Phi_{\rm ext}|\ll|\Phi|\to0, f\to f_0(v)$, where f_0 is an arbitrary smooth function, for example, a Maxwellian.

3. ADIABATIC APPROXIMATION

A very effective method for solving equations of the form of Eq. (4) is based on the adiabatic approximation. It has been used to treat the dynamics of charged particles in a potential well with slowly varying parameters[9] and in the field of a monochromatic wave,[7,10] and also the evolution of quasimonochromatic wave packets interacting with resonant particles.[11,12] In our case the main difference from the previous articles is connected with the presence of the last term in Eq. (4), which contains the inertial force.

Following Best[7] and Gurevich,[9] we transform in Eq. (4) to a new variable, the total energy of the electron $W = v^2/4 + \Phi(\xi,\tau)$:

$$\frac{\partial f_\pm}{\partial \tau} \pm (W-\Phi)^{1/2} \frac{\partial f_\pm}{\partial \xi} + \left(\frac{\partial \Phi}{\partial \tau} \mp (W-\Phi)^{1/2} \frac{du}{d\tau} \mp (W-\Phi)^{1/2} \frac{\partial \Phi_{ext}}{\partial \xi}\right) \frac{\partial f_\pm}{\partial W} = 0, \quad (6)$$

where the $\pm$ sign corresponds to the particles moving from left to right and right to left, respectively. For slow variation of the wave parameters ($\partial/\partial\tau \sim \Phi_{ext} \ll 1$) the distribution function can be represented in the form of the two leading terms of the expansion in $\partial/\partial\tau$

$$f_\pm = F_\pm(W,\tau) + \tilde{F}_\pm(W,\xi,\tau), \quad |\tilde{F}_\pm| \ll |F_\pm|.$$

Then to first order in $\partial/\partial\tau$, according to Eq. (6), we have

$$\frac{\partial F_\pm}{\partial \tau} + \frac{\partial \Phi}{\partial \tau} \frac{\partial F_\pm}{\partial W} \mp (W-\Phi)^{1/2} \left(\frac{du}{d\tau} + \frac{\partial \Phi_{ext}}{\partial \xi}\right) \frac{\partial F_\pm}{\partial W} = \mp (W-\Phi)^{1/2} \frac{\partial \tilde{F}_\pm}{\partial \xi}. \quad (7)$$

For particles undergoing unbounded motion (untrapped particles), it follows from the conditions of the periodicity of the distribution function that

$$\frac{\partial F_\pm}{\partial \tau} \int_{-\pi/2}^{\pi/2} \frac{d\xi}{(W-\Phi)^{1/2}} + \frac{\partial F_\pm}{\partial W} \int_{-\pi/2}^{\pi/2} \frac{d\xi}{(W-\Phi)^{1/2}} \frac{\partial \Phi}{\partial \tau} \mp \pi \frac{du}{d\tau} \frac{\partial F_\pm}{\partial W} = 0.$$

The solution of this equation is an arbitrary function of the integral of motion

$$I_\pm = u \pm \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} d\xi (W-\Phi)^{1/2} \quad (8)$$

and according to the initial conditions

$$F_\pm = f_0(I_\pm). \quad (9)$$

Note that the form of the distribution function to lowest order in $\partial/\partial\tau$ does not depend on the nature of the source Φ_{ext}.

For trapped particles, taking into account the equality of the number of electrons moving in both directions, $F_+ = F_- \equiv F_0/2$, and taking into account the requirement of zero increase of f upon making the complete contour around the particle trajectory $\oint d\xi(\partial\tilde{F}/\partial\xi) = 0$, for the function $F_0 = F_+ + F_-$ we find from Eq. (7)

$$\frac{\partial F_0}{\partial \tau} \int_{-\xi_0}^{\xi_0} \frac{d\xi}{(W-\Phi)^{1/2}} + \frac{\partial F_0}{\partial W} \int_{-\xi_0}^{\xi_0} \frac{d\xi}{(W-\Phi)^{1/2}} \frac{\partial \Phi}{\partial \tau} = 0,$$

where $\pm\xi_0$ are the coordinates of the turning points satisfying the relation $W = \Phi(\pm\xi_0,\tau)$. By direct substitution it is easy to show that the solution of this equation is an arbitrary function $F_0 = F_0(J)$ of the adiabatic invariant

$$J = \frac{2}{\pi} \int_{-\xi_0}^{\xi_0} d\xi (W-\Phi)^{1/2}. \quad (10)$$

We find the actual from of F_0 by invoking the constancy of the total number of particles within one wavelength

$$\int_{-\pi/2}^{\pi/2} \frac{d\xi}{\pi} \int_\Phi^\infty \frac{dW(f_+ + f_-)}{(W-\Phi)^{1/2}} = 1. \quad (11)$$

