Quantization of nonabelian anomalous theories

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A regularization method, called dynamic quantization, is proposed for nonabelian gauge theories. It is shown that within the framework of dynamic quantization the nonabelian anomaly is in fact absent. In consequence the Weyl nonabelian theory, i.e., the theory of interacting Yang-Mills and Weyl fields, turns out to be self-consistent.

1. INTRODUCTION

In recent papers of the author[1,2] a regularization method for nonabelian gauge theories was proposed and named the dynamic quantization method. Here we expound the dynamic quantization method in an extended form and show that within the framework of this method nonabelian Weyl theories may be correctly quantized. These theories turn out to be relativistically and gauge invariant, hence unitary.

We emphasize that our quantization of nonabelian Weyl theories is different in principle from the quantization proposed by Fadeev and Shatashvili.[3]

We clarify briefly what we understand by dynamic and what by Feynman, or static, quantization methods. We interpret dynamic quantization as the solution of Heisenberg equations or time-dependent Schrödinger equations. Regularization of perturbation theory (PT) is carried out on the energies of intermediate states (see Ref. 4). In particular, regularization of fermionic degrees of freedom is accomplished by partially filling the Dirac sea. In this manner in the regularized theory the right and left Weyl fields are separated, if the gauge field is viewed as external. In the corresponding PT use is made of the retarded Green function $G^{\rm ret}(x)$, which vanishes for $x^0<0$. Dynamic quantization is adequate in those situations in which the physical vacuum is qualitatively restructured relative to the naive vacuum when the interaction is turned off (for example in chromodynamics), or when a stable vacuum is altogether absent. This idea is due to V. N. Gribov.[5] In contrast, in Feynman quantization it is assumed that the physical vacuum differs little from the naive one and elementary excitations carry quantum numbers of bare fields. Under that assumption one obtains a PT which makes use of causal Green functions. The location of the poles of the causal Green function permits rotation of the contour of integration in the complex plane of the variable k^0 in such a way that the calculation of S-matrix elements can be performed in Euclidean space.

We note that in Feynman quantization the nonabelian anomaly is determined unambiguously.[6]

2. THE REGULARIZED FERMION TRANSITION AMPLITUDE AND THE ANOMALY

We shall describe a special technique used in evaluating the anomaly which is equivalent to dynamic quantization.

In four-dimensional Minkowski space we consider a theory with the action $S=S_A+S_\varphi$, where

$$S_A=\frac{1}{4e^2}\int d^4x\,\mathrm{tr}\,F_{\mu\nu}{}^2,\qquad F_{\mu\nu}=[\nabla_\mu,\nabla_\nu],$$

$$S_\varphi=\int d^4x\,\varphi^+(i\nabla_0+i\sigma^i\nabla_i)\varphi.$$

In what follows we have $\nabla_\mu=\partial/\partial x_\mu+A_\mu$, $A_\mu=A_\mu{}^a t^a$, $\mathrm{tr}\,t^a t^b=-\delta^{ab}$, σ^i are Pauli matrices, Greek indices $\mu,\nu,\ldots=0,1,2,3$, while Latin indices $i,j,\ldots=1,2,3$. Let us denote by $Z_+\{A_\mu\}$ the fermion transition amplitude in a specified gauge field. Then the full transition amplitude in a finite time interval (t_0,t_1) can be represented symbolically in the form

$$K(t_1,t_0)=\int DA_\mu Z_+\{A_\mu\}\exp iS_A. \qquad (2.1)$$

The operation of integration over the gauge field is described below, and the quantity $Z_+\{A_\mu\}$ is determined according to Refs. 1 and 2 as follows: let $\{\varphi_N(x)\}$ be a complete orthonormal set of solutions of the right Weyl equation:

$$(i\nabla_0+i\sigma^i\nabla_i)\varphi_N(x)=0, \qquad (2.2)$$

$$\sum_N\varphi_N(t,\mathbf{x})\varphi_N{}^+(t,\mathbf{y})=\delta^{(3)}(\mathbf{x}-\mathbf{y}). \qquad (2.3)$$

Everywhere we write $(t,x)=(x)$. As a consequence of the completeness condition (2.3), arbitrary fields $\varphi(x)$ and $\varphi^+(x)$ can be expanded in the sets of functions $\{\varphi_N(x)\}$ and $\{\varphi_N{}^+(x)\}$ with time-dependent coefficients $\{a_N(t)\}$ and $\{\bar a_N(t)\}$ respectively. The set of these coefficients is viewed as a complete set of Grassmann degress of freedom of the system. We regularize the fermion amplitude in terms of these variables. To this end we discard the ultraviolet (in energy) tails in the expansions of the fields $\varphi(x)$ and $\varphi^+(x)$ in terms of $\varphi_N(x)$ and $\varphi_N{}^+(x)$. Further, a prime over the sign for summation or multiplication with respect to the index N means that the index does not run over values in the ultraviolet tail (regardless of the sign of the energy). We define regularized Fermi fields as follows:

$$\varphi(x)=\sum_N{}' a_N(t)\varphi_N(x),\qquad \varphi^+(x)=\sum_N{}'\bar a_N(t)\varphi_N{}^+(x). \qquad (2.4)$$

Let us break up the time interval into small segments of length $\varepsilon\to0$: $t_{r+1}=t_r+\varepsilon$. To satisfy unitarity the coordinate variables $\{a_N(t_r)\}$ are determined at the instant of time t_r, while the momentum variables $\{\bar a_N(t_r+\varepsilon/2)\}$ are determined at the instant of time $t_r+\varepsilon/2$. The regularized fermionic measure and amplitude are given by the formulas

$$(D\varphi^+D\varphi)=\mathrm{const}\cdot\prod_r\prod_N{}'\delta\bar a_N\left(t_r+\frac{\varepsilon}{2}\right)\delta a_N(t_r),$$

$$Z_+\{A_\mu\}=\int(D\varphi^+D\varphi)\exp(iS_\varphi). \qquad (2.5)$$

The Fermi-fields φ and φ^+ are considered regularized according to (2.4). It is assumed that at the end of all calculations one passes to the limit of removing the cut-off. Such a limiting transition makes sense in asymptotically free theories. In an abelian theory this regularization (with the fermion sea partial filled, then gradually filled up) is hardly correct in view of the strong interaction in the ultraviolet region. (This remark is due to V. N. Gribov.)

