Density of states of two-dimensional electrons in the presence of a magnetic field and a random potential in exactly solvable models

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Various models describing the motion of two-dimensional electrons in a magnetic field in the random point potential of substitutional impurities are considered. Analytical expressions are obtained for the averaged electron density of states $\langle\rho(E)\rangle$. The asymptotic forms of $\langle\rho(E)\rangle$ in the limits of a strong field, a low impurity concentration, or low energies are investigated in detail. A comparison is made with the expressions obtained for $\langle\rho(E)\rangle$ in the absence of a magnetic field.

1. INTRODUCTION

The discovery of the quantum Hall effect has stimulated the appearance of a large number of experimental and theoretical papers devoted to the investigation of the density of states of two-dimensional electron systems in a transverse magnetic field. A detailed review of these investigations is contained in Ref. 1. Starting from the work of Wegner,[1] analytical expressions have been found in many theoretical publications for the density of states, with structural defects modeled by the potential of a random field or by Poisson-distributed point potentials (see Refs. 3–7 and other papers). It is important to note that exact results have been obtained only for the lowest Landau level and for limiting cases of the correlation of the random field—white noise and fully correlated potentials (an exception is the case of the Lorentzian distribution, in which the restrictions indicated above are not necessary). It is clear that the models considered in the papers listed above describe extreme disorder in the two-dimensional structure. Clearly, it is of interest to investigate models (not yet considered in the literature) of systems of two-dimensional electrons with weaker disorder—substitutional disorder. In view of this, we consider in this paper the motion of a two-dimensional electron in a static field formed by an infinite system of isolated short-range random potentials located in the plane perpendicular to the magnetic field.

We note that in the cited papers two cases of the distribution of the random potentials were investigated—the Gaussian and the Lorentzian. The Gaussian distribution permits one to find either an exact expression for the averaged density of states in the ultraquantum limit or the asymptotic form for the high Landau levels. The Lorentzian distribution (the Lloyd model) makes it possible to obtain an exact analytical expression for the averaged density of states in the general case. This is why, in the present paper, the main attention is paid to the Lloyd model. This model is widely applied in the study of disordered systems (Refs. 8–11, etc.). When using the Lloyd model one must keep in mind the presence in it of a "long tail" of the density of states $\rho(E)$, requiring renormalization of $\rho(E)$ when the number of states

$$N(E)=\int_{-\infty}^{E}\rho(E)\,dE$$

is calculated; this question, however, lies outside the scope of our work. Incidentally, it will be seen from the following that the features of interest to us in the behavior of the function $\rho(E)$ in various limiting cases for the Lloyd model are analogous to results for the Gaussian distribution, which is free from the above-mentioned defect.

Below, the investigation of the averaged density of states of two-dimensional electrons in a transverse magnetic field and in the static field of isolated random impurities is carried out for the following cases: 1) for the Lloyd model with an arbitrary arrangement of independent random point potentials; 2) for the same model, but with independent point potentials located at the sites of an aribtrary lattice (substitutional impurities); 3) for independent random point potentials with an arbitrary distribution function in the limit of a strong magnetic field or a low impurity concentration; 4) for completely correlated random point potentials distributed arbitrarily and located at lattice sites. For all cases except 3) the limit of zero magnetic field is considered.

2. THE HAMILTONIAN AND GREEN FUNCTION

The models to be considered are specified by the Hamiltonian

$$H=H_0+U, \tag{1}$$

where the unperturbed Hamiltonian H_0 is the energy operator of a two-dimensional electron moving in the xy plane and in a constant and uniform magnetic field directed along the z axis:

$$H_0=\frac{1}{2m}\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)^2. \tag{2}$$

For the vector potential $\mathbf{A}$ we choose the symmetric gauge: $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$. The random potential U has the form

$$U=\sum_{\lambda\in\Lambda}\varepsilon_\lambda V(\mathbf{r}-\boldsymbol{\lambda}), \tag{3}$$

where $V(\mathbf{r})$ is a nonrandom short-range potential concentrated near the origin (the potential of a single impurity), and Λ is the set of points at which the impurities are concentrated. The coupling constants ε_λ are assumed to be identically distributed random quantities (not necessarily independent).

In the following we denote the cyclotron frequency by ω_c, the magnetic length by l_M, and quantum of magnetic flux by Φ_0. We denote the quantity $eB/2\pi\hbar\omega_c$ by ξ; $|\xi|$ is the number of quanta of magnetic flux through unit area. Later we shall need an expression for the Green function

$$G_E^0(\mathbf{r},\mathbf{r}')=\langle\mathbf{r}|(H_0-E)^{-1}|\mathbf{r}'\rangle$$

of the operator H_0. For $\mathbf{A}\neq 0$ we have[12]:

$$G_E^0(\mathbf{r},\mathbf{r}')=\frac{m}{2\pi\hbar^2}\Gamma(1/2-E/\hbar\omega_c)\exp[-i\pi\xi(xy'-yx')-\pi|\xi|(\mathbf{r}-\mathbf{r}')^2/2]\Psi(1/2-E/\hbar\omega_c,1;\pi|\xi|(\mathbf{r}-\mathbf{r}')^2). \quad (4)$$

Here $\Gamma(x)$ is Euler's gamma function, and $\Psi(a, c; x)$ is the confluent hypergeometric function. For $\mathbf{A} = 0$, as is well known,

$$G_E^0(\mathbf{r},\mathbf{r}')=\frac{m}{\pi\hbar^2}K_0[(-2mE/\hbar^2)^{1/2}|\mathbf{r}-\mathbf{r}'|], \quad (5)$$

where K_0 is the MacDonald function.