Assuming without loss of generality that $\Phi_{min} = 0$ and $\Phi_{max} = A$, we reverse the order of integration and then, transforming to an integral over $I_\pm$ and J respectively for the untrapped ($W > A$) and trapped ($0 < W < A$) particles, we rewrite Eq. (11) in the form

$$\int_{u+R}^{\infty} dI_+ f_0(I_+) + \int_{-\infty}^{u-R} dI_- f_0(I_-) + \int_0^R dJ F_0(J) = 1, \quad (12)$$

where the instantaneous phase velocity of the wave is uniquely determined by its amplitude $u = u(A)$[2], and the value of the invariant J on the separatrix in the phase plane separating the trajectories of the untrapped and the trapped particles ($W = A$) is denoted by

$$R = R(A) = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} d\xi (A-\Phi)^{1/2}.$$

We introduce the complex function $\tilde{u} \equiv \tilde{u}(R) = u[A(R)]$. Then, differentiating Eq. (12) with respect to R, we find F_0 in explicit form:

$$F_0(R) = f_0(\tilde{u}+R) + f_0(\tilde{u}-R) + \frac{d\tilde{u}}{dR}[f_0(\tilde{u}+R) - f_0(\tilde{u}-R)].$$

Thus, the distribution function of the trapped particles is equal to

$$F_0(J) = f_0(\tilde{u}+J) + f_0(\tilde{u}-J) + \frac{d\tilde{u}}{dJ}[f_0(\tilde{u}+J) - f_0(\tilde{u}-J)], \quad (13)$$

where $\tilde{u} = \tilde{u}(J)$. This expression, which takes into account all of the peculiarities of entrainment of the particles trapped by a wave with slowly varying amplitude, phase, and shape, can be of use in a number of applied problems.

4. STRUCTURE OF THE QUASISTATIONARY WAVE

Let us go on to the solution of the Poisson equation. We represent the electrostatic potential of the wave in the form $\Phi = A\alpha(\xi,A)$, where $A = \Phi_{max}$ ($\Phi_{min} = 0$), and the function α, which varies between the limits 0 and 1, describes the shape of the wave while satisfying the boundary conditions

$$\xi = 0, \quad \alpha = 0, \quad \partial\alpha/\partial\xi = 0, \quad (14a)$$

$$\xi = \pm\pi/2, \quad \alpha = 1, \quad \partial\alpha/\partial\xi = 0. \quad (14b)$$

Replacing the integration variable in Eq. (5) by $W = v^2/4 + \Phi$, we rewrite Eq. (5) in the form

$$A \frac{\partial^2 \alpha}{\partial \xi^2} = 1 - \int_{A\alpha}^{\infty} \frac{dW f(W,A)}{(W-A\alpha)^{1/2}}, \quad f = f_+ + f_-. \quad (15)$$

In contrast with Ref. 3, in our case the electron distribution function depends on two variables and, moreover, according to Eqs. (9) and (13), up to the determination of the profile of the wave in explicit form it is, generally speaking, a functional of $\alpha(\xi,A)$. It is not hard to see that $\alpha(\xi,A)$ is an even function about the maxima ($\xi = \pm\pi/2$) and the minimum ($\xi = 0$) of the potential. Therefore it is sufficient to consider the behavior of α in the interval of monotonicity $(0,\pi/2)$.