The regularized measure $(D\chi + D\chi)$ and transition amplitude $Z_-\{A_\mu\}$ for the left Weyl field are defined in exactly the same way. In that case the field $\chi(x)$ is expanded in terms of a complete orthonormal set of solutions of the left Weyl equation $(i\nabla_0 - i\sigma^i\nabla_i)\chi p = 0$ with coefficients $\{b_p(t)\}$. Let $\psi(x)$ denote the Dirac field. The regularized Dirac measure is defined in a natural way:

$$(D\bar{\psi}D\psi) = (D\varphi^+D\varphi)(D\chi^+D\chi). \tag{2.6}$$

Since

$$S_\psi = \int d^4x\, \bar{\psi} i\gamma^\mu \nabla_\mu \psi = S_\varphi + S_\chi,$$

it follows from (2.6) that the regularized Dirac transition amplitude $Z\{A_\mu\}$ factors:

$$Z\{A_\mu\} = Z_+\{A_\mu\} Z_-\{A_\mu\}.$$

It is assumed that regularization preserves charge parity in the Dirac theory. To that end it is sufficient for the charge conjugation operation to establish a one-to-one correspondence between the solutions $\varphi_N(x)$ and $\chi_P{}^+(x)$ in the expansions of the fields (2.4) and the analogous expansions for the fields $\chi(x)$ and $\chi^+(x)$. The charge-conjugation operation has the form

$$\varphi^C(x) = -\sigma^2\tilde{\chi}^+(x), \quad \chi^C(x) = \sigma^2\tilde{\varphi}^+(x), \quad A_\mu{}^C(x) = -\tilde{A}_\mu(x). \tag{2.7}$$

Here the $\sim$ symbol denotes transposition. The necessary one-to-one correspondence is established thanks to the fact that the function $\varphi_P^C(x) = -\sigma^2\tilde{\chi}_P{}^+(x)$ satisfies the right Weyl equation $(i\nabla_0^C + i\sigma^i\nabla_i^C)\varphi_P{}^C = 0$, where $\nabla_\mu^C = \partial/\partial x^\mu + A_\mu^C$ and the set of functions $\{\varphi_P^C, \varphi_P^{+C}\}$ form a complete orthonormal set.

Let the capital Greek letter Ξ denote either the index N or P and the Dirac spinor ψ_Ξ denote either $\binom{\varphi_N}{0}$ or $\binom{0}{\chi_P}$. We then have the expansion

$$\psi(x) = \sum_\Xi{}' \eta_\Xi(t)\psi_\Xi(x),$$

where $\eta\Xi(t)$ equals either $\partial_N(t)$ or $b_P(t)$.

A. First of all we shall study the general dependence of the fermion amplitude on the gauge field. The fermion wave functionals depend in our representation on the variables $\bar{a}_N$. Since the fermion Hamiltonian in the variables $\bar{a}_N$, a_N vanishes, the amplitude takes on the very simple form:

$$Z_+\{A_\mu\} = \exp \sum_N{}' \bar{a}_N(t_1) a_N(t_0). \tag{2.8}$$

To discover the gauge-field dependence of the amplitude the coordinates $\{\bar{a}_N(t_1)\}$ in the moving frame $\{\varphi_N^+(t_1,x)\}$ should be expressed in terms of the coordinates $\{\bar{a}_N^{(0)}(t_1)\}$ in the stationary frame $\{\varphi_N{}^+(t_0,x)\}$. This connection is established with the help of the equation

$$\sum_N \bar{a}_N(t_1)\varphi_N{}^+(t_1,\mathbf{x}) = \sum_N \bar{a}_N^{(0)}(t_1)\varphi_N{}^+(t_0,\mathbf{x}). \tag{2.9}$$

To begin with we calculate the change in the amplitude for infinitesimal change in the gauge field. Assume that $A'_\mu = A_\mu + \delta A_\mu$ and the function φ'_N with the boundary condition $\varphi'_N(t_0,x) = \varphi_N(t_0,x)$. To first order in the field δA_μ we have $((x') = (t',x))$

$$\varphi_N'(x) = \varphi_N(x) - i\sum_M \varphi_M(x)\varkappa_{MN}(t),$$

$$\varkappa_{MN}(t) = -i\int_{t_0}^{t} dt' \int d^3x\, \varphi_M{}^+(x')\sigma^\mu \delta A_\mu(x')\varphi_N(x'). \tag{2.10}$$

We have introduced the notation $\sigma^\mu = (1,\sigma^i)$. From the equation

$$\sum_N \bar{a}_N(t)\varphi_N{}^+(x) = \sum_N \bar{a}_N'(t)\varphi_N{}^{+\prime}(x)$$

we find

$$\bar{a}_N'(t_1) = \bar{a}_N(t_1) - i\sum_M{}' \bar{a}_M(t_1)\varkappa_{MN}(t_1). \tag{2.11}$$

Substituting into the amplitude

$$Z_+\{A_\mu + \delta A_\mu\} = \exp\sum_N{}' \bar{a}_N'(t_1)a_N(t_0)$$

expressions (2.11) we obtain

$$\delta Z_+\{A_\mu\} = -i\sum_{MN}{}' \bar{a}_M(t_1)\varkappa_{MN}(t_1)a_N(t_0)Z_+\{A_\mu\}. \tag{2.12}$$

The current has the form

$$J_+{}^{\mu a} = \sum_{MN}{}' \bar{a}_M a_N \varphi_M{}^+ i t^a \sigma^\mu \varphi_N. \tag{2.12'}$$