It is convenient to consider (and this will be done below) the limiting case $V(\mathbf{r})\to\delta(\mathbf{r})$; for the existence of a nontrivial limit for H it is necessary that the coupling constants ε_λ be infinitesimals of the order of $(\ln R)^{-1}$ as $R\to 0$ [R is the effective range of the potential $V(\mathbf{r})$].[13] In this case the expression for the Green function G_E of the operator H can be obtained in explicit form[13]:

$$G_E(\mathbf{r},\mathbf{r}')=G_E^0(\mathbf{r},\mathbf{r}')-\sum_{\lambda,\mu\in\Lambda}[Q(E)+T]_{\lambda\mu}^{-1}G_E^0(\mathbf{r},\lambda)G_E^0(\mu,\mathbf{r}'). \quad (6)$$

In the equality (6) the infinite matrix $Q(E) = [Q_{\lambda\mu}(E)]_{\lambda,\mu\in\Lambda}$ has the form

$$Q_{\lambda,\mu}(E)=\begin{cases} G_E^0(\lambda,\mu), & \lambda\neq\mu,\\ -\dfrac{m}{2\pi\hbar^2}[\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|)], & \mu=\lambda,\quad \mathbf{A}\neq 0,\\ -\dfrac{m}{2\pi\hbar^2}\ln(-mE\varkappa/2\hbar^2), & \lambda=\mu,\quad \mathbf{A}=0. \end{cases} \quad (7)$$

Here $\psi=(\ln\Gamma)'$ and $\varkappa$ is a parameter of dimensions l^2, characterizing a zero-range potential (the "effective depth" of a well of zero radius; for the physical meaning of the corresponding quantity in the three-dimensional case, see Ref. 14). In (6), T is a diagonal matrix $T=[\tau_\lambda\delta_{\lambda\mu}]_{\lambda,\mu\in\Lambda}$, in which τ_λ are identically distributed random quantities. It can be assumed (by renormalizing the quantity $\varkappa$, if this is required) that $\langle\tau_\lambda\rangle = 0$ (the angular brackets here and below denote averaging of the random quantity).

To find the elements of the inverse matrix $(Q+T)^{-1}$ we shall make use of the so-called "supersymmetry trick"[15,16]:

$$[Q(E)+T]_{\lambda\mu}^{-1}=i\int z_\lambda^* z_\mu\exp\Big[i\sum_{\lambda,\mu\in\Lambda}Q_{\lambda\mu}(E)(z_\lambda^* z_\mu+\theta_\lambda^*\theta_\mu)+i\sum_\lambda\tau_\lambda(z_\lambda^* z_\lambda+\theta_\lambda^*\theta_\lambda)\Big]Dz\,D\theta, \quad (8)$$

$$Dz=\prod_\lambda dz_\lambda\,dz_\lambda^*,\quad D\theta=\prod_\lambda d\theta_\lambda\,d\theta_\lambda^*.$$

In formula (8) z_λ are complex (commuting) variables and θ_λ are Grassmann (anticommuting) variables.

The expression (8) in the case of independent τ_λ makes it possible to perform the averaging over the random realizations of the potential in explicit form:

$$\langle[Q(E)+T]_{\lambda\mu}^{-1}\rangle=i\int z_\lambda^* z_\mu\exp\Big[i\sum_{\lambda,\mu}Q_{\lambda\mu}(E)(z_\lambda^* z_\mu+\theta_\lambda^*\theta_\mu)\Big]\prod_\lambda g(z_\lambda^* z_\lambda+\theta_\lambda^*\theta_\lambda)Dz\,D\theta. \quad (9)$$

Here,

$$g(x)=\int_{-\infty}^{\infty}e^{itx}p(t)\,dt$$

is the characteristic function of the probability distribution $p(t)dt$ of the random quantity τ_λ.