Taking Eq. (14a) into account and repeating the calculations in Ref. 3, we reduce Eq. (15) to the equation of a nonlinear oscillator

$$\frac{1}{2}\left[\frac{\partial}{\partial\xi}\alpha(\xi,A)\right]^2+\mathscr{V}(\alpha,A)=0 \tag{16}$$

with effective potential $\mathscr{V}$ defined by

$$\frac{A}{2}(A\mathscr{V}+\alpha)$$

$$=\int_0^\infty dW f(W,A)W^{1/2}-\int_{A\alpha}^\infty dW f(W,A)(W-A\alpha)^{1/2}. \tag{17}$$

Since Eq. (16) is solvable by quadratures, the problem of determining the profile of the wave can be taken to be solved if the function $\mathscr{V}(\alpha,A)$ is determined explicitly.

In the derivation of Eqs. (16) and (17) the small nonstationary corrections to the distribution function $\tilde{F}_\pm \sim \partial/\partial\tau\to 0$ were neglected. Therefore in the calculation of the right-hand side of Eq. (17) for f, the expression [see Eqs. (9) and (13)]

$$f(W,A)=f_++f_-=\begin{cases} f_0(I_+)+f_0(I_-), & W>A \\ F_0(J), & W<A \end{cases}, \tag{18}$$

should be used, where the adiabatic invariants (8) and (10) can be represented with the help of Eq. (16) in the form of a functional of $\mathscr{V}(\alpha,A)$

$$I_\pm=u\pm\frac{4}{\pi}\int_0^1\frac{d\alpha(W-A\alpha)^{1/2}}{[-2\mathscr{V}(\alpha,A)]^{1/2}},\qquad J=\frac{4}{\pi}\int_0^{W/A}\frac{d\alpha(W-A\alpha)^{1/2}}{[-2\mathscr{V}(\alpha,A)]^{1/2}}.$$

For the potential $\mathscr{V}$, Eqs. (14) and (16) lead to the obvious restriction

$$\frac{2}{\pi}\int_0^1\frac{d\alpha}{[-2\mathscr{V}(\alpha,A)]^{1/2}}=1. \tag{19}$$

In addition to this, according to Eq. (17) and as a result of Eq. (14b) we find the following restriction on f

$$\int_0^\infty dW f(W,A)W^{1/2}-\int_A^\infty dW f(W,A)(W-A)^{1/2}=\frac{A}{2}. \tag{20}$$

Equations (17)–(20) completely determine the structure of the wave, i.e., the form of $\mathscr{V}(\alpha,A)$ and $F(W,A)$ for arbitrary amplitudes, and also the nonlinear dispersion law $\omega=\omega(k,A)$ for a plasma with an arbitrary perturbed distribution function.[3)] The solution of system (17)–(20) in the general case can be found with the help of numerical methods. In what follows we will dwell in more detail on the important case of small amplitudes ($A\to 0$).

Let us expand the phase velocity and the potential in a series in half-integer powers of the amplitude

$$u(A)=u_0+u_1A^{1/2}+u_2A+\ldots, \tag{21a}$$

$$\mathscr{V}(\alpha,A)=\mathscr{V}_0(\alpha)[1+2A^{1/2}\mathscr{V}_1(\alpha)+2A\mathscr{V}_2(\alpha)+\ldots]. \tag{21b}$$

Leaving out the details of the calculation of the right-hand side of Eq. (17), we present the final expression for the leading term in Eq. (21b)

$$\mathscr{V}_0(\alpha)=P_1\alpha(\alpha-2C_1)$$

$$+16f_0'u_1\left\{\int_0^1 d\varkappa\varkappa^2Z(\varkappa)-\int_{\alpha^{1/2}}^1 d\varkappa\varkappa(\varkappa^2-\alpha)^{1/2}Z(\varkappa)\right.$$

$$\left.-\frac{1}{6}[1-(1-\alpha)^{3/2}]\right\}, \tag{22}$$

where we have introduced the following notation:

$$P_1=\mathscr{P}\int_{-\infty}^\infty\frac{dV}{V-u_0}\frac{\partial f_0}{\partial V},\qquad f_0'=\left(\frac{\partial f_0}{\partial V}\right)_{V=u_0},$$

$$C_1=\frac{2}{\pi}\int_0^1\frac{d\alpha\,\alpha}{(-2\mathscr{V}_0)^{1/2}} \tag{23}$$

$$Z(\varkappa)=\int_0^{\varkappa^2}\frac{d\alpha(\varkappa^2-\alpha)^{1/2}}{(-2\mathscr{V}_0)^{1/2}}\Bigg/\int_0^1\frac{d\alpha(1-\alpha)^{1/2}}{(-2\mathscr{V}_0)^{1/2}}$$