Consequently it is seen from (2.12) and (2.10) that

$$\delta\langle Z_+\{A_\mu\}\rangle_{\Lambda\Sigma} = i\int d^4x\, \delta A_\mu{}^a(x)\langle J_+{}^{\mu a}(x) Z_+\{A_\mu\}\rangle_{\Lambda\Sigma}. \tag{2.13}$$

Everywhere $\langle\ldots\rangle_{\Lambda\Sigma}$ denotes the evaluation of matrix elements with respect to the fermion states $\langle\Lambda|$ and $|\Sigma\rangle$, which depend on the variables $\{a_N\}$ and $\{\bar{a}_N\}$ respectively. It should be remembered that in terms of the variables $\{a_N,\bar{a}_N\}$ the Hamiltonian equals zero and the operator $Z_+\{A_\mu\}$ is diagonal in the basis $\{|\Lambda\rangle, |\Sigma\rangle,\ldots\}$.

Let us see what Eq. (2.12) gives in the case $\delta A_\mu = \nabla_\mu u$. Substituting $\delta A_\mu = \nabla_\mu u$ into (2.12) and integrating by parts, we find

$$\delta Z_+ = -i\int_{t_0}^{t_1} dt \int d^3x\, u^a(x) \times\left[\nabla_\mu \sum_{MN}{}' \bar{a}_M(t_1)a_N(t_0)\varphi_M{}^+(x)\sigma^\mu i t^a \varphi_N(x)\right] Z_+. \tag{2.14}$$

In evaluating the expression in square brackets in (2.14) one should take care as the operator ∇_μ acts on a singular quantity. To eliminate the ambiguity we use time point-splitting of the functions in (2.14), $\varphi_M{}^+(t,x) \to \varphi_M{}^+(t+\varepsilon,x)$ and go to the limit $\varepsilon \to +0$ in the answer. With such an approach the calculation simplifies because the prime on the summation sign in (2.14) may be omitted: the ultraviolet tail drops out automatically. Using the Weyl equations in the gauge $A_0 = 0$ we reduce (2.14) to the form

$$\delta Z_+\{A_\mu\} = -iZ_+\{A_\mu\} \int d^4x\, u^a(x) \times \Big[\sum_{MN} \bar{a}_M(t_1) a_N(t_0) \varphi_M{}^+(x+\varepsilon)\, \varepsilon \dot{A}_i(x) \sigma^i i t^a \varphi_N(x) \Big]. \quad (2.15)$$

The final answer is easily written in any gauge as a result of covariance considerations. The overdot always denotes the derivative $\partial/\partial t$. Upon averaging with respect to states with filled Dirac sea only negative frequency functions from the ultraviolet region in (2.15) contribute to the anomaly. Indeed, in the ultraviolet region the states have the form $|\rangle \sim \Pi'_{(-N)} \bar{a}_N$, where the symbol $\Pi'_{(-N)}$ denotes multiplication of $\bar{a}_N$ with numbers N corresponding to negative frequency functions $\varphi_N(x)$. But for $\varepsilon \to +0$ the negative frequency part of the ultraviolet tail drops out automatically from the summation in (2.15), since under the Euclidean rotation $\varepsilon \to -i\varepsilon$ the quantity $\varphi_N(t,\mathbf{x}) \varphi_N(t - i\varepsilon, \mathbf{x})$ is found to be proportional to $\exp(-\varepsilon E_N)$, where $E_N \to +\infty$.

In evaluating the matrix elements $\langle [...] \rangle_{\Lambda\Sigma}$, where $[...]$ is the square bracket in (2.15), there arise Green's functions of the form

$$G_{\Lambda\Sigma}(x, y) = -i \langle T\psi(x)\overline{\psi}(y)\rangle_{\Lambda\Sigma},$$

where T is the time-ordering symbol. Here $\psi(x)$ and $\bar{\psi}(y)$ are nonregularized Fermi-fields, obeying the usual commutation relations and the Dirac equation $i\gamma^\mu \nabla_\mu \psi = 0$. Therefore such Green's functions satisfy the equations

$$i\gamma^\mu \nabla_\mu G_{\Lambda\Sigma}(x, y) = \delta_{\Lambda\Sigma} \delta^{(4)}(x-y),$$

from which it is seen that for $\Lambda \neq \Sigma$ the element $G_{\Lambda\Sigma}(x,y)$ has no singularities for $x = y$ and that $\langle [...] \rangle_{\Lambda\Sigma} \sim \delta_{\Lambda\Sigma}$. Setting $G_\Lambda = G_{\Lambda\Lambda}$ and going over to the continuum we get from (2.15)

$$\delta Z_+\{A_\mu\} = \frac{i}{4} Z_+\{A_\mu\} \Big[-\int d^4x\, u^a(x) \times \varepsilon\, \mathrm{tr} G_\Lambda(x, x+\varepsilon)(1-\gamma^5)\gamma^i \dot{A}_i(x) t^a + \text{c.c.} \Big]. \quad (2.16)$$

Taking the real part of the expression in square brackets is explained by comparison with the original expression (2.14). Direct calculation shows that only the term proportional to γ^5 contributes to (2.16).

We write out just that part of G_Λ that is relevant in (2.16) for $\varepsilon \to +0$:

$$G_\Lambda(x, x+\varepsilon) = -\gamma^0 (8\pi^2 \varepsilon)^{-1} \sigma^{\mu\nu} F_{\mu\nu} + \text{irrelevant terms}.$$

Substituting this in (2.16) we obtain

$$\delta_u Z_+\{A_\mu\} = -i\mathcal{A}_+\{u\} Z_+\{A_\mu\},$$

$$\mathcal{A}_+\{u\} = -\frac{i}{32\pi^2} \int d^4x\, \varepsilon^{\mu\nu\lambda\rho}\, \mathrm{tr}\, u F_{\mu\nu} F_{\lambda\rho}. \quad (2.17)$$

Comparing (2.17) and (2.13) we find

$$\int d^4x\, u^a \nabla_\mu J_+{}^{\mu a} = \mathcal{A}_+\{u\}. \quad (2.18)$$

In this fashion the regularization considered here brings about the covariant value for the nonabelian anomaly for the "correct" current [see (2.13) and (2.18)].