3. THE LLOYD MODEL

We shall consider the case of the Lorentzian distribution

$$p(t)=\frac{1}{\pi}\frac{\alpha}{t^2+\alpha^2},\quad g(x)=\exp(-\alpha|x|),\quad(\alpha>0). \quad (10)$$

In this case the formula (9) takes the form

$$\langle[Q(E)+T]_{\lambda\mu}^{-1}\rangle=i\int z_\lambda^* z_\mu\exp\Big[i\sum_{\lambda\mu}(Q_{\lambda\mu}(E)+i\alpha\delta_{\lambda\mu})\times(z_\lambda^* z_\mu+\theta_\lambda^*\theta_\mu)\Big]Dz\,D\theta=[Q(E)+i\alpha]_{\lambda\mu}^{-1}. \quad (11)$$

We seek the density of states $\rho(E)$ from the formula

$$\rho(E)=(\pi S)^{-1}\,\mathrm{Sp}\,\mathrm{Im}\,G_{E+i0}, \quad (12)$$

where S is the area of the system. We note first that for the unperturbed Hamiltonian H_0 the density of states, as is well known, is equal to

$$\rho_0(E)=|B|\Phi_0^{-1}\sum_{l=0}^{\infty}\delta(E-E_l), \quad (13)$$

where $E_l=(l+\frac{1}{2})\hbar\omega_c$ are the Landau levels. From (6), (11), and (12) we find that for $E\neq E_l$

$$\langle\rho(E)\rangle=\frac{\alpha}{\pi S}\sum_{\lambda,\mu\in S}[Q^2(E)+\alpha^2]_{\lambda\mu}^{-1}\int_S G_E^0(\mathbf{r},\lambda)G_E^0(\mu,\mathbf{r})\,d\mathbf{r}. \quad (14)$$

Since for real E the matrix $Q(E)$ is Hermitian, for $\alpha>0$ the matrix $[Q^2(E)+\alpha^2]^{-1}$ exists and is positive-definite. Consequently, taking into account that the functions $G_E^0(\mathbf{r},\lambda)$ form a Riesz basis,[17] we find that $\langle\rho(E)\rangle>0$ for all $E\neq E_l$. By virtue of the analytic dependence of Q and G on E, the function $\langle\rho(E)\rangle$ is analytic in E. These conclusions agree with the results of Wegner[2] for a white-noise potential.

A more detailed analysis of the expression (14) is possible if we assume that Λ is a lattice (not necessarily rectangular) constructed on vectors λ_1 and λ_2; thus, Λ consists of points of the form $\lambda = n_1\lambda_1 + n_2\lambda_2$, where n_1 and n_2 are integers. We shall denote by S_Λ the area of a unit cell of the lattice Λ. In addition, we assume that the system of coordinates x, y is chosen so that the vectors $\mathbf{e}_x$ and λ_1 are collinear. In order that group-theoretical arguments can be invoked in the analysis of (14) it is convenient to assume that the number of flux quanta of the field $\mathbf{B}$ through a unit cell of the lattice Λ (i.e., $\eta = S_\Lambda \xi$) is a rational number: $\eta = N/M$ (Refs. 18, 19). Below we shall discuss in detail the case $M = l$; i.e., $\eta = N$ is an integer. The case of arbitrary M can be reduced to the case $M = l$ by coarsening of the lattice Λ (Ref. 19); the formulas obtained for an integer flux η can be carried over, with certain modifications that will be discussed in the Conclusion, to the case of any rational η. We note that in the limit of large fields B or low impurity concentrations $n_i = S_\Lambda^{-1}$ it is obvious that η can be assumed to be an integer.

First let $N \neq 0$ (i.e., $B \neq 0$); if this is so, we can go over to the so-called qk-representation of Zak,[20] which, by analogy with the case $N = 0$, we shall call simply the quasimomentum representation. The states in this representation are described by quantum numbers $q_1 \in [0,1)$, $q_2 \in [0, |N|^{-1})$ [the vector $\mathbf{q} = (q_1, q_2)$ is the quasimomentum], the Landau-level number $l = 0, l,...,$ and the number $j = 0,1,...,|N| - 1$. These numbers essentially coincide with the quantum numbers of Wannier.[21] In the quasimomentum representation the averaged Green function has the form

$$\langle G_E(\mathbf{q}, l, j; \mathbf{q}', l', j')\rangle = (E_l - E)^{-1}\delta(\mathbf{q}-\mathbf{q}')\delta_{ll'}\delta_{jj'}$$
$$-|N|^{-1}(\tilde{Q}(\mathbf{q}, E)+i\alpha)^{-1}\frac{d(\mathbf{q}, l, j)}{E_l - E}\frac{d(\mathbf{q}', l', j')}{E_{l'} - E}\delta(\mathbf{q}-\mathbf{q}'). \tag{15}$$

Here we have used the notation

$$\tilde{Q}(\mathbf{q}, E) = \sum_{n_1, n_2=-\infty}^{\infty} \exp\left[-2\pi i\left(q_1 n_1 + N q_2 n_2 + \frac{N}{2} n_1 n_2\right)\right]$$
$$\times Q_{n_1\lambda_1+n_2\lambda_2,0}(E), \tag{16}$$

$$d(\mathbf{q}, l, j) = \sum_{m=-\infty}^{\infty} \exp\left[2\pi i m\left(q_2 + \frac{j}{|N|}\right)\right]$$
$$\times A(q_1+m, l)\, u_l\left(\lambda_{2y}\frac{q_1+m}{N l_M}\right), \tag{17}$$

$$A(z, l) = (l_M \lambda_{1x} \pi^{1/2} 2^l l!)^{-1/2} \exp(iz^2 \xi \lambda_{2x}\lambda_{2y}/2N^2),$$
$$u_l(x) = \exp(-x^2/2) H_l(x), \tag{18}$$

where H_l is a Hermite polynomial. The derivation of formula (15) from (6) and (11) is analogous to that of the formula (3.21) in Ref. 22 [in which a slightly different normalization of the function $d(\mathbf{q},l,j)$ was used]. We note that in Ref. 22 only the case of a rectangular lattice was considered, with $\lambda_{2x} = 0$ and with $A(z, l)$ independent of z. The Landau eigenfunctions used in the sum (17) and possessing the necessary translation properties under magnetic translations through vectors of the lattice Λ are indicated in Ref. 23.

