Motion of quasiperiodic structures depinned by an external field

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A study is made of inertia-free dynamics of one-dimensional harmonic chains in a periodic potential in the presence of a homogeneous longitudinal field, which constitutes a model describing the slip of charge density waves and the motion of a vortex lattice in a profiled superconducting film. A variant of a model with a piecewise-parabolic potential, introduced by Aubry, is used to obtain the exact solution for steady-state motion of a chain. The nature of behavior near the mobility threshold can be used to classify steadily moving structures as commensurate, incommensurate, and almost commensurate, each characterized by its own dependence of the velocity of motion on the applied force. A description is given of the motion of a solitary phase defect in a commensurate periodic structure.

The spatial coherence effects are manifested by a wide range of physical systems such as quasi-one-dimensional crystals with a space charge wave,[1–5] monatomic films on crystalline substrates,[6,7] Abrikosov vortex lattices formed in a thin profiled superconducting film,[8] etc. We shall consider systems in which there are two types of interaction, each of which tends to impose its own periodicity; they can be frequently regarded as the interaction of elements of a structure with one another, on the one hand, and the interaction with an external potential, on the other.

In the case of a corrugated superconducting film (Fig. 1) a transverse magnetic field applied transverse to the film tends to form a regular triangular vortex lattice in which the separation between the rows is

$$a=(^3/_4)^{1/4}(\Phi_0/B)^{1/2},$$

where B is the magnetic induction and $\Phi_0 = hc/2e$ is a magnetic flux quantum. The presence of a one-dimensional corrugation with a period b means that, for energetic reasons, the vortices are located where the film thickness is least so that the period of the vortex structure in the direction, of the corrugation tends to become commensurate with b.

Another example is a system with a charge density wave. In this case the "intrinsic" period of a structure is $a = \pi\hbar/p_F$, where p_F is the Fermi momentum. In view of the discrete nature of the lattice, the system tends to form a structure in which charge modulation along the direction of the wave vector of the charge density wave can have a period commensurate with the period of the lattice along a given direction of b.

In both these examples the application of an external force can set in motion a modulated structure in the form of a charge density wave or a vortex lattice. An Abrikosov lattice can be set in motion simply by passing a current j directed along corrugation grooves in a film (Fig. 1). The vortices then experience a Lorentz force equal to $f = c^{-1}\Phi_0 j$ and directed transverse to the current. A charge density wave subjected to an electric field also experiences a force directed along the field and proportional to its intensity.

It is known from a general theory of incommensurate structures[9,10] that in the systems under consideration we can expect either commensurate structures which are periodic or incommensurate quasiperiodic structures. The response to an applied force can vary. We can have a situation when force, no matter how weak, can set the system in motion or the motion of a structure may begin only when the force exceeds a certain threshold value. In the latter case it is usual to describe the structure as pinned. Commensurate structures are always pinned and incommensurate structures are pinned only if the applied potential is sufficiently high. Each quasiperiodic incommensurate structure is characterized by its own critical value of the applied potential, beginning from which a structure becomes pinned.

If the applied force exceeds a threshold value, a structure travels at some velocity v. In the case of a charge density wave this implies the appearance of an electric current proportional to v. The motion of vortices in a superconducting film creates an electric field proportional to v and directed along the current. The present paper is concerned with how the velocity v of a structure depends on the applied force f. In both applications mentioned above such a dependence is regarded as a current-voltage characteristic, i.e., as the dependence of the current on the voltage in the case of a charge density wave and of the voltage the current in the case of a superconducting film.

Aubry[11] proposed a model which can be solved exactly and which predicts commensurate and incommensurate

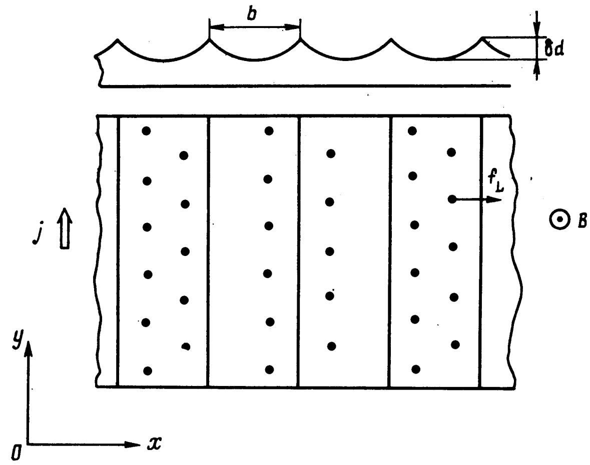

FIG. 1. Vortex structure moving along the x axis in a current-carrying corrugated superconducting film. A magnetic field B is directed at right-angles to the plane of the film. Each vortex experiences a Lorentz force $f_l = c^{-1}\Phi_{0j}$.

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structures. In view of the specific nature of the potential in this model all the structures (both commensurate and incommensurate) are pinned irrespective of the strength of the applied potential. However, the pinning force may depend strongly on this potential. Aubry obtained explicitly all the structures representing the ground state of his model for any ratio of the "internal" period to the period of the potential.

In the first section of the present paper we shall formulate the Aubry model and describe structures corresponding to commensurate and incommensurate ground states, and also structures with solitary defects. In the second section we shall derive expressions relating the velocity of steady-state motion of a structure to the applied force. In the third section we shall study the dependence of the force on the periodicity of the structure. In the fourth section we shall consider the motion of a solitary in a periodic structure. In the fifth section we shall discuss the motion of an incommensurate quasiperiodic structure in near-threshold fields. The results obtained will be discussed in the conclusions.

1. MODEL: GROUND STATE

We shall consider a one-dimensional chain in which each site is associated with a real quantity φj, where j is the site number. It is usual to consider models with an energy functional of the type

$$E(\{\varphi_j\})=\sum_j L(\varphi_j,\varphi_{j+1}), \quad (1.1)$$

where the function $L(x, y)$ is a discrete analog of a Lagrangian and it has the following properties: $L(x, y)$ is continuous, periodic in the sense

$$L(x+1,\ y+1)=L(x,\ y),$$

and satisfies the condition

$$\partial^2 L/\partial x\partial y<C<0.$$

In particular, these properties are exhibited by functionals of the type

$$L(\varphi_1,\ \varphi_2)={}^1/_2(\varphi_2-\varphi_1)^2+{}^1/_2\lambda[V(\varphi_1)+V(\varphi_2)] \quad (1.2)$$

with a periodic potential $V(\varphi+1)=V(\varphi)$.

The most popular potential is

$$V_{FK}(\varphi)=(1-\cos 2\pi\varphi)/4\pi^2, \quad (1.3a)$$

which corresponds to the familiar Frenkel–Kontorova model.[12] Aubry was able to obtain an exact solution by choosing the following potential:

$$V_A(\varphi)={}^1/_2[\varphi-\mathrm{Int}(\varphi+{}^1/_2)]^2, \quad (1.3b)$$

where Int x is the integral part of x.

We can formulate the task of minimization of the function (1.1) for a chain of length which tends to infinity, subject to the following additional condition superimposed on the configuration $\{\varphi j\}$:

$$\lim_{N\to+\infty,N'\to-\infty}\frac{\varphi_N-\varphi_{N'}}{N-N'}=\Phi. \quad (1.4)$$

Obviously, a minimum corresponds to one of the stationary configurations characterized by

$$\partial E/\partial\varphi_j=0. \quad (1.5)$$

The following theorem can be proved[10] about the ground state of the model described by Eq. (1.1) for a given value of ϕ: this ground state corresponds to one of the configurations from a set parametrized by a unique function $g(x)$ with the following properties: $g(x)$ is a nondecreasing function of x, $g(x_1)\geqslant g(x_2)$ for $x_1>x_3$ and $g(x+1)=g(x)$ or $g(x)=x+h(x)$, where $h(x)=h(x+1)$ is a periodic function of x. Then, all the configurations of the type

$$\varphi_j^0=g(j\Phi+\beta) \quad (1.6)$$

correspond to the ground state and the parameter β labels such configurations.