and $\mathscr{P}$ is the principal value symbol. The first term in Eq. (22) is the contribution of the nonresonant particles ($W\gg A$). The terms which contain integrals are due to the trapped electrons. And, finally, the remaining term is the contribution of the untrapped resonant particles ($W\gtrsim A$). For now the quantity u_1 will play the role of a free parameter. However, the value of u_1 will be completely determined after the calculation of the next correction to $\mathscr{V}(\alpha,A)$, which as will be clear in what follows also leads to a unique dispersion law $\omega_0=\omega_0(k)$ even as $A\to 0$, in contrast with well-known results.[1]

Running ahead, we note that the self-consistent solution of the next correction (proportional to $A^{1/2}$) is impossible in the case of zero phase shift, i.e., it is necessary that $u_1\neq 0$ (see also Refs. 5–8). Then according to Eq. (22) a quasistationary wave with arbitrarily small amplitude is not harmonic. The exception consists of the case $f_0'=0$, when Eq. (22) leads to a dispersion relation in Vlasov form, which in the standard notation has the form $P_1=2$, and the potential corresponds to a sinusoidal wave

$$\mathscr{V}_0(\alpha)=U(\alpha)=2\alpha(\alpha-1),$$

and correspondingly

$$C_1={}^1/_2,\ \alpha=\sin^2\xi,\ \Phi=(A/2)(1-\cos 2\xi).$$

The anharmonicity of the wave becomes especially strong if its phase velocity is comparable with the thermal velocity of the electrons $V_{\text{ph}0}\sim V_T$. The nature of this anharmonicity is closely connected with the increase of the number of electrons trapped in the potential wells which accelerate (in the laboratory system, slow down) in the process of excitation by the wave. The physical mechanism of the given phenomenon is revealed in the fact that as the phase velocity of the wave varies with growth of its amplitude, the resonant particles give a contribution to the perturbation of the density of the order of A, as well as one of the order of $A^{3/2}$. The deviation from harmonicity of the rigorous solution to some extent explains the difficulties of going over from the Van Kampen modes to the BGK waves.[3] In this connection we note that both approaches lead to the Vlasov dispersion relation only for $f_0'=0$.

Let us consider in more detail a typical situation when the number of resonant particles in the tail of the distribution is small and $f_0'\ll 1$ (the case of the Maxwellian distribution for $V_{\text{ph}0}\gtrsim 3V_T$). Under these conditions the wave is close to harmonic and Eq. (22) can be solved by the method of successive approximations. Taking into account Eqs. (14b) and (19) and the expression for C_1 from Eq. (23), we find

$$\mathscr{V}_0(\alpha)=U(\alpha)+16f_0'u_1\Big[aU(\alpha)+(1-\alpha)\int_0^1 d\varkappa\,\varkappa^2 Z_0(\varkappa)$$

$$-\int_{\alpha^{1/2}}^1 d\varkappa\,\varkappa(\varkappa^2-\alpha)^{1/2}Z_0(\varkappa)-\frac{1}{3}(1-\alpha)(1-(1-\alpha)^{1/2})\Big], \tag{24}$$

where

$$U(\alpha)=2\alpha(\alpha-1),\qquad a=\frac{2}{\pi}\Big[\int_0^1 d\varkappa\,\varkappa Z_0(\varkappa)E(\varkappa)-\frac{1}{3}\Big],$$

$$Z_0=E(\varkappa)-(1-\varkappa^2)K(\varkappa)$$

and $K(\varkappa)$ and $E(\varkappa)$ are complete elliptic integrals with modulus $\varkappa$ of the first and second kind, and the phase velocity u_0 is determined by the dispersion relation

$$P_1=2(1+16f_0'u_1a). \tag{25}$$

In this approximation[4] ($f_0'\ll 1$), calculating the right-hand side of Eq. (17) with accuracy up to terms in $A^{5/2}$, we can determine the form of the first correction to the potential $\mathscr{V}$ [see Eq. (21b)]

$$\mathscr{V}_1(\alpha)=4f_0''\Big\{b-\frac{1}{\alpha}\int_0^1 d\varkappa\varkappa^2\left(\frac{2Z_0}{\pi}\right)^2$$