We shall denote by $G_E(\mathbf{q})$ the operator with the kernel (15), acting for a fixed $\mathbf{q}$ on the variables l and j; also, let

$$\rho(\mathbf{q}, E) = \pi^{-1}\,\mathrm{Sp}\,\mathrm{Im} G_{E+i0}(\mathbf{q}). \tag{19}$$

Obviously,

$$\langle\rho(E)\rangle = C\int_0^1\int_0^{1/|N|} \rho(\mathbf{q}, E)\,dq_1\,dq_2, \tag{20}$$

where C is a normalization constant independent of the parameter $\varkappa$. In the limit $\varkappa \to 0$ we obtain

$$G_E \to G_E^0(\mathbf{q}, l, j; \mathbf{q}', l', j') = (E_l - E)^{-1}\delta(\mathbf{q}-\mathbf{q}')\delta_{ll'}\delta_{jj'},$$

where $G_E^0(\ ...\ )$ is the quasimomentum representation of the Green function G_E^0. Since

$$\rho_0(\mathbf{q}, E) \equiv \pi^{-1}\,\mathrm{Sp}\,\mathrm{Im}\, G^0_{E+i0}(\mathbf{q})$$
$$= \lim_{\delta\to 0} \pi^{-1}|N|\sum_{l=0}^{\infty}\delta^{-1}[(E-E_l)^2+\delta^2] = |N|\sum_{l=0}^{\infty}\delta(E-E_l), \tag{21}$$

we have

$$\int_0^1\int_0^{1/|N|}\rho_0(\mathbf{q}, E)\,dq_1\,dq_2 = \sum_l \delta(E-E_l). \tag{22}$$

The constant C is determined from the equality

$$C\int \rho_0(\mathbf{q}, E)\,d\mathbf{q} = \rho_0(E),$$

from which, taking (13) and (22) into account, we obtain

$$C = |B|\Phi_0^{-1} = |N| S_\Lambda^{-1}.$$

Now, from (15) and (20) for $E \neq E_l$, we derive the formula

$$\langle\rho(E)\rangle = \frac{\alpha}{\pi S_\Lambda}\sum_{j,l}(E-E_l)^{-2}\int_0^1\int_0^{1/|N|}\frac{|d(\mathbf{q}, l, j)|^2\,dq_1\,dq_2}{[\tilde{Q}(\mathbf{q}, E)]^2+\alpha^2}. \tag{23}$$

Since $\tilde{Q}(\mathbf{q},E)$ is a function period in q_2 with period $|N|^{-1}$, summing over $j = 0,...,|N| - 1$ we obtain $(E \neq E_l)$

$$\langle\rho(E)\rangle = \frac{\alpha}{\pi S_\Lambda}\sum_{l=0}^{\infty}(E-E_l)^{-2}\int_0^1\int_0^1\frac{|d(\mathbf{q}, l)|^2\,dq_1\,dq_2}{[\tilde{Q}(\mathbf{q}, E)]^2+\alpha^2}, \tag{24}$$

where the function $d(\mathbf{q}, l)$ is defined by the equality

$$d(\mathbf{q}, l) = \sum_{m=-\infty}^{\infty}\exp(2\pi i m q_2) A(q_1+m, l)\, u_l\left(\lambda_{2y}\frac{q_1+m}{N l_M}\right). \tag{25}$$

To investigate the behavior of $\langle\rho(E)\rangle$ for $E \neq E_l$ we consider the quantity

$$\sigma = \sum_l (E-E_l)^{-2}\int_0^1\int_0^1 |d(\mathbf{q}, l)|^2\,d\mathbf{q}. \tag{26}$$

Using (18), (25), and the orthogonality of the Hermite functions, we easily obtain

$$\int_0^1\int_0^1 |d(\mathbf{q},l)|^2 d\mathbf{q}=|N|S_\Lambda^{-1}. \qquad (27)$$

Since

$$\sum (E-E_l)^{-2}=(\hbar\omega_c)^{-2}\psi'(1/2-E/\hbar\omega_c),$$

we have

$$\sigma=m(2\pi\hbar^3\omega_c)^{-1}\psi'(1/2-E/\hbar\omega_c). \qquad (28)$$

Since the function $\tilde{Q}$ is periodic and continuous in $\mathbf{q}$, the following bounds are valid:

$$c_1(E)\leqslant|\tilde{Q}(\mathbf{q},E)|\leqslant c_2(E), \qquad (29)$$

where $0\leqslant c_1(E)\leqslant c_2(E)<\infty$ and $c_i(E)$ depends only on E. Using (24), (28), and (29), we obtain for $E\neq E_l$ the bounds

$$\frac{\alpha}{\alpha^2+c_2^2(E)}\leqslant\frac{4\pi^3\hbar^4|N|}{m^2\psi'(1/2-E/\hbar\omega_c)}\langle\rho(E)\rangle\leqslant\frac{\alpha}{\alpha^2+c_1^2(E)}. \qquad (30)$$