In the case of rational values of $\Phi=M/L$, where M and L are integers, the ground state exhibits the property of periodicity: $\varphi^0_{j+L}=\varphi^0_j+M$. Then, in the interval [0; 1] the function $g(x)$ has exactly L discontinuities separated by $1/L$ and in the intervals between the discontinuities we have $g(x)=\text{const}$. This form of $g(x)$ corresponds to a unique ground state of the system, apart from the substitution

$$\varphi_j^0\to\tilde{\varphi}_j^0=\varphi^0_{j+l}+m.$$

The configuration $\{\varphi^d_j\}$ minimizing the functional (1.1) corresponds to an elementary topological phase defect above a commensurate ground state with $\Phi=M/L$ if

$$\varphi_j^d=\begin{cases} g(j\Phi+\beta), & j\to-\infty\\ g(j\Phi+\beta\pm1/L), & j\to+\infty\end{cases}. \quad (1.7)$$

The plus sign in Eq. (1.7) corresponds to a dilatation defect and the minus sign corresponds to a compression defect. For example, if $\Phi=0$ and $g(x)=\text{Int}\ x$, the ground state is the configuration $\varphi^0_j=m$, where m is an integer, whereas in the case of an elementary defect, the ground state is the following kink configuration:

$$\varphi_j^d=\begin{cases} m, & j\to-\infty\\ m\pm1, & j\to+\infty\end{cases}. \quad (1.7a)$$

The ground state with $\Phi=1/N$ can be regarded as a structure composed of equidistant kinks separated by a distance N. An elementary dilatation (compression) defect in such a structure is the configuration in which the distance is $N\pm1$ for one of the pairs of the neighboring kinks. The ground state with $\Phi=1/(N_1+1/N_2)$ can then be regarded as a sequence of defects separated by the same distance N_1N_2+1 above a structure with $\Phi=1/N$.

This can be generalized as follows: the ground state with a given rational value Φ_s of the type

$$\Phi_s=M_{s+1}/L_{s+1}=1/(N_1+1/(N_2+\ldots+1/N_s)\ldots) \quad (1.8)$$

represents a sequence of defects separated by the same distance L_{s+1} above a structure characterized by

$$\Phi=\Phi_{s-1}=1/(N_1+1/(N_2+\ldots+1/N_{s-1})\ldots).$$

The ground state with an irrational value of Φ can be regarded as a hierarchy of defect structures in a sequence governed by an expansion of Φ in the form of an infinite continued fraction.[13]

These ideas are, strictly speaking, valid only if the dimensions of defects are small compared with the separation between them, which is true when the potential λ is sufficiently strong. If the potential V in the model of Eq. (1.2) is sufficiently smooth, we can show[10] that for low values of λ the ground state with an irrational value of Φ corresponds to an analytic function of $g(x)$ without any discontinuities. This means that a continuous degeneracy of the ground state with a Goldstone parameter β [see Eq. (1.6)] occurs, so that the threshold force is zero.

In the model with a potential (1.3b) the critical value of the potential force λ_c inducing a transition from one incommensurate structure to another is zero for all the irrational Φ. The ground state is always pinned and it corresponds to a finite or an infinite hierarchy of defects. The function $g(x)$ can then be found exactly[11]:

$$g_A(x)=\frac{1-r}{1+r}\sum_{l=-\infty}^{\infty} r^{|l|}\,\mathrm{Int}\left(x+l\Phi+\frac{1}{2}\right), \qquad (1.9)$$

where

$$r=1+\frac{\lambda}{2}-\left(\lambda+\frac{\lambda^2}{4}\right)^{1/2}. \qquad (1.10)$$

If $\lambda \ll 1$, then $r\approx 1-\lambda^{1/2}$. For irrational values of Φ, we find that $g_A(x)$ has a dense set of discontinuities at the following points: $x=\frac{1}{2}+l\Phi+m$.

2. STEADY-STATE MOTION

We shall now assume that each site in a chain experiences an external force f which displaces φ_j from its equilibrium position. For the sake of simplicity, we shall consider an inertialess (instantaneous-response) system for which the equations of motion contain only first derivatives with respect to time:

$$\dot\varphi_j=-\partial E/\partial\varphi_j+f. \qquad (2.1)$$

In the case of a vortex lattice in a superconducting film the motion of vortices is always inertialess.[14] Here, f is proportional to the current of the Lorentz force. Neglect of the inertia of a charge density wave is also often justified.[15,16] In the latter case the quantity f represents the electric field.

Substituting $E(\{\varphi_j\})$ in the form of Eqs. (1.1), (1.2), and (1.3b), into Eq. (2.1), we obtain the equation of motion for the Aubry model:

$$\dot\varphi_j=\varphi_{j+1}+\varphi_{j-1}-(2+\lambda)\varphi_j+\lambda m_j+f, \qquad (2.2)$$

where

$$m_j=\mathrm{Int}(\varphi_j+{}^1/_2). \qquad (2.3)$$

We shall seek the solution of equations of the form (2.2) corresponding to steady-state motion of a chain as a whole at some velocity v subject to the condition that the phase shift is maintained during subsequent times at infinitely distant sites of the chain, i.e., when the parameter Φ of Eq. (1.4) remains constant. We can expect that this type of motion occurs when the force f is appplied adiabatically.

By analogy with the Aubry solution, in the case of the ground state of the chain we shall assume that[11]

$$m_j(t)=\mathrm{Int}(j\Phi+vt+{}^1/_2). \qquad (2.4)$$

It is then necessary to demonstrate *a posteriori* that this selection of $m_j(t)$ is self-consistent, i.e., we can show that the solution of the system of equations (2.2) with $m_j(t)$ in the form of Eq. (2.4) satisfies the condition (2.3). We shall show later that this leads to coupling v and f.

The solution of the system (2.2) with m_j in the form of Eq. (2.4) will be described by

$$\varphi_j(t)=j\Phi+vt+\psi_j(t). \qquad (2.5)$$

Substituting Eqs. (2.4) and (2.5) into Eq. (2.2), we obtain

$$\dot\psi_j+(2+\lambda)\psi_j-\psi_{j-1}-\psi_{j+1}=f-v-\lambda d(j\Phi+vt), \qquad (2.6)$$

where $d(x)$ is

$$d(x)=x-\mathrm{Int}\left(x+\frac{1}{2}\right)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\exp(2\pi in(x+{}^1/_2))}{2\pi in}. \qquad (2.7)$$

The right-hand side of Eq. (2.6) contains a function which is periodic in time and osci''ates at the frequency $\omega=2\pi v$. Consequertly, $\psi_j(t)$ should be sought in the form

$$\psi_j(t)=\sum_{n=-\infty}^{\infty}\psi_j^{(n)}e^{2\pi invt}. \qquad (2.8)$$

The equations for $\psi_j^{(n)}$, obtained from Eqs. (2.6)–(2.8), are of the form

$$(2+\lambda)\psi_j^{(0)}-\psi_{j+1}^{(0)}-\psi_{j-1}^{(0)}=f-v,$$

$$(2+\lambda+2\pi inv)\psi_j^{(n)}-\psi_{j-1}^{(n)}-\psi_{j+1}^{(n)}=\lambda\frac{\exp[2\pi in(j\Phi+{}^1/_2)]}{2\pi in}, \qquad n\neq 0. \qquad (2.9)$$

It follows from Eqs. (1.4) and (2.5) that the solutions of the system (2.9) which are finite in the limit $j\to\pm\infty$ can be found using the Green's functions which decrease at $j\to\pm\infty$:

$$\psi_j^{(0)}=\frac{1}{\lambda}(f-v),$$

$$\psi_j^{(n)}=\frac{\lambda}{2\pi in}\frac{r_n}{1-r_n^2}\sum_{l=-\infty}^{\infty}r_n^{|l|}\exp\{2\pi in[\Phi(j+l)+{}^1/_2]\}, \qquad n\neq 0; \qquad (2.10)$$

$$r_n=1+\frac{\lambda}{2}+\pi inv-\left[(\lambda+2\pi inv)\left(1+\frac{\lambda+2\pi inv}{4}\right)\right]^{1/2}. \qquad (2.11)$$

Summing the series in Eq. (2.10) over and substituting the result in Eq. (2.8) and then into Eq. (2.5), we obtain

$$\varphi_j(t)=\lambda^{-1}(f-v)+g(j\Phi+vt), \qquad (2.12)$$

$$g(x)=x+\lambda\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\exp[2\pi in(x+{}^1/_2)]}{2\pi in(\lambda+4\sin^2\pi n\Phi+2\pi inv)}. \qquad (2.13)$$

In the Appendix A it is shown that the function $g(x)$ of Eq. (2.13) has the following properties:

1) if $v\to 0$, then $g(x)\to g_A(x)$, where $g_A(x)$ is the corresponding function of the ground state [Eq. (1.9)];

2) if $v\neq 0$, then $g(x)$ is a continuous monotonically increasing function of x;

3) $g(x+1)=g(x)+1$.