B. We shall now obtain the anomaly by another method, applicable to evaluations of functional integrals. This method was widely used by R. Feynman in deriving various Ward identities (Ref. 7, chapter 7). The author noted in 1978 that in the case when the regularized functional measure is not invariant under a certain continuous transformation an anomalous Ward identity results.[8] This method was used for the first time in Ref. 9 to derive the axial vector anomaly. Thereafter the method was rediscovered and developed by Fujikawa.[10]

Let us make an infinitesimal transformation of the fields in the form

$$\varphi(x) \to \varphi'(x) = (1+u(x))\varphi(x) - \sum_N{}'' \varphi_N(x) c_N{}^u(t),$$

$$\varphi^+(x) \to \varphi^{+\prime}(x) = \varphi^+(x)(1-u(x)) - \sum_N{}'' \bar{c}_N{}^u(t) \varphi_N{}^+(x),$$

$$c_N{}^u(t) = \int d^3x\, \varphi_N{}^+(x) u(x) \varphi(x), \quad (2.19)$$

where $u(x) = u^a(x) t^a$ are transformation parameters and the symbol $\Sigma_N{}''$ denotes summation over the ultraviolet tail so that

$$\sum_N = \sum_N{}' + \sum_N{}''.$$

It should be remembered that the functions entering Eqs. (2.19) are taken at the points t_r and $t_r + \varepsilon/2$ respectively, but $u(t_r,\mathbf{x}) = u(t_r + \varepsilon/2, \mathbf{x})$ because the fields $\varphi(t_r,\mathbf{x})$ and $\varphi^+(t_r + \varepsilon/2, \mathbf{x})$ are transformed by one and the same parameter. The last terms on the right hand sides of (2.19) are necessary to keep the transformations under consideration from taking the fields $\varphi'(x)$ and $\varphi^{+\prime}(x)$ outside the limits of the regularized spaces with bases $\{\varphi_N(x)\}'$ and $\{\varphi_N^+(x)\}'$ respectively. Here $\{\varphi_N\}'$ means that the basis does not contain functions from the ultraviolet tail. In the transition amplitude (2.5) we make the change of variables according to (2.19). Let us follow the resultant change in the measure. In terms of integration variables (2.19) takes on the form

$$a_N'(t_r) = a_N(t_r) + \sum_M{}' c^u_{NM}(t_r) a_M(t_r),$$

$$\bar{a}_N'\Big(t_r + \frac{\varepsilon}{2}\Big) = \bar{a}_N\Big(t_r + \frac{\varepsilon}{2}\Big) - \sum_M{}' \bar{a}_M\Big(t_r + \frac{\varepsilon}{2}\Big) c^u_{MN}\Big(t_r + \frac{\varepsilon}{2}\Big),$$

$$c^u_{MN}(t) = \int d^3x\, \varphi_M{}^+(x) u(x) \varphi_N(x). \quad (2.20)$$

The Jacobian of the transformation from the variables $\{a_N, \bar{a}_N\}$ to the variables $\{\bar{a}'_N, a'_N\}$ accurate to first order in the parameter u has the form

$$\left[1+\sum_r\sum_N{}'\left(c^u_{NN}\left(t_r+\frac{\varepsilon}{2}\right)-c^u_{NN}(t_r)\right)\right]. \qquad (2.21)$$

Since a change in the variables does not change the integral the contributions from the change in the measure and the action in (2.6) cancel. It is easy to see that the last term in (2.19) does not contribute to the variation of the action. However this simplification occurs only in the case when the field is expanded in solutions of the Weyl (Dirac) equation. (In this manner, in the evaluation of the anomaly by the finite-mode regularization method one may ignore the projection of the transformed fields on the original regularized space with basis $\{\varphi_N,\varphi^+_N\}'$.) We find for the variation of the action

$$S_{\varphi'}=S_\varphi-\int d^4x\, u^a\nabla_\mu J_+{}^{\mu a},$$

where the current $J^{\mu a}_+$ is defined in (2.12′). Using what has been said above we arrive at the following equation:

$$\int d^4x\, u^a\nabla_\mu J_+{}^{\mu a}=-i\sum_r\sum_N{}'\left(c^u_{NN}\left(t_r+\frac{\varepsilon}{2}\right)-c^u_{NN}(t_r)\right). \qquad (2.22)$$

Similarly we find in left Weyl theory

$$\int d^4x\, u^a\nabla_\mu J_-{}^{\mu a}=-i\sum_r\sum_P{}'\left(d^u_{PP}\left(t_r+\frac{\varepsilon}{2}\right)-d^u_{PP}(t_r)\right),$$

$$d^u_{PQ}(t)=\int d^3x\,\chi_P{}^+(x)u(x)\chi_Q(x). \qquad (2.23)$$

It is seen from the charge-conjugation properties of (2.7) that the sum of the right hand sides of Eqs. (2.22) and (2.23) equals zero. Indeed in the Dirac theory (2.6), which is explicitly even under charge conjugation, we have the equality $[u^C(x)=-\tilde u(x)]$

$$\langle Z\{A_\mu+\nabla_\mu u\}\rangle_{\Lambda\Sigma}=\langle Z^c\{A_\mu{}^c+\nabla_\mu{}^c u^c\}\rangle_{\Lambda\Sigma}{}^c, \qquad (2.24)$$

where $\langle\ldots\rangle^C_{\Lambda\Sigma}$ denotes taking matrix elements with respect to charge conjugate states. It follows from (2.24) and (2.13) that

$$\mathscr{A}_+\{u\}+\mathscr{A}_-\{u\}=\mathscr{A}_+{}^c\{u^c\}+\mathscr{A}_-{}^c\{u^c\}. \qquad (2.25)$$