$$+\frac{1}{\alpha(1-\alpha)}\int_{\alpha^{1/2}}^1 d\varkappa\varkappa(\varkappa^2-\alpha)^{1/2}\left(\frac{2Z_0}{\pi}\right)^2$$

$$+\int_0^1\frac{d\varkappa}{\varkappa^6}\Big[\left(\frac{2E}{\pi}\right)^2\frac{1}{\alpha}\left(\frac{(1-\alpha\varkappa^2)^{1/2}-\alpha(1-\varkappa^2)^{1/2}}{1-\alpha}-1\right)$$

$$-\frac{\varkappa^4}{8}\Big]-\frac{1}{8}\Big\}, \tag{26}$$

where

$$b=\frac{5}{24}-2\int_0^1 d\varkappa\,\varkappa\left(\frac{2Z_0}{\pi}\right)^2$$

$$\times\frac{2}{\pi}(K-E)-2\int_0^1\frac{d\varkappa}{\varkappa^6}\Big[\left(\frac{2E}{\pi}\right)^2\frac{2}{\pi}(K-E)-\frac{\varkappa^2}{2}\left(1-\frac{\varkappa^2}{8}\right)\Big],$$

and the first correction to the phase velocity (21), assuming the validity of Eqs. (19) and (20), is equal to

$$u_1=16f_0''b/P_2, \tag{27}$$

where

$$f_0''=\left(\frac{\partial^2 f_0}{\partial V^2}\right)_{V=u_0},\qquad P_2=\mathscr{P}\int_{-\infty}^{\infty}\frac{dV}{V-u_0}\frac{\partial^2 f_0}{\partial V^2}=\frac{\partial P_1}{\partial u_0}.$$

Expressions (16) and (24)–(27) determine the spatial dependence of the electrostatic potential of the quasistationary wave to first order in $A^{1/2}$. In the more general situation the profile of the potential can be found by using numerical methods for an exact determination of $\mathscr{V}_0$ from Eq. (22), the corresponding expression for $\mathscr{V}_1$, and the solution of Eq. (16).

5. THE NONLINEAR DISPERSION RELATION

Using Eqs. (25) and (27) and transforming back from the dimensionless quantities to the standard notation (3), it is easy to derive a dispersion relation of the form

$$1-\left(\frac{\omega_p}{k}\right)^2\mathscr{P}\int_{-\infty}^{\infty}\frac{dV\,\partial f_0/\partial V}{V-V_{ph}(A)}+256ab\left(\frac{\omega_p}{k}\right)^2\frac{\partial}{\partial V}\left(\frac{\partial f_0}{\partial V}\right)^2_{V=V_{ph}}$$

$$\Big/\mathscr{P}\int_{-\infty}^{\infty}\frac{dV\,\partial^2 f_0/\partial V^2}{V-V_{ph}}+16b\left(\frac{\omega_p}{k}\right)^3\left(\frac{\partial^2 f_0}{\partial V^2}\right)_{V=V_{ph}}(2A)^{1/2}=0,$$

$$V_{ph}(A)=\omega(k,A)/k. \tag{28}$$

This equation differs from the Vlasov equation by the presence of two terms which describe the contribution of the resonant particles. The last term of Eq. (28) describes the effect of the nonlinear frequency shift and is associated with the perturbation of the resonant electron density of order $A^{3/2}$. The third term, which does not disappear even in the limit of infinitesimally small amplitudes, deserves special attention, and, at the same time, has no analog in the linear theory. Its occurrence is due to the same cause as the anharmonicity of the wave (see Sec. 4) which does not disappear as $A\to 0$, i.e., the linear (in amplitude) perturbation of the resonant particle density. In its turn, the modulation of the resonant electron density, which is proportional to A, is associated with the asymmetric character of the trapping of the particles overtaking the wave and remaining behind it under conditions of varying phase velocity. This result, strange at first glance, allows, nevertheless, a simple, qualitative explanation.