When we allow for these properties of the function

$g(x)$, we may conclude that the sequence of φ_j in Eq. (2.12) satisfies the requirement

$$\mathrm{Int}(\varphi_j+{}^1/_2)=\mathrm{Int}(j\Phi+vt+{}^1/_2),$$

which follows from Eqs. (2.3) and (2.4) if

$$\lambda^{-1}(f-v)+g(-{}^1/_2)+{}^1/_2=0, \tag{2.14}$$

which relates the force f to the velocity v of the motion of a configuration. Substituting Eq. (2.13) into Eq. (2.14), we find that simple transformations give

$$f=\lambda^2 v\sum_{n=-\infty}^{\infty}\frac{1}{(\lambda+4\sin^2\pi n\Phi)^2+(2\pi n v)^2}. \tag{2.15}$$

If $\Phi = M/L$ is a rational number, then the series of Eq. (2.13) for $g(x)$ can be transformed into a finite sum of L terms. Then, Eq. (2.15) transforms into (see the Appendix A):

$$f=\frac{\lambda^2}{2L}\sum_{k=0}^{L-1}\frac{1}{\lambda+4\sin^2\pi k\Phi}\,\frac{1-t_k^2}{1-2t_k\cos(2\pi k/L)+t_k^2}, \tag{2.16}$$
$$t_k=\exp\left[-\frac{1}{Lv}(\lambda+4\sin^2\pi k\Phi)\right].$$

3. THRESHOLD FORCE

If we go to the limit $v\to\pm 0$ in Eq. (2.16), we find that the second factor in the sum becomes ± 1 and then the sum can be easily calculated, so that for

$$f_t^{(L)}=\lim f(v, M/L)$$

in the limit $v\to +0$ we have

$$f_t^{(L)}=\frac{(1-r)^3}{2r(1+r)}\,\frac{1+r^L}{1-r^L}. \tag{3.1}$$

If $\lambda \ll 1$, then using $r\approx 1-\lambda^{1/2}$, we can transform Eq. (3.1) to

$$f_t^{(L)}\approx\frac{\lambda^{3/2}}{4}\mathrm{cth}\left(\frac{L\lambda^{1/2}}{2}\right)\approx\begin{cases}\dfrac{\lambda}{2L}, & L\ll L_0\\[2mm] \dfrac{\lambda^{3/2}}{4}(1+2\exp(-L\lambda^{1/2})), & L\gg L_0\end{cases}, \tag{3.2}$$

where $L_0 = 2\lambda^{-1/2}$. We can find f_t for irrational values of Φ by going to the limit $L\to\infty$ in Eqs. (3.1) and (3.2), which gives

$$f_t^{(\infty)}\equiv f_t=(1-r)^3/2r(1+r)\approx\lambda^{3/2}/4,\qquad \lambda\ll 1. \tag{3.3}$$

Physically $f_t^{(L)}$ is the weakest force which has to be applied to the chain sites so that the configuration begins to move. Its origin is as follows. Let us discuss the ground state with the configuration $\{\varphi_j^{(0)}\}$. We shall introduce

$$\xi_j=\varphi_j^{(0)}-\mathrm{Int}(\varphi_j^{(0)}+{}^1/_2),$$

where $\varphi_j^{(0)}$ is the deviation, at a site j, from a minimum of the potential $V(\varphi)$ [Eq. (1.3b)]. A configuration $\{\Phi_j^{(0)}\}$ is described by Eqs. (1.6) and (1.9). Since $g_A(x)$ has a discontinuity at $x = 1/2$, we have $\xi_j \leqslant \xi_{\max} < 1/2$, where

$$\xi_{max}=g_A\left(\frac{1}{2}-0\right)=\frac{1}{2}-\frac{1}{2}\,\frac{1-r}{1+r}\,\frac{1+r^L}{1-r^L}.$$

If the applied force obeys $f<f_t$, it follows from the local quadratic nature of the potential of Eq. (1.3b) that the whole change in the configuration reduces to a homogeneous displacement of all φ_j by an amount f/λ. As the force reaches a value at which the condition $\xi_j < 1/2$ is no longer satisfied at any site, then φ_j jumps from this site to a neighboring potential well and this triggers the slide of the other sites.

The threshold force decreases when the order of commensurability L of a structure is reduced and reaches its smaller value given by Eq. (3.3) for incommensurate structures. This value does not vanish for any positive value of λ, in contrast to the case of smooth potentials.[11]

For specific systems the parameter Φ can be given a clear physical meaning. For example, in the case of a superconducting film it is simply the vortex lattice period which is inversely proportional to the square root of the applied field, whereas in the case of a charge density wave the parameter Φ is the number of electrons per site. The dependence of the threshold field on Φ is clearly unrealistic $f_t(\Phi) = f_t$ of Eq. (3.3) applies to all irrational values of Φ, whereas $f_t(M/L) = f_t^{(L)}$ of Eq. (3.1) applies to all rational values of Φ. However, in the case of real systems a configuration moves also when $f<f_t$. In the case of a superconducting film this is due to the thermal creep of magnetic flux[17] and in the case of a charge density wave it is associated with the current of "above-condensate" electrons[18] and/or the thermal drift of solitary defects.[19] Therefore, the observed threshold force is not $f(+0, \Phi)$, but more probably the value of f at some finite threshold velocity v_t, but this aspect is outside the scope of the present paper.

We shall consider f as a function of Φ for a fixed (small) value of v. The results of a numerical calculation using Eq. (2.16) are presented in Figs. 2 and 3. If $v\neq 0$, then f is a smooth function of Φ and the graph of this function shows a finite number of peaks corresponding to certain rational values of Φ.

In the Appendix B we obtain the following expansion for $f(v, \Phi)$:

$$f(v,\Phi)=f_0(v)+\sum_{l=1}^{\infty}f_l(v,l\Phi), \tag{3.4}$$

$$f_0(v)=\frac{\lambda^2}{4\pi}\int_{-\pi}^{\pi}\frac{d\xi}{\lambda+4\sin^2\xi/2}\,\mathrm{cth}\left[\frac{1}{2v}\left(\lambda+4\sin^2\frac{\xi}{2}\right)\right], \tag{3.5}$$

$$f_l(v,x)=\frac{\lambda^2}{2\pi}\int_{-\pi}^{\pi}d\xi\,\frac{\cos l\xi}{\lambda+4\sin^2(\xi/2)}\times\frac{\mathrm{ch}[v^{-1}(x-\mathrm{Int}\,x-{}^1/_2)(\lambda+4\sin^2(\xi/2))]}{\mathrm{sh}[(2v)^{-1}(\lambda+4\sin^2(\xi/2)]}. \tag{3.6}$$

It is shown there that if $v \ll \lambda \ll 1$ and $l \gg 1/\lambda^{1/2}$, we can approximate $f_l(x)$ as follows:

$$f_l(x)\approx\begin{cases}\dfrac{\lambda^{3/2}}{2}\exp(-l\lambda^{1/2}), & |d(x)|<x_0 l\\[2mm] 0, & |d(x)|>x_0 l\end{cases}, \tag{3.7}$$

where $d(x)$ of Eq. (2.7) represents the proximity of x to an integer; we also have