Here $\mathscr{A}_\pm\{u\}$ denotes the right-hand-sides of Eqs. (2.22) and (2.23) respectively and $\mathscr{A}_\pm{}^C\{u^c\}$ denotes the same quantities formed from charge conjugate fields. Using Eq. (2.7) and the equality $u^C=-\tilde u$ it is easy to see that the right-hand-side of Eq. (2.25) is equal to the negative of the left-hand-side, i.e., that both sides of this equation vanish. This means that our regularization preserves the gauge invariance of the Dirac theory:

$$\nabla_\mu J^{\mu a}=0,\quad J^{\mu a}=\bar\psi\gamma^\mu it^a\psi=J_+{}^{\mu a}+J_-{}^{\mu a}.$$

Using this, it is easy to find that in continuum notation (in the $A_0=0$ gauge) the nonabelian anomaly has the following form:

$$\mathscr{A}_+\{u\}=-\frac{i}{4}\int d^4x\,\mathrm{tr}\,u(x)\gamma^5\frac{\partial}{\partial t}\sum_\Xi{}'\psi_\Xi(x)\psi_\Xi{}^+(x). \qquad (2.26)$$

Just as before we use the shift in the time variable $\psi_\Xi{}^+(x)\to\psi_\Xi{}^+(x+\varepsilon)$, where the 4-vector $\varepsilon=(\varepsilon,0,0,0)$ tends to zero, in order to correctly evaluate the derivative $\partial/\partial t$. One should evaluate separately the contribution from the positive- and negative-frequency parts of the functions ψ_Ξ. We calculate first the negative-frequency contribution. Let $\varepsilon>0$. Then

$$i\sum_{(-\Xi)}\psi_\Xi(x)\bar\psi_\Xi(x+\varepsilon)=G_\Lambda(x,x+\varepsilon).$$

With the help of the Dirac equation the negative-frequency contribution to (2.26) can be rewritten as follows:

$$\mathscr{A}_+\{u\}=\frac{1}{4}\int d^4x\,u^a(x)\,\mathrm{tr}\,(t^a\gamma^5\gamma^i\nabla_i{}^xG_\Lambda(x,x+\varepsilon))$$
$$+\frac{1}{4}\int d^4x\,u^a(x)\,\mathrm{tr}\,(\varepsilon A_i(x)t^a\gamma^i\gamma^5G_\Lambda(x,x+\varepsilon)). \qquad (2.27)$$

It is easily understood that taking the positive-frequency contribution into account results in adding to the right-hand-side of Eq. (2.27) its complex conjugate. The sum of the first term in (2.27) and its complex conjugate value vanishes for $\varepsilon\to+0$. In this manner comparison of (2.27) with (2.16) and (2.17) again results in the answer (2.18).

3. SELF-CONSISTENCY EQUATIONS

The value (2.18) of the nonabelian anomaly is in contradiction with the Wess-Zumino self-consistency equation.[11] It is obvious that (2.18) does not satisfy that equation, although expression (2.18) is the variation of the fermionic transition amplitude with respect to the gauge transformation. To remove this paradox we look into the way the Wess-Zumino equation comes about.

Let L_u be the generator of an infinitesimal gauge transformation in space-time with parameter $u(x)$:

$$[L_u,\ A_\mu(x)]=\nabla_\mu u(x). \qquad (3.1)$$

The set of generators L_u form a Lie algebra with commutation relations

$$[L_u,\ L_v]=L_{[u,\,v]}. \qquad (3.2)$$

Let us define the action of the operators $\{L_u\}$ on the amplitude $Z_+\{A_\mu\}$ by $L_uZ_+\{A_\mu\}=Z_+\{A_\mu+\nabla_\mu u\}$. We then have

$$L_uZ_+\{A_\mu\}=-i\int d^4x\,u^a\nabla_\mu J_+{}^{\mu a}Z_+\{A_\mu\}. \qquad (3.3)$$

It immediately follows from Eqs. (3.2) and (3.3) that the expression (2.18) satisfies the Wess-Zumino equation

$$L_u\mathscr{A}_+\{v\}-L_v\mathscr{A}_+\{u\}=\mathscr{A}_+\{[u,\ v]\}. \qquad (3.4)$$

When (2.18) is substituted into (3.4) the latter equation is not satisfied.

To remove the resultant paradox one should try to change the self-consistency equation.

Equation (3.4) is a consequence of the three equations (3.1)–(3.3). It is obvious that (3.1) and (3.3) cannot be changed. The only possibility is to change Eq. (3.2) by introducing a Schwinger term.

Let us denote by l the totality of operators $\{L_u,\mathscr{A}_+\{u\}\}$, acting on fermionic amplitudes, where

$\mathscr{A}_+\{u\}$ is given by Eq. (2.17) and it acts by multiplication. The manifold l becomes a Lie algebra if we assume the following system of commutation relations:

$$[L_u, L_v]=L_{[u,v]}-i\mathscr{A}_+\{[u, v]\},$$
$$[L_u, \mathscr{A}_+\{v\}]=\mathscr{A}_+\{[u, v]\}, \quad [\mathscr{A}_+\{u\}, \mathscr{A}_+\{v\}]=0. \tag{3.5}$$

We shall postulate that the commutation relations (3.5) are realized in the action of the operators from the manifold l on fermionic amplitudes. It is easy to verify that the Jacobi identity holds for the algebra (3.5). It now follows from Eqs. (3.1), (3.3), and (3.5) that the new self-consistency equation has the form

$$L_u\mathscr{A}_+\{v\}-L_v\mathscr{A}_+\{u\}$$
$$=\mathscr{A}_+\{[u, v]\}-\frac{1}{32\pi^2}\int d^4x\, \varepsilon^{\mu\nu\lambda\rho}\operatorname{tr}[u, v]F_{\mu\nu}F_{\lambda\rho}. \tag{3.6}$$

The anomaly (2.17) satisfies Eq. (3.6).