We recall that

$$N=BS_\Lambda\Phi_0^{-1}=Bn_i^{-1}\Phi_0^{-1},$$

where n_i is the concentration of impurities. From (4) and (16) we obtain the asymptotic form

$$\tilde{Q}(\mathbf{q},E)\sim-\frac{m}{2\pi\hbar^2}[\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|)]. \qquad (31)$$

In (31) we have assumed that one of the following conditions is fulfilled:

1) $E\neq E_l$ and n_i are fixed, and $B\to\infty$;
2) $E\neq E_l$ and B are fixed, and $n_i\to 0$;
3) B and n_i are fixed, and $E\to-\infty$.

Taking into account the equality $\xi=B\Phi_0^{-1}$, from (30) and (31) we obtain the following asymptotic form, valid in each of the cases 1)–3):

$$\langle\rho(E)\rangle\sim\frac{m^2\Phi_0 n_i}{4\pi^2\hbar^4|B|}\times\frac{\alpha\psi'(1/2-E/\hbar\omega_c)}{\pi\left\{\alpha^2+\dfrac{m^2}{4\pi^2\hbar^4}[\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|)]^2\right\}}. \qquad (32)$$

In particular, by fixing the value $E=l\hbar\omega_c$ lying midway between the Landau levels E_{l-1} and E_l, we obtain

$$\langle\rho(E)\rangle\sim|B|^{-1}(1+\text{const}\ln|B|)^{-2}\quad(B\to\infty,\ n_i\ \text{fixed}); \qquad (33)$$

$$\langle\rho(E)\rangle\sim n_i\quad(n_i\to 0,\ B\ \text{fixed}). \qquad (34)$$

In the limit $\alpha\to 0$ the Lloyd model goes over into a deterministic model of the Kronig-Penney type, describing the motion of a two-dimensional Landau electron in a periodic point potential. In this case the energy of the lth Landau level lies near the root $\tilde{E}_l$ of the equation

$$\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|)=0 \qquad (35)$$

($\tilde{E}_l$ is the root of this equation nearest to and to the left of E_l), and spreads into a band lying between the levels E_{l-1} and E_l (Refs. 13, 22). From the formula (24), taking the limit $\alpha\to 0$ we find that the density of states in the deterministic model for $E\neq E_l$ is given by the expression

$$\rho(E)=S_\Lambda^{-1}\sum_{l=0}^{\infty}(E-E_l)^{-2}\int_0^1\int_0^1\delta(\tilde{Q}(\mathbf{q},E))|d(\mathbf{q},l)|^2 dq_1 dq_2 \qquad (36)$$

[in complete correspondence with the fact that the bands of the spectrum in this model are determined by the equation $\tilde{Q}(\mathbf{q},E)=0$]. We note that the presence in (33) of the factor $(1+\text{const}\cdot\ln|B|)^{-2}$ is explained precisely by the shift of the Landau level to $\tilde{E}_l$. Having fixed $\tilde{E}_l$ in (32), we obtain the asymptotic form

$$\langle\rho(E)\rangle\sim|B|^{-1}.$$

The question of the behavior of $\langle\rho(E)\rangle$ at the points E_l requires special investigation and, because of the presence of the δ-function singularities in $\rho_0(E)$, cannot be solved by taking the limit $E\to E_l$. The analysis performed in Ref. 22 for the deterministic model shows that for each fixed quasimomentum the multiplicity of E_l in the spectrum of H_0 is equal to $|N|$, and decreases by unity when the potential of one point impurity, placed in each Wigner-Seitz cell, is added to H_0. From the method given in Ref. 13 for constructing the eigenfunctions of the operator H_0 that occur simultaneously in the spectrum of H it follows that they are the same for any distribution of the parameters $\varkappa$ over the sites of the lattice Λ. From this and from (21) it follows that the complete form of $\langle\rho(E)\rangle$ is

$$\langle\rho(E)\rangle=(|N|-1)\sum_{l=0}^{\infty}\delta(E-E_l)+\frac{\alpha}{\pi S_\Lambda}\sum_{l=0}^{\infty}(E-E_l)^{-2}\int_0^1\int_0^1\frac{|d(\mathbf{q},l)|^2 dq_1 dq_2}{[\tilde{Q}(\mathbf{q},E)]^2+\alpha^2}. \qquad (37)$$

The coefficient of the Dirac comb in (37) agrees with the analogous coefficient in Ref. 3, obtained for point impurities with a Poisson distribution. Finally, we note that the complete form of $\langle\rho(E)\rangle$ in the deterministic model is obtained by replacing the second term in (37) by the expression in the right-hand side of (36).