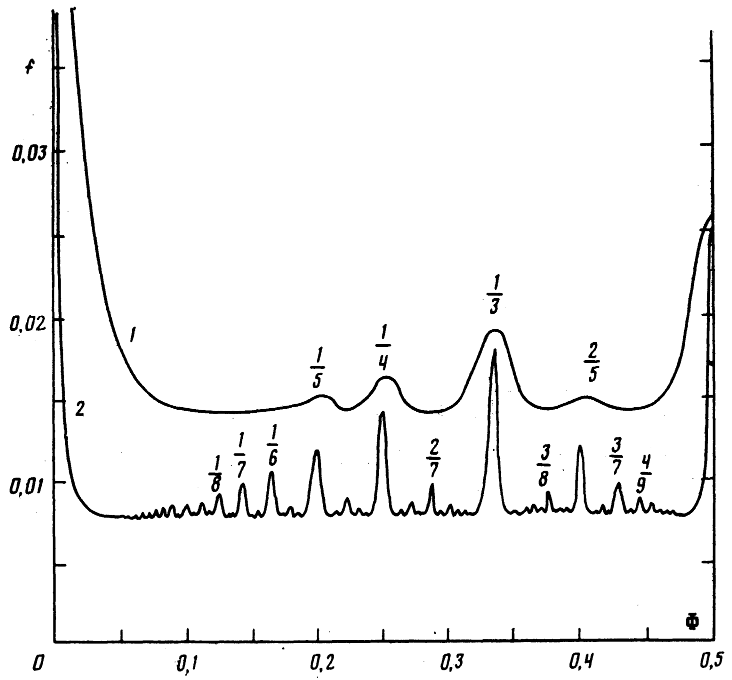

FIG. 2. Dependences $f(\Phi)$ plotted for $\lambda = 0.1$; the values; of v are 10^{-2} for curve 1 and 10^{-3} for curve 2.

$$x_0 = v/2\lambda^{1/4}. \tag{3.8}$$

The approximation represented by Eq. (3.7) is valid if $x_0 l \ll 1/2$, whereas in the opposite case we find that $f_l(x)$ differs little from a constant.

It follows from the above discussion that $f_1(v, l\Phi)$ considered as a function of Φ has maxima at the points $\Phi = m/l$, where m is an integer, provided $l < (2x_0)^{-1}$. The width of the maximum is x_0. However, not all these maxima appear in $f(v, \Phi)$ of Eq. (3.4). Obviously, $f(v, \Phi)$ has maxima for rational values of $\Phi = M/L$, where $L < L_c(v)$ and $L_c(v)$ should be found from the condition that there be no overlap between neighboring maxima which are separated by distances of the order of L_c^{-2}, i.e., the relationship $x_0 \propto L_c^{-2}$ should be obeyed and hence we should have

$$L_c(v) \propto \lambda^{1/8}/v^{1/2}. \tag{3.9}$$

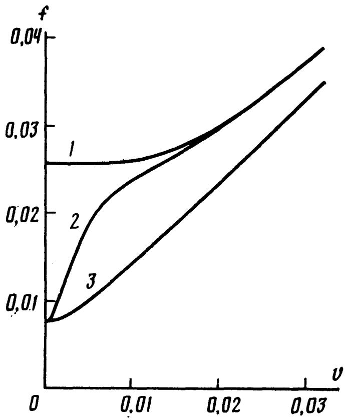

FIG. 3. Dependences $f(v)$ for structures with $\Phi = 1/2$ (curve 1), $\Phi = 27/55$ (curve 2), and $\Phi = 34/55$ (curve 3). The force of the potential λ is 0.1. These dependences illustrate respectively the cases of commensurate, almost commensurate, and incommensurate structures.

4. MOTION OF A SOLITARY DEFECT

We shall consider the motion of a solitary defect in a commensurate structure such that

$$\Phi = M/L = 1/(N_1 + 1/(N_2 + \ldots + 1/N_s)\ldots). \tag{4.1}$$

We shall modify Eq. (2.16) noting that the summation sign includes an expression which is periodic with a period L in the summation index k. Consequently, we can replace k with $k\tilde{M}$, where $\tilde{M}$ is any integer which is not a multiple of L. We can always select M in such a way that $\tilde{M}M = PL \pm 1$, where P is an integer. After this substitution Eq. (2.16) becomes

$$f = \frac{\lambda^2}{2L} \sum_{k=0}^{L-1} \frac{1}{\lambda + 4\sin^2(\pi k/L)} \frac{1 - \tilde{t}_k^2}{1 - 2\tilde{t}_k \cos 2\pi k\tilde{\Phi} + \tilde{t}_k^2}$$
$$\tilde{t}_k = \exp\left[-\frac{1}{Lv}\left(\lambda + 4\sin^2\frac{\pi k}{L}\right)\right], \tag{4.2}$$

where $\tilde{\Phi} = \tilde{M}/L$. In the Appendix it is shown that the expansion of Φ as a chain fraction gives

$$\tilde{\Phi} = \tilde{M}/L = L_s/L_{s+1} = 1/(N_s + 1/(N_{s-1} + \ldots + 1/N_1)\ldots). \tag{4.3}$$

Here, L_{r+1} is the denominator of a continued fraction consisting of the first r terms of the continued fraction of the number Φ.

In the structure with $\Phi = M/L$ we form a lattice of defects separated by equal distances, going over from Φ to Φ_N, where

$$\Phi_N = M_N/L_N = 1/(N_1 + 1/(N_2 + \ldots + 1/(N_s + 1/N)\ldots). \tag{4.4}$$

Then, Φ in Eq. (4.2) transforms to $\Phi_{\tilde{N}}$, where

$$\tilde{\Phi}_N = 1/(N + 1/(N_s + \ldots + 1/N_1)\ldots) = 1/(N + \tilde{\Phi}) = L/L_N. \tag{4.5}$$

We then have one elementary defect per L_N sites and the phase shift at each defect is $1/L$. Therefore, when the average rate of change in the phase at a defect is equal to v, the velocity of defects along a chain is $LL_N v$. If we go to the limit $N\to\infty$ (and also $L_N\to\infty$) and $v\to 0$, where $LL_N v=\beta=$ const, we are then dealing with the case of a solitary defect traveling at a velocity β. The sum with respect to k in Eq. (4.2) then changes to an integral with respect to $x=2\pi k/L_N$:

$$f(\beta)=\frac{\lambda^2}{2}\int_{-\pi}^{\pi}\frac{dx}{2\pi}\frac{1}{\lambda+4\sin^2(x/2)}\frac{1-t^2(x)}{1-2t(x)\cos Lx+t^2(x)},$$
$$t(x)=\exp\left[-\frac{L}{\beta}\left(\lambda+4\sin^2\frac{x}{2}\right)\right]. \qquad (4.6)$$

In discussing the limiting cases we shall expand the integrand in Eq. (4.6) in powers of $t(x)$:

$$f(\beta)=\frac{\lambda^2}{2}\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\frac{dx}{2\pi}\frac{t^{|l|}(x)e^{ilLx}}{\lambda+4\sin^2(x/2)}. \qquad (4.7)$$

Equation (4.7) readily yields f in the limits $\beta\to 0$ and $\beta\to\infty$:

$$f(+0)=f_t\approx\lambda^{3/2}/4,\quad \lambda\ll 1, \qquad (4.8)$$

$$f(+\infty)=f_t^{(L)}\approx\frac{\lambda^{3/2}}{4}\operatorname{cth}\frac{L\lambda^{1/2}}{2}. \qquad (4.9)$$

Having differentiated Eq. (4.7) with respect to β, we obtain

$$\beta^2\frac{\partial f}{\partial\beta}=\lambda^2 L\sum_{l=1}^{\infty}l\exp\left(-\frac{2+\lambda}{\beta}lL\right)I_{lL}\left(\frac{2lL}{\beta}\right), \qquad (4.10)$$

where $I_p(z)$ is a modified Bessel function. Assuming that $\beta\ll 1$, we shall replace the $I_p(z)$ functions with their asymptotic forms, assuming that $p^2+z^2\gg 1$ (Ref. 20):

$$\beta^2\frac{\partial f}{\partial\beta}\approx\frac{\lambda^2 L}{2}\sum_{l=1}^{\infty}\left(\frac{l\beta}{\pi}\right)^{1/2}\left(1+\frac{\beta^2}{4}\right)^{-1/4}\exp[-lLh(\beta)],$$
$$h(\beta)=-\frac{(4+\beta^2)^{1/2}}{\beta}+\frac{2+\lambda}{\beta}+\operatorname{arsh}\frac{\beta}{2}. \qquad (4.11)$$

The function $h(\beta)$ has a minimum at $\beta=\beta_0$, where

$$\beta_0=2(\lambda+\lambda^2/4)^{1/2}. \qquad (4.12)$$

We shall only discuss the case when $\lambda\ll 1$ so that $\beta_0=2\lambda^{1/2}$ and $h(\beta_0)=\lambda^{1/2}$. At large values of L, when $Lh(y_0)=L\lambda^{1/2}\gg 1$, the sum over l of Eq. (4.11) can be limited to the first term with $l=1$. However, if $L\lambda^{1/2}\ll 1$, then in a certain range of velocities β, including $\beta_0=2\lambda^{1/2}$, we have to sum the whole series of Eq. (4.11).