The relations (3.5) show that $Z_+\{A_\mu\}$ is not a uniquely defined quantity on the space of fields A_μ, which we shall denote by $\mathscr{H}$. Let $\mathscr{O}$ denote the space of classes of gauge-equivalent Yang-Mills fields and $\mathscr{G}$ the group of gauge transformations in the space $\mathscr{H}$. Then the space $\mathscr{H}$ may be viewed as a fiber bundle with fiber $\mathscr{G}$ and base $\mathscr{O}$. (In fact it is assumed that $\mathscr{H}$ includes not all fields but only those on which $\mathscr{G}$ acts freely (see Ref. 12).) The first of the relations (3.5) shows that going round a closed contour in the fiber $\mathscr{G}$ results in a change in the amplitude $Z_+\{A_\mu\}$ + multiplication by a certain phase $\exp(i\mathscr{A}_+\{[u,v]\})$. Let us introduce the operators

$$Q\{\xi^\mu\}=\int d^4x\, \xi^{\mu a}\{\delta/\delta A^{a\mu}\}.$$

As a result of what has been said we have

$$[Q\{\xi^\mu\}, Q\{\eta^\nu\}]=\Lambda\{\xi^\mu, \eta^\nu\}\neq 0. \tag{3.7}$$

Indeed, in the special case $\xi_\mu = \nabla_\mu u$ we have $Q\{\nabla_\mu u\} = L_u$ and the relation (3.7) should coincide with (3.5).

The right-hand-side of (3.7) is easily reestablished for ξ^μ, η^μ, satisfying $\nabla_\mu \xi^\mu = \nabla_\mu \eta^\mu = 0$:

$$\Lambda\{\xi^\mu, \eta^\nu\}=-2i(-\underset{x}{\operatorname{tr}})(-\underset{y}{\operatorname{tr}})\int d^4x\, d^4y[\xi^\sigma(x), \eta_\sigma(x)]$$
$$\times D(x, y)\nabla_\mu J_+{}^\mu(y). \tag{3.8}$$

Here $D(x,y)$ is the causal Green's function for the scalar field $\lambda^a(x)$, transforming according to the adjoint representation of the gauge group and satisfying the equation of motion $\nabla_\mu{}^2\lambda = 0$:

$$D^{ab}(x, y)=i\langle T\lambda^a(x)\lambda^b(y)\rangle.$$

We have

$$\nabla_\mu{}^2 D(x, y)=\delta^{(4)}(x-y). \tag{3.9}$$

It is the causal Green's function that should be used in Eq. (3.8) for which the equality

$$D^{ab}(x, y)=D^{ba}(y, x),$$

is valid, since the amplitude (2.8) is invariant under time reversal. If any other Green's function were used the equation (3.7) would violate this fundamental property of the amplitude (2.8). The last equation will be used below in a substantial way.

To verify (3.8) we take $\xi^\mu = \nabla^\mu u, \eta^\mu = \nabla^\mu v$, so that $\nabla_\mu{}^2 u = \nabla_\mu{}^2 v = 0$ and therefore $\nabla_\mu{}^2[u,v] = 2[\nabla_\mu u, \nabla^\mu v]$. From here and from Eqs. (3.9) and (2.18) we find that in this case the right-hand-side of (3.8) equals $(-i\mathscr{A}_+\{[u,v]\})$. This agrees with (3.5) since

$$Q\{\nabla^\mu u\}=L_u.$$

Let us now construct a space in which the amplitude $Z_+\{A_\mu\}$ is single-valued. Let ω be an infinitesimal horizontal path in the space of fields, whose projection $\overline{\omega}$ onto the base $\mathscr{O}$ is closed. If $A^\mu(x;s), 0\leqslant s\leqslant 1$ is the path ω and $\delta A^\mu = (\partial A^\mu/\partial s)\delta s$, then $\nabla_\mu \delta A^\mu = 0$ along the path ω. It is easy to evaluate the change in the field A_μ along the path ω in a general form (see Ref. 13):

$$\delta\delta A_\mu(x)=-2\nabla_\mu(-\underset{y}{\operatorname{tr}})\int d^4y\, D(x, y)\,\delta A_\nu(y)\wedge\delta A^\nu(y),$$
$$\nabla_\mu\delta A^\mu=0. \tag{3.10}$$

Here Λ is the exterior multiplication sign.

Assertion 1. $\oint\overline{\omega}\delta Z_+\{A_\mu\} = 0$ where the integrand is given according to (2.13).

Let us write out the integrand 1-form (2.13):

$$i\int d^4x\, \delta A_\mu{}^a\langle J_+{}^{\mu a}Z_+\rangle.$$

The integral along the closed contour $\overline{\omega}$ can be transformed according to Stokes' theorem into an integral over the infinitesimal area σ bounded by the loop $\overline{\omega}(\partial s = \overline{\omega})$:

$$i\int d^4x\oint_{\overline{\omega}}\delta A_\mu{}^a\langle J_+{}^{\mu a}Z_+\rangle=i\int d^4x\int_\sigma \delta(\delta A_\mu{}^a\langle J_+{}^{\mu a}Z_+\rangle).$$

From here it is seen that the integral is made up of two contributions. The first contribution has the form

$$i\int d^4x\int_\sigma \delta\delta A_\mu{}^a\langle J_+{}^{\mu a}Z_+\rangle$$

and after transferring the operator ∇_μ [see (3.10)] it is reduced to a simple form expressible in terms of the anomaly:

$$\frac{1}{16\pi^2}(-\underset{x}{\operatorname{tr}})(-\underset{y}{\operatorname{tr}})\int d^4x\, d^4y\, \varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x)$$
$$\times D(x, y)\,(\delta A_\nu(y)\wedge\delta A^\nu(y))\langle Z_+\rangle, \quad \nabla_\mu\delta A^\mu=0.$$