Let us compare the process of trapping of the electrons in the potential wells of a wave of growing amplitude with fixed phase velocity (in the absence of a nonlinear frequency shift) and with decreasing phase velocity. We will consider the perturbation of the distribution function of the trapped particles, making use of the expansion of the unperturbed distribution function near the resonance

$$f_0\approx f_0(V_{ph})+(V-V_{ph})f_0'(V_{ph}).$$

For a fixed value of V_{ph} in the trapping region the well-known distribution in the form of a spatially modulated plateau develops.[13] The form of the distribution function in the $\xi=0$ plane, which corresponds to the minimum of the potential well, where the perturbation of $f-f_0$ is at its maximum, is shown in Fig. 1 by the dashed line. The particles are trapped symmetrically with respect to the point $V=V_{ph}$, and the perturbation of the trapped particle density is

$$n\sim\int dV(f-f_0)=0.$$

However, if the phase velocity of the wave decreases with growth of the amplitude, the number of trapped particles moving with velocity $V\approx V_{ph}(A)$ is less than its value at the endpoints of the trapping region $V=V_{ph}\pm V_E$ ($V_E\sim A^{1/2}$ is the halfwidth of the trapping region) by the quantity

$$\Delta f=|f_0'(V_{ph})\Delta V_{ph}|=|f_0'(V_{ph})[V_{ph}(A)-V_{ph}(0)]|,$$

since these particles were trapped earlier as $A\to 0$ and, correspondingly, at a larger value of the phase velocity $V_{ph}(0)$. With increase of the amplitude these particles become entrained by the wave (shown in Fig. 1 by an arrow) and always remain at the bottom of the well. The qualitative form

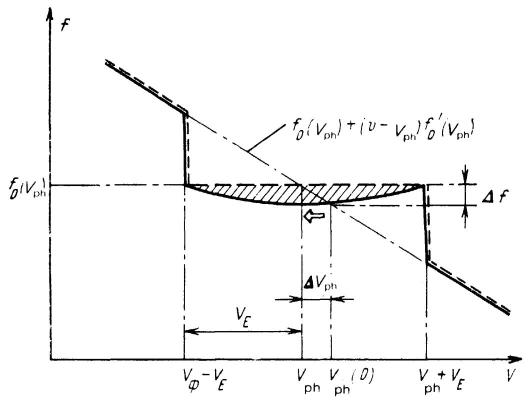

FIG. 1.

of the distribution function is shown in the figure by a solid line. Decrease of the density of the trapped electrons is numerically equal to the area of the hatched region and is equal in order of magnitude to $V_E\Delta f$. In our case the wave velocity grows as a result of the nonlinear variation of the phase, whereby

$$\Delta V_{\text{ph}}\sim A^{1/2}f_0''(V_{\text{ph}})$$

[see Eq. (27)]. Then the perturbation of the trapped particle density, arising as a result of the asymmetry of trapping, is found to be proportional to the first power of the amplitude [see Eq. (28)]

$$n\approx V_E f_0'\Delta V_{\text{ph}}\sim Af_0'f_0''=A\frac{\partial}{\partial V}\left(\frac{\partial f_0}{\partial V}\right)^2_{V=V_{\text{ph}}}$$

A rigorous calculation of the perturbation of the electron density, taking into account the untrapped electrons and the spatial dependence

$$n[\alpha(\xi)]=A\partial\mathscr{V}(\alpha,A)/\partial\alpha,$$

can be carried out by a simple differentiation of Eqs. (22), (24), and (26).

Thus, the nonlinear phase shift, which by itself is due to the perturbation of the density $n_{\text{res}}\sim A^{3/2}$, entails the generation of harmonics of the resonant electron density in its leading term $n_{\text{res}}\sim A$, which is also reflected in the third term in Eq. (28). We note that this effect is highly nonlinear and in principle cannot be described within the linear approximation. In this sense the standard approach of preliminary linearization of the kinetic equation is not sufficiently systematic since it does not take into account the strong perturbation of the particle trajectories near the resonance. For the same reason the question of the contribution of the resonant particles to the dispersion relation in the linear theory of stationary waves is necessarily circumvented by integration of the expression containing the resonant denominator $(\omega-kV)^{-1}$ in the principal-value sense, and the role of the resonant particles is thereby in effect ignored. On the other hand, the method proposed in Ref. 3 and taken as a basis above (the rigorous solution of the nonlinear equations with subsequent taking of the limit $A\to 0$) allows one to correctly determine the contribution of the resonant particles to the dispersion properties of the plasma.