We note one further interesting consequence of the formula (32). Since $\psi(z)$ has simple poles at the points $0, -1,\ldots$, and the residues at these poles are pairwise different,[24] the limits of $\langle\rho(E)\rangle$ at E_l and E_{l+1} are also different, indicating asymmetry of the graph $\langle\rho(E)\rangle$ about the Landau levels.

4. LIMIT OF A LARGE FIELD OR LOW IMPURITY CONCENTRATIONS

Even when the quantities τ_λ have a Gaussian distribution, the direct application of formula (9) leads to complications analogous to the difficulties in the calculation of correlation functions in the φ_2^4 lattice model. However, when the limit $B\to\infty$ (or $n_i\to 0$) is taken the calculations are simplified. In this case we can neglect the off-diagonal elements of the matrix $[Q+T]^{-1}$ and for $E\neq E_l$ obvious transformations give for $\langle\rho(E)\rangle$ the expression

$$\langle\rho(E)\rangle=S^{-1}\sum_{\lambda\in S}p\left[\frac{m}{2\pi\hbar^2}(\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|))\right]\times\int_S G_E{}^0(\mathbf{r},\lambda)G_E{}^0(\lambda,\mathbf{r})\,d\mathbf{r}. \quad (38)$$

The integrals in formula (38) do not depend on λ, and calculation of them in the quasimomentum representation leads to the expression (28). Since the result of dividing the number of terms in the sum (38) by S is the impurity density n_i, and

$$\omega_c=2\pi\hbar|B|/m\Phi_0,$$

we have

$$\langle\rho(E)\rangle=\frac{m^2\Phi_0 n_i}{4\pi^2\hbar^4|B|}p\left[\frac{m}{2\pi\hbar^2}(\psi(1/2-E/\hbar\omega_c)+\ln(\pi\varkappa|\xi|))\right]\psi'(1/2-E/\hbar\omega_c). \quad (39)$$

We note that in the derivation of the formula (39) it was not assumed that the quantities τ_λ are independent. If the τ_λ have a Lorentzian distribution, (39) obviously coincides with (32).

We shall consider, in particular, a Gaussian distribution for τ_λ:

$$p(t)=(2\pi w)^{-1/2}\exp(-t^2/2w). \quad (40)$$

In this case the density of states $\langle\rho(E)\rangle$ has sharp peaks near the shifted Landau levels $\tilde{E}_l$. With increasing (but small) $|E-\tilde{E}_l|$, $\langle\rho(E)\rangle$ falls in accordance with the law

$$\langle\rho(E)\rangle\propto\exp[-\gamma(E-E_l)^2],$$

where

$$\gamma=\frac{1}{2w}\left[\frac{m}{2\pi\hbar^3\omega_c}\psi'(1/2-E_l/\hbar\omega_c)\right]^2.$$

For $E\to E_l$

$$\langle\rho(E)\rangle\propto(E-E_l)^{-2}\exp[-\mathrm{const}(E-E_l)^{-2}], \quad (41)$$

i.e., $\langle\rho(E)\rangle\to 0$. At the same time, here too the asymmetry of the spread-out Landau level is preserved, since $\tilde{E}_l$, generally speaking, does not lie at the midpoint of the interval (E_{l-1},E_l). Using the $z\to+\infty$ asymptotic form $\psi(z)\propto\ln z$, we see that for $E\to-\infty$ the tail of the function $\langle\rho(E)\rangle$ has the asymptotic form

$$\langle\rho(E)\rangle\sim(1/2-E/\hbar\omega_c)^{-1}\exp\{-\mathrm{const}[\ln(1/2-E/\hbar\omega_c)]^2\}. \quad (42)$$

If $E=\tilde{E}_l$, for any distribution $p(t)$ the following asymptotic forms are valid:

$$\langle\rho(E)\rangle\propto|B|^{-1},\quad B\to\infty;\quad\langle\rho(E)\rangle\infty n_i,\quad n_i\to 0.$$

With allowance for the $z\to-\infty$ asymptotic form of $\psi(z)$ (Ref. 24) the formula (39) shows that the form of the spread-out Landau levels with large values of l depend only on the distribution of the random quantities τ_λ and does not depend on their correlation. For a Gaussian distribution this property was noted in Ref. 1.

5. CASE OF FULLY CORRELATED POTENTIALS

For centered random quantities τ_λ having a second moment, this case implies that τ_λ are equal random quantities ($\tau_\lambda=\tau$); we shall use this in the following. In addition, we assume that Λ is a lattice and that the flux η is an integer: $\eta=N$. From the quasimomentum representation of the Green function G_E [see formula (15), in which $i\alpha$ must be replaced by τ], we immediately obtain

$$\langle\rho(E)\rangle=(|N|-1)\sum_{l=0}^{\infty}\delta(E-E_l)+S_\Lambda^{-1}\sum_{l=0}^{\infty}(E-E_l)^{-2}\int_0^1\!\!\int_0^1 p(\tilde{Q}(\mathbf{q},E))|d(\mathbf{q},l)|^2\,dq_1\,dq_2. \quad (43)$$

We note that for the Lorentzian distribution we again arrive at the formula (24); the lack of dependence of $\langle\rho(E)\rangle$ on the correlation in the Lloyd model in the general case was noted in Ref. 11. Analysis of the expression (43) in the limit of large B (or small n_i) returns us to the formula (39), which was discussed above.