(An analogous and nonvanishing contribution to the change in the amplitude is present in Feynman quantization.) The second contribution arises from second differentiation of the amplitude $Z_+\{A_\mu\}$, whose first derivative equals $\{J_+{}^{\mu a}Z_+\rangle$. Here the integral over the area σ receives a contribution from the antisymmetric part of the second derivative, which is expressible through the right-hand-side of (3.7). Using (3.7) and (3.8) we find the second contribution:

$$-\frac{1}{16\pi^2}(-\underset{x}{\mathrm{tr}})(-\underset{y}{\mathrm{tr}})\int d^4x\, d^4y\,(\delta A_\nu(x)\wedge\delta A^\nu(x))$$
$$\times D(x,y)\varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}(y)F_{\lambda\rho}(y)\langle Z_+\rangle,\quad \nabla_\mu\delta A^\mu=0.$$

[In Feynman quantization the analog of the second contribution is absent as the right-hand-side of Eq. (3.7) vanishes in that case.] We see that the sum of the two contributions equals zero.

Although assertion 1 was established for infinitesimal paths it is easily generalized, using results of Ref. 12, to arbitrary paths, at least for gauge groups SU(N). Assertion 1 means in fact that fermionic amplitudes are correctly defined on the space of orbits $\mathcal{O}$. According to (2.17) and (3.5) the manifold of generators $\{\tilde{L}_\mu\}$, where $\tilde{L}_u = L_u + i\mathscr{A}_+\{u\}$, forms a Lie subalgebra with commutation relations $[\tilde{L}_u,\tilde{L}_v] = \tilde{L}_{[u,v]}$ and $\tilde{L}_u Z_+\{A_\mu\} = 0$. Let us denote by $\mathscr{L}$ the Lie group corresponding to the algebra (3.5), and by $\mathscr{G}'$ its subgroup, corresponding to the subalgebra $\{\tilde{L}_u\}$. The group $\mathscr{G}'$ and its algebra $\{\tilde{L}_\mu\}$ are in an obvious way isomorphic to the gauge group $\mathscr{G}$ and its algebra $\{L_u\}$, correspondingly ($L_u \rightleftarrows \tilde{L}_u$). One may therefore construct the fiber bundle $\mathscr{H}'$ with the structure group $\mathscr{G}'$ over the base $\mathcal{O}$, associated with $\mathscr{H}$. These bundles are isomorphic as the space $\mathscr{H}'$ differs from $\mathscr{H}$ only in the replacement of the structure group $\mathscr{G}$ by $\mathscr{G}'$ according to the specified isomorphism.

This means that the amplitude $Z_+\{A_\mu\}$ is defined to be single-valued and continuous on the space $\mathscr{H}'$. Moreover the amplitude is *invariant* under the action of the group $\mathscr{G}'$, and Eq. (2.13) is valid in the event that $\delta A_\mu \neq \nabla_\mu\delta\alpha$.

In conclusion we note that restriction of the amplitude $Z_+\{A_\mu\}$ to the space $\mathscr{H}'$ means in fact a certain change in the passage to the limit of removing the cut-off. Indeed, let us look at our equations from a different point of view. We carry out the gauge transformation $\delta A_\mu = \nabla_\mu\delta\alpha$, $\delta\varphi_N = -\delta\alpha\varphi_N$, $\delta\varphi_N{}^+ = \varphi_N{}^+\delta\alpha$, $\delta\alpha(x)\to 0$ for $t\to t_0$ or $t\to t_1$. At the same time the variables $\{a_N(t_0),\bar{a}_N(t_1)\}$ are unchanged and therefore the amplitude (2.8) is left invariant by the action of the gauge group. Such an interpretation of the equations corresponds to direct evaluation (without time point-splitting) of the derivative $\partial/\partial t$ in the square bracket in (2.14). This is correct as the quantity in the square bracket is regularized. In this manner we again arrive at gauge invariance of the amplitude $Z_+\{A_\mu\}$. This change in the calculations corresponds to the passage from the algebra $\{L_u\}$ to the algebra $\{\tilde{L}_u\}$. Our work shows that the two ways of performing the calculations are equivalent.

4. RELATIVISTIC INVARIANCE

It is now obvious that the integration $\int DA_\mu$ in (2.1) should proceed over the space $\mathscr{H}'$.

Let us clarify the question of relativistic invariance of the system. This question is technically solved analogously to the question of gauge anomaly: we subject the fields to an infinitesimal localized Lorentz transformation and evaluate the corresponding anomaly arising due to noninvariance of the measure. Since relativistically invariant methods for the regularization of the fluctuations of the gauge fields exist, the possible anomaly is contained in the integral (2.5).

In the transition amplitude (2.1) let us carry out a variation of the fields in the form

$$\delta\varphi=-\omega_\nu{}^\mu x^\nu\nabla_\mu\varphi-{}^1/_2\omega_{\mu\nu}\sigma^{\mu\nu}\varphi,$$
$$\delta A_\mu=-\omega_\nu{}^\lambda x^\nu F_{\mu\lambda},$$
$$\sigma^{\mu\nu}=-\sigma^{\nu\mu},\quad \sigma^{0i}={}^1/_2\sigma^i,\quad \sigma^{ij}=-{}^1/_2 i\varepsilon_{ijk}\sigma^k. \qquad (4.1)$$

Here $\omega_{\mu\nu}(x) = -\omega_{\nu\mu}(x)$ are the transformation parameters. Since for $\partial_\lambda\omega_{\mu\nu} = 0$ the action of the system is left invariant, some conservation law corresponds to the transformations (4.1) and can be violated by a Lorentz anomaly. Analogously to the derivation of the gauge anomaly, we find (in the dangerous case $\omega^{ij} = 0$, $\omega^{0i}(x) = \omega^i(x)$):

$$\delta S=\frac{1}{4}\int d^4x\,\omega^i\frac{\partial}{\partial t}\sum_{\Xi}{}'\Big\{x_0\frac{i}{2}[\psi_\Xi{}^+\gamma^5\nabla_i\psi_\Xi-\text{c.c.}]$$
$$-x^i\frac{i}{2}[\bar{\psi}_\Xi\gamma^j\nabla_j\gamma^5\psi_\Xi-\text{c.c.}]\Big\}. \qquad (4.2)$$