Relation (28) differs from the well-known results of Refs. 1–3, which contain an indeterminacy (the absence of a unique dependence $\omega=\omega(k)$, the arbitrariness of the choice of the resonant particle distribution function, and the indeterminacy of their contribution to the dielectric constant), which is a consequence of the absence of information on the character of the evolution of the wave in the given stationary state. This indeterminacy can be eliminated, however, by solution of the evolutionary problem. In fact, as was shown above, specifying the means of excitation of the stationary wave (here adiabatic) leads to a completely determined dispersion law $\omega=\omega(k,A)$. The solution of Eq. (28) has the form

$$\omega(k,A)=\Omega_0(k)+256abk\frac{\partial}{\partial V}\left(\frac{\partial f_0}{\partial V}\right)^2_{V=\Omega_0/k}\Bigg/\left(\mathscr{P}\int_{-\infty}^{\infty}\frac{dV\,\partial^2 f_0/\partial V^2}{V-\Omega_0/k}\right)^2$$

$$+16b\omega_b\left(\frac{\partial^2 f_0}{\partial V^2}\right)_{V=\Omega_0/k}\Bigg/\mathscr{P}\int_{-\infty}^{\infty}\frac{dV\,\partial^2 f_0/\partial V^2}{V-\Omega_0/k},\qquad(29)$$

where $\Omega_0(k)$ is the solution of the Vlasov dispersion equation

$$\varepsilon_0(\Omega_0,k)=1-\left(\frac{\omega_p}{k}\right)^2\mathscr{P}\int_{-\infty}^{\infty}\frac{dV\,\partial f_0/\partial V}{V-\Omega_0/k}=0,\qquad(30)$$

$\omega_b=(ek^2\Phi_0/m)^{1/2}$ is the characteristic frequency of the oscillations of the trapped particles in the potential wells of the wave, and $\Phi_0=(\varphi_{\max}-\varphi_{\min})/2$.

6. MOMENTUM AND ENERGY OF THE WAVE

Knowing the form of the distribution function (18), it is possible by direct calculation to determine the momentum of the wave and its energy. In the fixed reference system associated with the ions at rest the momentum density and the kinetic energy density of the electrons, averaged over one wavelength, are equal to

$$\left\langle\frac{P}{n_0 m v_0}\right\rangle=u+\tilde{P},\qquad\left\langle\frac{2W_{\text{K}}}{n_0 m v_0^2}\right\rangle=u^2+2u\tilde{P}+\langle\tilde{W}\rangle,$$

where the angular brackets denote the average

$$\langle\ldots\rangle=\int_{-\pi/2}^{\pi/2}\frac{d\xi}{\pi}(\ldots)=\frac{2}{\pi}\int_0^1\frac{d\alpha}{[-2\mathscr{V}(\alpha,A)]^{1/2}}(\ldots),$$

and the dimensionless momentum and energy of the electrons in the system associated with the wave are given by

$$\tilde{P}=2\int_A^{\infty}dW(f_+-f_-),\qquad\tilde{W}=4\int_{A\alpha}^{\infty}dW(f_++f_-)(W-A\alpha)^{1/2}.$$

The trapped particles do not contribute to the momentum $\tilde{P}$ since for them $f_+=f_-=F_0/2$ (see Sec. 3). Calculation, accurate to terms of order A^2 within the restrictions invoked in the derivation of Eqs. (24) and (26) ($V_{\text{ph}0}>3V_T$), leads to the result

$$\left\langle\frac{P}{n_0 m v_0}\right\rangle=-\frac{64}{9\pi}f_0'A^{3/2}+\frac{1}{2u_0}\frac{\partial}{\partial u_0}(u_0\varepsilon_0)A^2,$$

$$\left\langle\frac{2W_{\text{K}}}{n_0 m v_0^2}\right\rangle-\left\langle\frac{2W_{\text{K}}}{n_0 m v_0^2}\right\rangle_{A=0}=-\frac{128}{9\pi}u_0 f_0'A^{3/2}$$