We shall begin with the case of high orders of commensurability characterized by $L\gg 1/\lambda^{1/2}$. We then have $f(\infty)=f_t^{(L)}$, which is governed by the second formula in Eq. (3.2). Limiting the sum in Eq. (4.11) to the term with $l=1$, and integrating with respect to β using Eqs. (4.8) and (4.9), we obtain the following results for different limiting cases:

$$\frac{f(\beta)-f_t}{f_t}=\begin{cases}2\left(\frac{\beta}{\pi\lambda L}\right)^{1/2}\exp\left(-\frac{\lambda L}{\beta}-L\beta\right), & \beta\ll 2\lambda^{1/2};\\ 2\exp(-L\lambda^{1/2})-4\left(\frac{\lambda}{\pi\beta^3 L}\right)^{1/2}\exp\left(-\frac{\lambda L}{\beta}-L\beta\right), & 2\lambda^{1/2}\ll\beta\ll 1;\\ 2\exp(-L\lambda^{1/2})-\frac{4\lambda^{1/2}}{(L+1)!}\left(\frac{\beta}{L}\right)^{-L-1}\times\exp\left(-\frac{2+\lambda}{\beta}L\right), & \beta\gg 1.\end{cases} \qquad (4.13)$$

We shall now consider the cases of low orders of commensurability when $L\ll\lambda^{-1/2}$. We then have

$$f(\infty)=f_t^{(L)}=\lambda/2L\gg\lambda^{3/2}/4=f_t=f(0).$$

In the range $\beta\ll 1$ the function $h(\beta)$ of Eq. (4.11) is

$$h(\beta)=\lambda/\beta+{}^1/_4\beta. \qquad (4.14)$$

If $\beta\ll\lambda L$ or $\beta\gg 4/L$, we can—as in the case of large values of L—limit Eq. (4.11) to just the first term. However, the inequality $Lh(\beta)\ll 1$ is satisfied within the interval $\lambda L\ll\beta\ll 4/L$, so that the sum over l in Eq. (4.11) can be replaced by an integral with respect to dl. When Eq. (4.11) transformed in this way is integrated with respect to β, we find that simple operations yield

$$f(\beta)=\begin{cases}\frac{\lambda^{3/2}}{4}+\frac{\lambda}{2}\left(\frac{\beta}{\pi L}\right)^{1/2}\exp\left(-\frac{\lambda L}{\beta}\right), & \beta\ll\lambda L;\\ \frac{\lambda^{1/2}\beta}{4L}\left(1+\frac{\beta^2}{4\lambda}\right)^{-1/2}, & \lambda L\ll\beta\ll 4/L;\\ \frac{\lambda}{2L}-\frac{\lambda^2}{(\pi L\beta^3)^{1/2}}\exp\left(-\frac{1}{4}L\beta\right), & 4/L\ll\beta\ll 1;\\ \frac{\lambda}{2L}-\frac{\lambda}{(L+1)!}\left(\frac{L}{\beta}\right)^{L+1}\exp\left(-\frac{2L}{\beta}\right), & \beta\gg 1.\end{cases} \qquad (4.15)$$

It is worth noting that the interval $\lambda L\ll\beta\ll 4/L$ can be split into two parts: the range of linear dependence $f(\beta)$:

$$f(\beta)=\lambda^{1/2}\beta/4L,\quad \lambda L\ll\beta\ll 2\lambda^{1/2}, \qquad (4.16)$$

and the range where it reaches the value $\lambda/2L$, beginning from $\beta\sim 2\lambda^{1/2}$.

Generally, the motion of a solitary defect is characterized by the following properties. The mobility threshold of the defect is equal to the mobility threshold of an incommensurate structure. The asymptotic behavior of $f(\beta)$ in the limit $\beta\to 0$ or $f\to f_t+0$ is described by $f\propto\exp(-\lambda L/\beta)$ which is the same expression as that describing the dependence $f(\beta)$ in the case of a single-site system in a potential of the form (1.3b). If the defect is formed in a structure with a high order of commensurability, $L\gg 1/\lambda^{1/2}$, then such a dependence is observed almost to the values of $f=f_t^{(L)}$, corresponding to the mobility threshold of the structure in which the defect is formed [see Eq. (4.13)]. In a structure with a low order of commensurability the velocity of a defect becomes proportional to the applied force as soon as this force begins to exceed significantly the threshold value of the force f_t for the given defect [see Eq. (4.16)]. Naturally, in this case the velocity of the defect again tends to infinity when the force increases to the value $\lambda/2L$, which is the threshold for a defect-free structure.

5. MOTION OF AN INCOMMENSURATE CHAIN

In the case of a commensurate structure with a period L, i.e., when $\Phi = M/L$, the asymptote of $f(v)$ in the limit $v\to 0$ [corresponding to $f\to f_t^{(L)}+0$] is of the form

$$f-f_t^{(L)}\propto\exp(-\lambda/Lv).$$

In this section we shall consider the behavior of incommensurate structures at low velocities v. With this in mind, for a given L we shall approximate the irrational number Φ by the nearest, rational number $\Phi_L = M_L/L$ bearing in mind that we should later go to the limit $L\to\infty$, $\Phi_L\to\Phi$. Within the limits of this approximation, we can use Eq. (4.2) in which we have to substitute

$$\tilde{\Phi}=\tilde{\Phi}_L=\tilde{M}_L/L,$$

where $\tilde{M}_L$ is governed by the condition

$$\tilde{M}_L M_L=PL\pm 1.$$

Expanding Eq. (4.2) as a series in powers of $\tilde{t}_k$ and differentiating with respect to v, we find that

$$v^2\frac{\partial f}{\partial v}=\frac{\lambda^2}{2L^2}\sum_{l=-\infty}^{\infty}|l|\exp\left(-\frac{\lambda|l|}{Lv}\right) \tag{5.1}$$

$$\times\sum_{k=0}^{L-1}\exp\left(-\frac{4|l|}{Lv}\sin^2\frac{\pi k}{L}+2\pi ikl\tilde{\Phi}_L\right).$$

In Eq. (5.1) we shall go over from summation over k to integration with respect to $x = 2\pi k/L$ using the relationship

$$\sum_p\delta\left(x-\frac{2\pi p}{L}\right)=\frac{L}{2\pi}\sum_m\exp(-iLmx).$$

Then, we find from Eq. (5.1) that

$$v^2\frac{\partial f}{\partial v}=\frac{\lambda^2}{L}\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}l\exp\left(-\frac{2+\lambda}{Lv}l\right)I_{l\tilde{M}_L-mL}\left(\frac{2l}{Lv}\right). \tag{5.2}$$

We shall show below that in the limit $L\to\infty$ the greatest contribution to the sum over l in Eq. (5.2) comes from terms with a specific value of $l/L = x_1$. Therefore, in the case of sufficiently low values of v we can use the familiar asymptotes of $I_p(z)$ when $p^2+z^2\gg 1$ (Ref. 20):

$$v^2\frac{\partial f}{\partial v}\approx\frac{\lambda^2}{2}\left(\frac{v}{\pi}\right)^{1/2}\sum_{l=1}^{\infty}x_l^{1/2}\left[1+\left(\frac{vy_l}{2x_l^2}\right)^2\right]^{-1/4}$$

$$\times\exp\left\{-\frac{2+\lambda}{v}x_l+\frac{2x_l}{v}\left[1+\left(\frac{vy_l}{2x_l^2}\right)^2\right]^{1/2}-\frac{y_l}{v}\operatorname{arsh}\frac{vy_l}{2x_l^2}\right\}. \tag{5.3}$$