Here the operator $\partial/\partial t$ acts only on the functions ψ_Ξ and ψ_Ξ for the same reason as in Sec. 2 [see (2.19) and following]. It is easy to see that the right-hand-side of (4.2) equals zero. Indeed, it follows from the properties of charge conjugation (2.7) that the right-hand-side of (4.2) is odd under charge conjugation; moreover it is gauge invariant. But it is impossible to construct from gauge fields a local entity with these properties. Analogous conclusions hold for arbitrary values of the parameter $\omega_{\mu\nu}$, as well as for arbitrary translations. Therefore we have

$$\partial_\lambda\eta^{\lambda\mu\nu}=0,\quad \partial_\lambda\Theta^{\lambda\mu}=0,$$
$$\eta^{\lambda\mu\nu}=x^\mu\Theta^{\lambda\nu}-x^\nu\Theta^{\lambda\mu}+i\varphi^+\sigma^\lambda\sigma^{\mu\nu}\varphi, \qquad (4.3)$$

where $\Theta^{\mu\nu}$ is the energy-momentum tensor.

We note that if the integration $\int DA_\mu$ proceeds over the space $\mathscr{H}'$, then the accompanying gauge transformation of the Yang-Mills field in (4.1) plays no role, as the amplitude $Z_+\{A_\mu\}$ is invariant under the action of the group $\mathscr{G}'$.

It follows from (4.3) that the operators

$$P^\mu=\int d^3x\,\Theta^{0\mu},\qquad M^{\mu\nu}=\int d^3x\,\eta^{0\mu\nu}$$

are conserved. Moreover one obtains from (4.3), from the commutation relations $[P^\mu,P^\nu] = 0$ and from the space rotation group the missing commutation relations of the Poincare algebra for the operators $P^\mu, M^{\mu\nu}$ (see Refs. 14, 2).

5. CONCLUSION

We may now draw the following conclusion:

Assertion 2. The nonabelian Weyl theory is relativistically and gauge invariant, provided the fermionic amplitudes are viewed in the space $\mathscr{H}'$ and the integration $\int DA_\mu$ in (2.1) is understood to be over the orbit[1)] space $\mathcal{O}$.

To develop the perturbation theory for calculating transition amplitudes use can be made of the Fadeev-Popov trick. In (2.1) should be inserted $Z_+\{A_\mu\}$, calculated according to the formulated rules. In contrast to Feynman PT, there will be present in the expansion of $Z_+\{A_\mu\}$ the retarded Green's function $G^{\text{ret}}(x)$. Since the theory is gauge-invariant, renormalizability can be established with the help of generalized Ward identities. Variation of the variables in the integral (2.1) of the form $\delta A_\mu = \nabla_\mu\delta\alpha$ leads to the identity $0=0$, while variation under the condition $\delta A_\mu \neq \nabla_\mu\delta\alpha$ yields $\nabla^\nu F_{\mu\nu} = e^2 J_\mu^\perp$, where

$$J_\mu^\perp(x)=J_\mu(x)-\nabla_\mu(-\underset{y}{\mathrm{tr}})\int d^4y\, D^{ret}(x,y)\nabla_\nu J^\nu(y),$$

$$\nabla_\mu^2 D^{ret}(x,y)=\delta^{(4)}(x-y),\quad D^{ret}(x,y)=0\quad(x^0<y^0).$$

Let us formulate the result in Hamiltonian language. Regularization of the theory should preserve gauge invariance. This can be achieved if the fermionic states are taken from an incompletely filled Dirac sea. Thereafter one must solve either the time-dependent Schrödinger equation, or the Heisenberg equation with constraints. The 't Hooft and Veltman dimensional regularization method is likely to turn out to be a useful tool for the regularization of PT in solving the Schrödinger or Heisenberg equations because it preserves gauge invariance.

Let $\chi_u = \int d^3x u^a(-\nabla_i(\delta/\delta A_i^a)+\varphi^+ t^a\varphi)$ be the generators of gauge transformations. Then $[\chi_u, \chi_v] = \chi_{[u,v]} + S$, where s is a Schwinger term, which is expressed covariantly in terms of gauge field variables. If Lorentz invariance is preserved (according to Assertion 2 this is possible), then the term s is necessarily local. The most general expression for such a term is

$$s=\int d^3x\,\mathrm{tr}\{c_1\varepsilon_{ijk}(u\nabla_k v-v\nabla_k u)F_{ij}+c_2[u\nabla_k v+(\nabla_k v)u-v\nabla_k u-(\nabla_k u)v]E_k+c_3[u,v]\nabla_i E_i\},\qquad E_i=-i\delta/\delta A_i$$

However $c_3=0$, since the given formula can be obtained in PT, it is valid for all groups, and in the SU(2) case the Schwinger term is absent. It is easy to verify that $c_1=c_2=0$ follows from the Jacobi identity $[\chi_u,[\chi_v,\chi_\omega]]+\ldots=0$.

Similarly, one establishes from kinematic considerations that

$$[P^\mu,\chi_u]=0,\qquad [M^{\mu\nu},\chi_u]=0.$$

It is therefore possible and consistent with relativistic invariance to impose the constraints $\chi_u\approx 0$. In addition to the Weyl equation the following are present among the Heisenberg equations of motion:

$$\dot{A}_i=e^2E_i+\nabla_i A_0,\qquad \nabla_0 E_i^a=(1/e^2)\nabla_j F_{ji}+\varphi^+\sigma^i i t^a\varphi.$$

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[1)]It can be shown that in dynamic quantization global anomalies[2] are also absent.

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