6. THE CASE B=0

For the unperturbed Hamiltonian H_0 in this case, as is well known, the density of states $\rho_0(E)$ has the form

$$\rho_0(E)=\begin{cases}m/2\pi\hbar^2, & E\geqslant 0,\\ 0, & E<0.\end{cases} \quad (44)$$

In the case of the full Hamiltonian H the function $\langle\rho(E)\rangle$ is also given by different analytical expressions for $E>0$ and $E<0$. For simplicity we shall confine ourselves to the case $E<0$, making it possible to investigate the behavior of $\langle\rho(E)\rangle$ for $B\to 0$ in the tail of the lowest Landau level.

Suppose first that the τ_λ are independent and have a Lorentzian distribution (the two-dimensional Lloyd model; for three-dimensional electrons this model was investigated in Ref. 9). Then, as before, the formula (14) is valid. For a more detailed investigation of this formula we go over to the quasimomentum representation. We denote by Γ the lattice dual to Λ: For $\gamma\in\Gamma$ and $\lambda\in\Lambda$ the quantity $\gamma\cdot\lambda$ is a multiple of 2π. We denote by S_Γ the area of the Brillouin zone (the unit cell of the lattice Γ). In analogy with the three-dimensional case,[25,26] for the quasimomentum representation of the averaged Green function $\langle G_E\rangle$ we obtain, after simple but cumbersome transformations, the expression

$$\langle G_E(\mathbf{q},\gamma;\mathbf{q}',\gamma')\rangle=\frac{2m}{\hbar^2}\frac{\delta(\mathbf{q}-\mathbf{q}')\delta_{\gamma\gamma'}}{(\mathbf{q}+\gamma)^2+k^2}-\left(\frac{m}{\pi\hbar^2}\right)^2 S_\Gamma[\tilde{Q}(\mathbf{q},E)+i\alpha]^{-1}\frac{1}{(\mathbf{q}+\gamma)^2+k^2}\frac{\delta(\mathbf{q}-\mathbf{q}')}{(\mathbf{q}'+\gamma')^2+k^2}. \quad (45)$$

Here $\mathbf{q}$ is the quasi momentum, $\gamma\in\Gamma$, $k^2=-2mE/\hbar^2$, and the function $\tilde{Q}(\mathbf{q},E)$ is given by the equality

$$\tilde{Q}(\mathbf{q},E)=\sum_{\lambda\in\Lambda}\exp(-i\lambda\mathbf{q})Q_{\lambda 0}(E). \quad (46)$$

From (45), taking into account that $S_\Gamma = (2\pi)^2 S_\Lambda^{-1}$, we obtain

$$\langle\rho(E)\rangle = \frac{m^2\alpha}{\pi^3\hbar^4 S_\Lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_1\,dq_2}{([\tilde{Q}(\mathbf{q},E)]^2+\alpha^2)(\mathbf{q}^2+k^2)}. \quad (47)$$

Taking into account the $z\to\infty$ asymptotic form of $K_0(z)$ (Ref. 24) and the formula (7), for $E\to-\infty$ we obtain

$$\tilde{Q}(\mathbf{q},E)\sim\frac{m}{2\pi\hbar^2}\ln\left(-\frac{mE\varkappa}{2\hbar^2}\right). \quad (48)$$

Making use of the $z\to+\infty$ asymptotic form $\psi(z)\propto\ln z$, from (32) and (47) we easily find that for arbitrary B the corresponding density of states $\langle\rho(E)\rangle$ satisfies the relation

$$\langle\rho(E)\rangle(\langle\rho(E)\rangle|_{B=0})^{-1}\to 1 \quad \text{as} \quad E\to-\infty. \quad (49)$$

Thus, the tail of $\langle\rho(E)\rangle$ in the Lloyd model is asymptotically independent of the quantities B and n_i.

In the case of a low concentration of impurities and arbitrarily correlated τ_λ, all distributed by the same law with probability density $p(t)$, the analog of the formula (38) is the following expression:

$$\langle\rho(E)\rangle = S^{-1}\sum_{\lambda\in S} p\left(\frac{m}{2\pi\hbar^2}\ln(-mE\varkappa/2\hbar^2)\right)\times\int G_E{}^0(\mathbf{r},\lambda)G_E{}^0(\lambda,\mathbf{r})\,d\mathbf{r}. \quad (50)$$

By virtue of the resolvent identity $dR_E/dE = R_E^2$ and the translational invariance of the function $G_E^0(\mathbf{r},\mathbf{r}')$, all the integrals in (50) are equal to the expression

$$\lim_{\mathbf{r}\to 0}\partial G_E{}^0(\mathbf{r},0)/\partial E.$$

Using the equality[24]

$$\lim_{x\to 0}\partial K_0(ax)/\partial a = -a^{-1},$$

we obtain

$$\int G_E{}^0(\mathbf{r},\lambda)G_E{}^0(\lambda,\mathbf{r})\,d\mathbf{r} = m/2\pi\hbar^2|E|.$$

Hence we have

$$\langle\rho(E)\rangle = \frac{mn_i}{2\pi\hbar^2|E|}p\left[\frac{m}{2\pi\hbar^2}\ln(-m\varkappa E/2\hbar^2)\right], \quad (51)$$

which agrees with (39) for $B\to 0$ and large $|E|$ ($E<0$).