In Eq. (5.2) m of the sum over m only the term with $m=\mathrm{Int}(l\tilde{\Phi}_L+1/2)$ is retained, so that $l\tilde{\Phi}_L - m\to d(l\tilde{\Phi}_L)$ and the notation $y_l = |ld(l\tilde{\Phi}_L)|$ is introduced. In the sum over in Eq. (5.3) we shall separate the terms which correspond to the maximum value of the argument of the exponential function. We note that at $x_l = x_l^{(0)}$, where

$$x_l^{(0)}=(3/4)^{1/4}y_l^{1/2}v^{1/2}/\lambda^{1/4}, \tag{5.4}$$

the expression in the argument of the exponential function has a sharp minimum in terms of the variable $x_1 = l/L$. Its relative width is $(v/y_l\lambda^{3/2})^{1/4}$ and this width is small if $v\ll\lambda^{3/2}$. The value of the argument of the exponential function in Eq. (5.3) is $-(4/3)^{3/4}y_l^{1/2}\lambda^{3/4}/v^{1/2}$ at $x_1 = x_l^{(0)}$. If y_0 is a minimum, then under given conditions characterized by $L\to\infty$, $l\to\infty$, and $l/L = x_0$, where x_0 is found from Eq. (5.4) assuming that $y = y_0$, and $y_1 = |ld(l\tilde{\Phi}_L)|$ the asymptotic form of $f(v)$ in the limit $v\to 0$ is

$$v^2\frac{\partial f}{\partial v}\propto\lambda^{15/8}(y_0v)^{1/4}\exp\left[-\left(\frac{4}{3}\right)^{3/4}y_0^{1/2}\frac{\lambda^{3/4}}{v^{1/2}}\right] \tag{5.5}$$

or

$$(f-f_t)\propto\lambda^{9/8}(y_0v)^{1/4}\exp\left[-\left(\frac{4}{3}\right)^{3/4}y_0^{1/2}\frac{\lambda^{3/4}}{v^{1/2}}\right], \tag{5.6}$$

where y_0 is governed by the behavior of the sequence

$$y_{l,L}=|ld(l\tilde{\Phi}_L)|, \tag{5.7}$$

for $L\to\infty$, $l\to\infty$, $l/L = x_0$. Since $x_0\to 0$ in the limit $v\to 0$ [see Eq. (5.4)], then in calculating y_0 we should go to the limit $x_0\to 0$ in dealing with the above sequence. A numerical coefficient has been dropped from Eqs. (5.5) and (5.6).

We shall select a rational approximation for Φ by a suitable fraction of the order of s, consisting of the first s denominators of an expansion of irrational Φ as an infinite continued fraction:

$$\Phi=1/(N_1+1/(N_2+\ldots). \tag{5.8}$$

From the point of view of the $y_{l,L}$ = min criterion, the values of l should be selected from the set $L_{k+1}^{(s)}$ of denominators of suitable fractions of order k for the number $\Phi_{(s)} = L_s/L_{s+1}$ (a suitable fraction of order k for the number Φ is the continued fraction consisting of the first k elements N_i of a continued fraction of the number Φ). In the Appendix C it is shown that for $s\to\infty$ and $k\to\infty$, but with $s-k$ finite, we have

$$|L_{k+1}^{(s)}d(L_{k+1}^{(s)}\tilde{\Phi}^{(s)})|\to(1/\tilde{\Phi}^{(s-k)}+\Phi_{s-k})^{-1}, \tag{5.9}$$

$$L_{k+1}^{(s)}/L_{s+1}\to\Phi_0\Phi_1\ldots\Phi_{s-k-1}, \tag{5.10}$$

where

$$\Phi_n=1/(N_{n+1}+1/(N_{n+2}+\ldots)\ldots, \tag{5.11}$$

$$\tilde{\Phi}^{(n)}=1/(N_n+1/(N_{n-1}+\ldots+1/N_1)\ldots). \tag{5.12}$$

An asymptote of the type given by Eq. (5.6) clearly applies only if for a given irrational number Φ the sequence of Eq. (5.9) has a lower limit. It is known that this property is exhibited by what are called quadratic irrationalities,[13] i.e., irrational numbers which are roots of quadratic equations with coefficients which are integers. Expansion of such Φ numbers as a continued fraction yields a periodic (beginning from a certain value r) sequence of denominators N_r. In this case we obviously have

$$y_0=\min_r(1/\tilde{\Phi}_r+\Phi_r)^{-1}, \tag{5.13}$$

where $\tilde{\Phi}_r$ and Φ_r are periodic chain fractions for which sequences of denominators have, respectively, the forms

$N_{r+1}, N_{r+2} \ldots N_{r+R}$ and $N_r, N_{r-1} \ldots N_{r-R}$, where R is the period of a sequence of denominators of the number Φ. We can show that the highest value of y_0 is obtained for the "golden mean" $\Phi = (5^{1/2}-1)/2$, corresponding to $y_0 = 1/5^{1/2}$.

The dependence (5.6) can also be represented in the form

$$f - f_t \approx \frac{\lambda^2}{2}\left[\frac{3}{\pi L_c(v)}\right]^{1/2} \exp\left[-\frac{\lambda}{L_c(v)v}\right], \tag{5.14}$$

where

$$L_c(v) = \left(\frac{3}{4}\right)^{1/4} y_0^{-1/2} \frac{\lambda^{1/4}}{v^{1/4}}. \tag{5.15}$$

The same $L_c(v)$ (apart from a factor of the order of unity) sets the limit of resolution of peaks in a graph of $f(\Phi)$ for a given value of v [see Eq. (3.9)]. The dependence of Eqs. (5.14) and (5.15) can generally be predicted on the basis of the following considerations. We shall assume a specific value of v. We shall approximate an irrational number Φ with a rational one $\Phi_L = M_L/L$. If L is not too large, we can assume that $f - f_L$ consists of two contributions, which are

$$(f_t^{(L)} - f_t) \propto \exp(-L\lambda^{1/2})$$

and

$$(f - f_t^{(L)}) \propto \exp(-\lambda/Lv).$$

The terms become comparable for

$$L \sim \lambda^{1/4}/v^{1/2} \sim L_c(v).$$

We can assume that $L_c(v)$ is the limit beyond which an increase in the order of rational approximation L does not result in a significant improvement in the accuracy of the calculation of $f - f_t$.

The dependence $f(v)$ reaches the asymptote described by Eq. (5.6) when

$$L_c(v) \gg L_0 \sim 1/\lambda^{1/2}.$$

The range of values of v where $f(v)$ is described by Eq. (5.6) exists also for the rational value $\Phi = M/L$ provided only that $L \gg L_0$. The limits of this range are set by the inequalities $L \gg L_c(v) \gg L_0$ or by

$$\lambda^{1/2}/L \ll v \ll \lambda^{3/2}. \tag{5.16}$$

The range of velocities where this structure behaves as commensurate is $v \ll \lambda^{1/2}/L$ and it becomes narrower or the order of commensurability of L increases.

6. CONCLUSIONS

We have obtained different dependences of the velocity v on the force f applied to various types of structures. We have distinguished the commensurate, incommensurate, and almost commensurate structures. Among the commensurate structures we have included those that correspond to the rational values of $\Phi = M/L$ with denominators which are not very large:.

$$L \ll L_0 \sim \lambda^{-1/2},$$

because if $L \gg L_0$ the dependence $v(f)$ has practically the same form as that in the case of irrational values of Φ.

In the case of commensurate structures the threshold force $f_t^{(L)}$ depends strongly on the order of commensurability L. If $v \ll \lambda/2L$ or if

$$f - f_t^{(L)} \ll f_t^{(L)},$$

then in Eq. (2.16) relating f and v we need retain only the term with $k = 0$, which gives rise to a dependence $v(f)$:

$$v \approx 2f_t^{(L)}\left[\ln\left(\frac{2f_t^{(L)}}{f - f_t^{(L)}}\right)\right]^{-1}. \tag{6.1}$$

If $f \gg f_t^{(L)}$, the dependence reaches the linear asymptote $v = f$.

In the case of incommensurate structures the threshold force is f_t. The relationship between the velocity and the force valid in the $f - f_t \ll f_t$ case is given by Eq. (5.6), which can be rewritten in the form

$$v \approx A f_t \ln^{-2}\left[\frac{f_t}{f - f_t} \Big/ \ln^{1/2}\left(\frac{f_t}{f - f_t}\right)\right], \tag{6.2}$$

where A is a number of the of order unity. The dependence $v(f)$ is of the form given by Eq. (6.2) when $v \ll \lambda^{3/2}$, i.e., when $f - f_t \ll f_t$, and its asymptotic form when $f \gg f_t$ is $v = f$.