$$+\left[\frac{\partial}{\partial u_0}(u_0\varepsilon_0)-1\right]A^2.$$

In this approximation the electrostatic energy of the wave is equal to

$$\langle 2W_E/n_0 m v_0^2\rangle = A^2.$$

Transforming back from dimensionless quantities to the standard notation, we find expressions for the total energy and the momentum of the wave

$$W = W_K + W_E = \frac{k^2\Phi_0^2}{16\pi}\frac{\partial}{\partial\Omega_0}(\Omega_0\varepsilon_0) - \frac{64}{9\pi} n_0 m \frac{\Omega_0}{k}\left(\frac{\partial f_0}{\partial V}\right)_{V=\Omega_0/k}\left(\frac{e\Phi_0}{m}\right)^{3/2}, \quad P = \frac{k}{\Omega_0} W, \tag{31}$$

where Ω_0 and ε_0 are determined by Eq. (30).

The first term in Eq. (31) has the usual form. The second term describes the contribution of the resonant particles which is due to the characteristic modulated-plateau deformation of the distribution function in the vicinity of the phase velocity of the wave.[13] This division of the energy of the wave into two "blocks" was noted in Ref. 14. It should be emphasized that the second term in Eq. (31) is of lower order in the wave amplitude, by virtue of which the store of energy carried by the resonant particles can play an important role in various problems of plasma radiophysics (see, e.g., Refs. 15 and 16).

7. CONCLUSION

As is well known, quite a wide class of plasma wave phenomena exist in which the oscillations evolve rather slowly. We have demonstrated above a method of kinetic description of such processes in the instance of perturbations of the simplest type—periodic Langmuir waves. A nonlinear dispersion law has been found for the quasistationary natural oscillations of a collisionless plasma. It has been established that for slow excitation of the wave there exists a unique coupling between the frequency and the wave vector, and the wave is not harmonic even in the limit of infinitely small amplitudes. The dispersion equation, nevertheless, differs from the Vlasov equation by the presence of a term which takes into account the contribution of the resonant electrons. This term [see Eq. (28)] was derived by a rigorous analysis of the self-consistent processes of trapping of the particles and variation of the phase velocity of the wave with growth of its amplitude. At the same time, the usual procedure of preliminary linearization of the kinetic equation does not permit an expansion of the solution in half-integer powers of the amplitude and thereby in effect ignores the effects of trapping.

The main virtue of the proposed approach thus consists in the possibility of correctly taking into account the contribution of the resonant particles to the dispersion characteristics of the plasma and their participation in processes of energy and momentum exchange with the wave. In addition, in spite of the fact that a large part of the results of the present article were obtained in the small-amplitude approximation, there are no fundamental obstacles (restrictions on the wave amplitude A or the form of f_0) to carrying out a more general investigation. The only limitation is the well-known requirement of adiabaticity of trapping of the particles by the potential wells,[9] i.e., slowness of variation of the wave parameters in the time scale of the characteristic period of motion of the particles.

It is well known that the wave-particle interaction can have an influence on the dynamics of the development of an entire list of wave processes in a plasma, such as the nonlinear interaction of waves, the development of parametric instabilities, etc. A study of these phenomena at the level of the kinetic description can be carried out within the framework of the adiabatic approximation for waves of finite amplitude (see Sec. 3), and in fact this is all the more to be preferred since the standard method of linearization of the kinetic equation is hardly a systematic approach for the study of processes in which resonant particles participate.

From a practical point of view the proposed approach can be useful in the investigation of wave processes in a laboratory plasma and in space plasmas, and also in problems of advanced technology.[17] In particular, on the basis of this method it is possible to investigate the dispersion properties of plasmas and the energy distribution of accelerated trapped particles, and to determine the detailed shape of the profile of the self-consistent electrostatic potential.

[1)]The ions for simplicity are assumed to be fixed.

[2)]Strictly speaking, we assume here a monotonically increasing wave amplitude. In addition, in the derivation of Eq. (12), note was taken of the inequality $|u - u_0| < R$, which can be easily verified *a posteriori* (see Sec. 4). Generalization to the case of attenuating waves does not require much work although here one must take account of the irreversible character of trapping of the particles.[11,12]

[3)]Here we do not touch on the question of the stability of such waves.

[4)]We do not present the more general expression for $\mathscr{V}_1(\alpha)$ because of its cumbersomeness.

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Translated by Paul Schippnick