Finally, for completely correlated τ_λ we obtain the following analog of the formula (43) for the case $B = 0$:

$$\langle\rho(E)\rangle = \frac{m^2}{\pi^2\hbar^4 S_\Lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p[\tilde{Q}(\mathbf{q},E)]}{\mathbf{q}^2+k^2}dq_1\,dq_2. \quad (52)$$

Comparison of the expressions (52) and (43) shows that the density of states (in the case of point potentials) in continuous in B down to zero. We note that the continuity of $\rho(E)$ in B for $B\neq 0$ for a broad class of potentials was proved by Simon.[27]

7. CONCLUSION

First we shall discuss the changes that must be introduced into our formulas for the lattice potentials in the case of an arbitrary rational flux through a unit cell of the lattice: $\eta = N/M$ ($M\geqslant 1$). With the use of the technique of Ref. 22 the expression (37) is modified as follows:

$$\langle\rho(E)\rangle = \min(|\eta|-1,0)\sum_{l=0}^{\infty}\delta(E-E_l)+\frac{\alpha}{\pi S_\Lambda}\sum_{l=0}^{\infty}(E-E_l)^{-2}\times\int_0^1\int_0^1\sum_{s,t=0}^{M-1}[(\tilde{Q}(\mathbf{q},E))^2+\alpha^2]_{st}^{-1}d_s(\mathbf{q},l)\,d_t^*(\mathbf{q},l)\,dq_1\,dq_2, \quad (53)$$

where $\tilde{Q}(\mathbf{q},E)$ is a matrix of order $M\times M$, the elements of which are given by the expression ($s, t = 0, 1, \ldots, M-1$):

$$\tilde{Q}_{st}(\mathbf{q},E) = \sum_{n_1,n_2=-\infty}^{\infty}\exp\left\{-2\pi i\left[(q_1+Ns)n_1+Nq_2n_2+\frac{N}{2}n_1n_2\right]\right\}Q_{n_1\lambda_1+n_2\lambda_2,(t-s)\lambda_2}(E), \quad (54)$$

while the functions $d_s(q,l)$ are determined by the formula

$$d_s(\mathbf{q},l) = \sum_{m=-\infty}^{\infty}\exp(2\pi imq_2)A(q_1+m+Ns,l)\times u_l[\lambda_{2y}(q_1+m+Ns/Nl_M)]. \quad (55)$$

The factor multiplying the Dirac comb in (53), in agreement with Refs. 13 and 22, shows that Landau levels appear in the spectrum of the Hamiltonian H only for $|\eta|>1$; in this case the multiplicity of each of the Landau levels for a fixed quasimomentum decreases by unity in comparison with that for the unperturbed Hamiltonian H_0. Crudely speaking, "weakness" of the point potential leads to the result that for $|\eta|>1$ some of the states are split off from the Landau level and the energy corresponding to them is spread out into a band. For $|\eta|\leqslant 1$ all the Landau levels are spread out into a band consisting of M magnetic sub-bands. This has been discussed in more detail in Refs. 13 and 22. We note that, as shown in Ref. 3, in the case of Poisson-distributed point scatterers with a Gaussian random coupling constant, the first term of the formula (53) also appears in the expression for $\langle\rho(E)\rangle$ at the lowest Landau level. The condition $|\eta|>1$ for Landau levels to appear in the spectrum of H was, in fact, noted by Ando.[28]

As shown by the formulas obtained above for $\langle\rho(E)\rangle$, this function depends analytically on E between Landau levels if $p(t)$ depends analytically on t. In this case $\langle\rho(E)\rangle$ can vanish only at isolated points (in the examples given in the paper, $\langle\rho(E)\rangle>0$ even for $E\neq E_l$). In the models considered, this eliminates the physically unreal steep declines of the density of states, with a peak of semielliptical shape, that have been obtained by numerical calculations using the method of the self-consistent Born approximation.[29]

Finally, it is interesting to note that for a single-point Lorentzian random impurity, situated at the point $\mathbf{r}_0$ the averaged Green function has the form

$$\langle G_E(\mathbf{r},\mathbf{r}')\rangle = [Q_{00}{}^2(E)+\alpha^2]^{-1}G_E{}^0(\mathbf{r},\mathbf{r}_0)G_E{}^0(\mathbf{r}_0,\mathbf{r}). \quad (56)$$

From this, as in the case of the three-dimensional model without a magnetic field,[9] it follows that for two-dimensional electrons (even in the presence of a magnetic field) the behavior of the tail of the function $\langle \rho(E) \rangle$ as $E \to -\infty$ in the Lloyd model is the same as for a single Lorentzian impurity. For a Gaussian random impurity this statement is no longer true [see formula (42)].

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