The almost commensurate structures represent a special case when

$$\Phi = M/L + \delta, \quad L \ll \lambda^{-1/2}.$$

Near the threshold when $f < f_t^{(L)}$, the structure moves as a result of motion of its internal defects relative to an immobile structure with $\Phi = M/L$. The dependence $f(v)$ is then described by Eqs. (4.15) and (4.16) where we have to substitute $\beta = v/\delta$. With the exception of a narrow range of values of f near f_t, the dependence in question is linear:

$$v = 4\delta L \lambda^{-1/2} v. \tag{6.3}$$

The linear regime corresponds to the motion along a chain of practically completely depinned defects. The limits of the interval $|\delta| < \Delta_L$ inside which structures behave as almost commensurate, relative to a given commensurate structure characterized by $\Phi = M/L$, are determined by the requirement that for $f \approx f_t^{(L)}$ the value of v determined from Eq. (6.3) should be less than $\lambda/2L$ and hence the width of the interval is $\Delta_L \sim \lambda^{1/2}/2L$.

In this model a dependence of the type

$$(f - f_t) \propto \exp(-A/v)$$

typical of commensurate structures at low values of v is a direct consequence of the fact that the potential of Eq. (1.3b) is not smooth. In fact, we can consider a single-site system for which the equation of motion is

$$\dot{\varphi} = -\frac{\partial V}{\partial \varphi} + f, \quad V(\varphi+1) = V(\varphi). \tag{6.4}$$

The threshold value of f is then

$$f_t = \max V'(\varphi).$$

For $f > f_t$, the average velocity v is found from

$$\frac{1}{v} = \int_0^1 \frac{d\varphi}{f - V'(\varphi)}. \tag{6.5}$$

Let us assume that $V'(\varphi)$ attains a maximum at the point φ_0.

In the case of a smooth potential we have

$$V'(\varphi)=f_t-A(\varphi-\varphi_0)^2$$

in the vicinity of φ_0, so that in the limit $f\to f_t+0$ the asymptotic form $v(f)$ has the form $v\propto(f-f_t)^{1/2}$. However, if φ_0 is a point where there is a kink in the potential, as in the case described by Eq. (1.3b), then the integral in Eq. (6.5) diverges logarithmically, so that

$$v\propto 1/|\ln(f-f_t)|.$$

It therefore follows that in systems with a smooth potential $V(\varphi)$ the behavior near the threshold differs considerably from the Aubry model in that it should be characterized by different critical exponents. However, in any case a pinned structure can be represented as a hierarchy of interacting defects of the soliton type. Consequently, the classification of possible structures as physically commensurate and incommensurate with a special type of above-threshold behavior for each class, should be of universal validity. Another universal feature should be the presence of almost commensurate structures which are realized in the form of chains of distant defects above a physically commensurate structure. In the case of these structures the dependence of the velocity v on the force f should have a linear region corresponding to an almost depinned motion of defects. Such problems as the establishment of critical dependences for systems with a smooth potential and identification of specific limits of classification of structures in such systems into physically commensurate and incommensurate will require a further study.

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APPENDIX A

The property 3 of the function $g(x)$ follows directly from the expansion described by Eq. (2.13):

$$g(x)=x+h(x),$$

where $h(x)$ is a periodic function.

We can demonstrate the properties 1 and 2 by expanding each term of the series (2.13) as a Fourier series in terms of $2\pi n\Phi$:

$$g(x)\equiv g(x,\Phi)=\sum_{l=-\infty}^{\infty} g_{|l|}(x+l\Phi), \tag{A1}$$

$$g_l(x)=\lambda\sum_{n\neq 0}\frac{r_n^{\,l}}{2\pi i n(r_n^{-1}-r_n)}\exp\left(2\pi i n\left(x+\frac{1}{2}\right)\right)+\frac{1-r}{1+r}r^l x. \tag{A2}$$

Here, r_n is described by Eq. (2.11). If $v=0$ and $r_n=r$, the sum over n in Eq. (A2) is readily calculated subject to (2.7), because

$$\sum_{n\neq 0}\frac{\exp[2\pi i n(x+{}^1/_2)]}{2\pi i n}=\operatorname{Int}\left(x+\frac{1}{2}\right)-x. \tag{A3}$$

Substituting Eq. (A3) into Eq. (A1), we can show that if $v=0$ then the function $g(x)$ is identical with $g_A(x)$ of Eq. (1.9).

We can prove the property 2 by differentiating the expression for $g(x)$ given by Eq. (2.13). We shall represent the summation variable n in the form $n=mL+k$, where $m=0;\ \pm1;\ \pm2\ldots$ and assume that $k=0;\,1;\,\ldots L-1$. We then have

$$g'(x)=\frac{\lambda}{Lv}\sum_{k=0}^{L-1}\exp(2\pi i k(x+{}^1/_2))\sum_{m=-\infty}^{\infty}\exp(2\pi i mL(x+{}^1/_2))\times\left[2\pi i m+2\pi i\frac{k}{L}+\frac{1}{Lv}(\lambda+4\sin^2\pi k\Phi)\right]^{-1}. \tag{A4}$$

Summation over m is carried out using

$$\sum_{m=-\infty}^{\infty}\frac{e^{2\pi i m x}}{2\pi i m+a}=\frac{\exp[-a(x-\operatorname{Int}x)]}{1-e^{-a}}. \tag{A5}$$

We then obtain a finite sum of the type

$$g'(x)=\frac{\lambda}{Lv}\sum_{k=0}^{L-1}\frac{1-t_k\cos(2\pi k/L)}{1-2t_k\cos(2\pi k/L)+t_k^2}\times\exp\left\{-\frac{1}{Lv}(\lambda+4\sin^2\pi k\Phi)[Lx-\operatorname{Int}(Lx)]\right\}, \tag{A6}$$

where t_k are defined as in Eq. (2.16). It follows directly from Eq. (A6) that $g'(x)$ is positive for rational values of $\Phi=M/L$. If $v\neq 0$, the series of Eq. (2.13) is absolutely convergent and hence $g(x)$ is continuous in x and Φ. This proves the continuity and monotonicity of $g(x)$ for any parameter Φ.

If $\Phi=M/L$, Eq. (2.16) can be derived from Eq. (2.15) by replacing the summation variable with $n=mL+k$, where $m=0;\ \pm1,\ \pm2\ldots$, and $k=0;\,1;\,\ldots L-1$; then, summation over m is readily carried out allowing for

$$\sum_{m=-\infty}^{\infty}\frac{1}{(a+2\pi i m)^2+b^2}=\frac{\sin b}{b}\,\frac{e^{-a}}{1-e^{-a}\cos b+e^{-2a}}. \tag{A7}$$

APPENDIX B

Substituting the expansion of Eq. (A1) into Eq. (2.14), we have

$$f(v,\Phi)=v-\frac{\lambda}{2}-\lambda\sum_l g_{|l|}\left(l\Phi-\frac{1}{2}\right), \tag{B1}$$

where the functions $g_l(x)$ are defined by Eq. (A2) and differentiation of this equation gives

$$g_l'\left(x-\frac{1}{2}\right)=\lambda\sum_{n=-\infty}^{\infty}\frac{r_n^{\,l}}{r_n^{-1}-r_n}e^{2\pi i n x}. \tag{B2}$$

In the calculation of the sum over n we shall use the expression

$$\sum_{n=-\infty}^{\infty}f(n)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}dy\,f(y)\,e^{2\pi i m y}, \tag{B3}$$

the application of which gives

$$g_l'\left(x-\frac{1}{2}\right)=\sum_m h_l(x-m), \tag{B4}$$

where the functions $h(x)$ modified by substitution of the integration variable

$$2\pi ivy=2z-2-\lambda$$

become

$$h_l(x)=\frac{\lambda}{2\pi iv}\exp\left(-\frac{2+\lambda}{v}x\right)\int_{-i\infty+1+\lambda/2}^{+i\infty+1+\lambda/2}dz \times\frac{(z-(z^2-1)^{1/2})^l}{(z^2-1)^{1/2}}\exp\left(\frac{2xz}{v}\right). \tag{B5}$$

The only singularity of the integrand in Eq. (B5) is a cut from $z=-1$ to $z=+1$. Closing of the contour in the left- or right-hand half-plane gives, depending on the sign of x,

$$h_l(x)=\theta(x)\frac{\lambda}{2\pi v}\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/2}}[(z+i(1-z^2)^{1/2})^l+(z-i(1-z^2)^{1/2})^l]\exp\left(\frac{2xz}{v}\right), \tag{B6}$$

where $\theta(x)$ is the Heaviside step function. Substituting Eq. (B6) into Eq. (B4) and allowing for the restriction imposed by the θ functions on the summation over m, we find that after replacement of the integration variable with $z=\cos\xi$, we have

$$g_l'\left(x-\frac{1}{2}\right)=\frac{\lambda}{v}\int_{-\pi}^{\pi}\frac{d\xi}{2\pi}\cos l\xi \times\frac{\exp[-v^{-1}(x-\mathrm{Int}\,x-{}^1/_2)(\lambda+4\sin^2(\xi/2))]}{\mathrm{sh}[(2v)^{-1}(\lambda+4\sin^2(\xi/2))]}. \tag{B7}$$

Integrating Eq. (B7) with respect to x and substituting the result in Eq. (A1) we obtain

$$g\left(x-\frac{1}{2}\right)=C-\sum_l \tilde{g}_{|l|}(x+l\Phi), \tag{B8}$$

$$\tilde{g}_l(x)=-\frac{1-r}{1+r}r^l\,\mathrm{Int}\left(x+\frac{1}{2}\right)+\lambda\int_{-\pi}^{\pi}\frac{d\xi}{2\pi}\frac{\cos l\xi}{\lambda+4\sin^2(\xi/2)} \times\frac{\exp[-v^{-1}(x-\mathrm{Int}\,x-{}^1/_2)(\lambda+4\sin^2(\xi/2))]}{\mathrm{sh}[(2v)^{-1}(\lambda+4\sin^2(\xi/2))]}, \tag{B9}$$

where C is the constant of integration. Substituting Eqs. (B8) and (B9) in Eq. (2.14) and finding C from the identity of the dependence $f(v,\Phi)$ in the special case when $\Phi=0$ with the dependence deduced from Eq. (2.16), we obtain Eqs. (3.4)–(3.6).

If $v\ll\lambda\ll1$, then the sinh function in the denominator of the integrand in Eq. (3.6) is replaced with an exponential function. Representing cosh in the numerator by a sum of two components and differentiating f_l with respect to x, we obtain

$$\frac{\partial f_l}{\partial x}=S_l(\mathrm{Int}\,x-x+1)-S_l(x-\mathrm{Int}\,x), \tag{B10}$$

$$S_l(y)=\frac{\lambda^2}{v}I_l\left(\frac{2y}{v}\right)\exp\left(-\frac{2+\lambda}{v}\right). \tag{B11}$$

Assuming that $y\gg v$, we shall use the familiar asymptotic form of the modified Bessel functions when $l^2+z^2\gg1$ (Ref. 20), so that

$$S_l(y)=\frac{\lambda^2}{v(2\pi)^{1/2}}[l^2+(2y/v)^2)^{-1/4} \times\exp\left\{[l^2+(2y/v)^2]^{1/2}-\frac{2+\lambda}{v}-l\,\mathrm{arsh}\frac{lv}{2y}\right\}. \tag{B12}$$

The function which occurs in Eq. (B12) in the argument of the exponential function has a maximum of width Δy at $y=y_b$ where allowing for $\lambda\ll1$, we find that

$$y_l=\frac{lv}{2\lambda^{1/2}}=lx_0,\qquad \frac{\Delta y}{y_l}=\frac{2}{l^{1/2}\lambda^{1/4}}. \tag{B13}$$

If $l\gg\lambda^{-1/2}$, the inequality $\Delta y\ll y_l$ is satisfied, so that $S_l(y)$ can be replaced with $A_l\delta(y-y_l)$, where the constant A_l is found by integration of Eq. (B11):

$$A_l=\int_0^\infty dy S_l(y)=\lambda\frac{1-r}{1+r}r^l\approx\frac{\lambda^{3/2}}{2}\exp(-l\lambda^{1/2}). \tag{B14}$$

Substituting S_l in the form of the δ functions into Eq. (B10) and integrating with respect to x allowing for $f_l(0)\approx A_1$, we obtain Eqs. (3.7) and (3.8).

APPENDIX C

The expansion of the number Φ $(0<\Phi<1)$ as a continued fraction can be written in the form of the following algorithm:

$$\Phi_k=1/(N_{k+1}+\Phi_{k+1}),\quad 0\leqslant\Phi_k<1;\quad \Phi_0\equiv\Phi. \tag{C1}$$

Applying Eq. (C1) k times, we obtain

$$\Phi=(M_{k+1}+M_k\Phi_k)/(L_{k+1}+L_k\Phi_k), \tag{C2}$$

whereas M_k and L_k in Eq. (C1) are described by the recurrence relationships

$$M_{k+1}=N_kM_k+M_{k-1},\ M_0=1,\ M_1=0;$$
$$L_{k+1}=N_kL_k+L_{k-1},\ L_0=0,\ L_1=1. \tag{C3}$$

If $\Phi^{(s)}=M/L$ can be represented by an s-term chain fraction, then $\Phi_s^{(s)}=0$, $M=M_{s+1}$, and $L=L_{s+1}$.

It readily follows from Eq. (C.3) that

$$M_{k+1}L_k-M_kL_{k+1}=-(M_kL_{k-1}-M_{k-1}L_k) \dots=(-1)^k(M_1L_0-M_0L_1)=(-1)^{k+1}. \tag{C4}$$

Hence, it follows that if $\Phi=\Phi^{(s)}$, the smallest integer such that

$$\tilde{M}\Phi^{(s)}=P\pm1/L_{s+1},$$

where P is an integer, is $\tilde{M}=L_s$. We then have

$$\tilde{\Phi}^{(s)}=\tilde{M}/L_{s+1}=L_s/L_{s+1}=L_s/(N_sL_s+L_{s-1}) =1/(N_s+L_{s-1}/L_s)=\dots=1/(N_s+1/(N_{s-1}+\dots+1/N_1)\dots). \tag{C5}$$

We shall consider the difference $L_{k+1}\Phi-M_{k+1}$ ...

ing Eqs. (C2) and (C4), we obtain

$$L_{k+1}\Phi-M_{k+1}=(-1)^k\Phi_k/(L_{k+1}+L_k\Phi_k), \quad (C6)$$

$$|L_{k+1}d(L_{k+1}\Phi)|=(1/\Phi_k+L_k/L_{k+1})^{-1}=(1/\Phi_k+\tilde{\Phi}^{(k)})^{-1}, \quad (C7)$$

where $\Phi^{(k)}$ is the approximation of Φ with a k-term chain fraction.

Let us assume that $L^{(s-k)}_{k+1}$ is the denominator of a k-term ($k<s$) approximation for the number $\tilde{\Phi}^{(s)}$, which in turn is an s-term approximation for an irrational number Φ. It then follows from Eq. (C7) that

$$|L^{(s-k)}_{k+1}d(L^{(s-k)}_{k+1}\tilde{\Phi}^{(s)})|=(1/\tilde{\Phi}^{(s-k)}+\Phi^{(k)}_{s-k})^{-1}. \quad (C.8)$$

If $s\to\infty$, $k\to\infty$, and fixed $s-k$, we find that Eq. (C8) becomes Eq. (5.9).

The left-hand side of Eq. (5.10) can be written in the form

$$L^{(s-k)}_{k+1}/L_{s+1}=L^{(s-k)}_{k+1}/L^{(s-k-1)}_{k+2}\dots L^{(s-j)}_{j+1}/L^{(s-j-1)}_{j+2}\dots L^{(1)}_{s}/L^{(0)}_{s+1}. \quad (C9)$$

The factors $L^{(s-j)}_{j+1}/L^{(s-j-1)}_{j+2}$ represent ratios of the denominators of two consecutive approximations of the number $\Phi^{(s)}$ with j-term and $(j+1)$-term chain fractions, and it follows from Eq. (C5) that

$$L^{(s-j)}_{j+1}/L^{(s-j-1)}_{j+2}=1/(N_{s-j}+1/(N_{s-j+1}+\dots+1/N_s)\dots), \quad (C10)$$

so that

$$L^{(s-k)}_{k+1}/L_{s+1}=\Phi^{(k+1)}_{s-k-1}\Phi^{(k+2)}_{s-k-2}\dots\Phi^{(s)}_0. \quad (C11)$$

If in Eq. (C11) we go to the limits $s\to\infty$ and $k\to\infty$, then for a fixed value of $s-k$ we obtain Eq. (5.10).

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