Nonlinear theory of flame-front instability

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A nonlinear equation in the form $u_t' + uu_x' + \hat{H}u_x' = 0$, which describes the instability of a planar flame front in the slow-combustion regime, is derived with account taken of quadratic corrections. Instability leads to a cellular structure of the front, in qualitative agreement with observations.

1. The instability of a burning gas is particularly important in rocket technology, since it can lead to detonation and undesirable explosions, for the prevention of which special measures are taken. In the present paper, however, we consider this problem in the classical purely hydrodynamic Landau-Zel'dovich formulation for a plane flame front (FF) in the regime of slow combustion and esentially subsonic velocities.

To be specific, we assume that a hot gas flows upward along the z axis, and that $z = 0$ and $z = a(t,x)$ are respectively the unperturbed and perturbed flame fronts. This problem was first considered in the linear approximation by Landau in 1944 (Ref. 1; see also Ref. 2), who obtained for perturbations in the form $a = a_0 e^{\gamma t}$ the growth rate

$$\gamma=|k|v_1^0\mu\nu,\quad \nu=(\varkappa-1)/(1+\mu),\quad \varkappa=(1+\mu-\mu^{-1})^{1/2}. \tag{1}$$

Here v_1^0 is the front velocity and $\mu = \rho_1/\rho_2$ is the mass-density ratio of the hot gas ahead of the front and behind it. Since $\mu > 1$, it follows from Ref. 1 that the front is unstable.

In practice, however, the FF is frequently stable, and to resolve the paradox it was resorted in many papers to viscosity, thermal conductivity, and diffusion,[3] and also to the influence of the bending of the front,[4] by making the formal substitution $v_1 \to v_1^0(1 - la_{xx}'')$, where l is an empirically chosen constant with the dimension of length.

In the present paper, remaining in the framework of Landau's purely hydrodynamic approach,[1] we attempt to take into account nonlinear effects. These were also taken into account earlier, particularly in a paper by Zel'dovich[5] (see also Refs. 6 and 7), where it was concluded on the basis of estimates that the nonlinearity exerts a stabilizing influence, but the estimates there are of qualitative character. It is shown in the present paper that if account is taken of quadratic corrections the FF instability is described by the equation

$$\psi_T'+\hat{H}\psi_X'=-\psi\psi_X', \tag{2}$$

which has heretofore not been derived. We introduce here the dimensionless variables $T = \gamma t$ and $X = kx$, while $\hat{H}$ denotes the Hilbert operator

$$\hat{H}f(X)=\pi^{-1}\fint (X'-X)^{-1}f(X')\,dX' \tag{3}$$

with an integral in the sense of principal value.

2. Equation (2) will be derived somewhat later. We note here that in view of its relative simplicity this equation may apparently be encountered also in other problems, and is by itself also of mathematical interest, since it is a member of a special group of equations containing the Hilbert operator $\hat{H}$. These equations arise in hydrodynamic problems with boundary conditions on a perturbed plane surface $z = a(t,x)$, if the flow of the medium in the volume is characterized by a potential φ that satisfies the Laplace equation $\nabla^2\varphi(t,x,z) = 0$. Examples of such equations are

$$\psi_t'+\hat{H}\psi_{xx}''=-\psi\psi_x, \tag{4}$$

$$\psi_t'+\hat{H}\psi_x'=\psi\hat{H}\psi, \tag{5}$$

$$\psi_t'+\hat{H}\psi_x'=\psi_x'\hat{H}\psi_x', \tag{6}$$

$$\psi_t'+\hat{H}\psi_x'=[\psi^2+(H\psi)^2]_x', \tag{7}$$

expressed in dimensionless variables. Recall that the known Koreteweg-de Vries equation $\psi_t' + \psi\psi_x' + \psi_{xxx}''' = 0$, which is widely used, is also partly similar to the group (2)–(6), but does not contain the Hilbert operator. Equation (4) is the well-known Benjamin-Ono equation, which has a particular solution in the form of a soliton. Equation (5), previously derived in Ref. 8, describes electron flows in a planar plasma layer and reduces for $w = \psi + i\hat{H}\psi$ to the form $iw_t' + w_x' = w^2/2$, which can be solved. Its simplified variant in the form $\psi_t' = \psi\hat{H}\psi$ was proposed in Ref. 9 as a model for a vortex in an incompressible liquid. The next equation, (6), was obtained by us earlier[10] and describes a classical tangential discontinuity (the Kelvin-Helmholtz instability), but with quadratic nonlinear corrections included. To solve (6) it is useful to introduce two new functions $\rho = 1 - \psi_x'$ and $v = -\hat{H}\psi_x'$, and then, with allowance for the identity $2\hat{H}(f\hat{H}f) = H(f)^2 - f^2$, we obtain from (6) the system

$$\rho_t'+(\rho v)_x'=0,\quad v_t'+vv_x'=\rho\rho_x', \tag{8}$$

which is fully integrable by the holograph method.[11] Finally, Eq. (7) which we include for subsequent comparison, can have a particular solution in the form of a traveling soliton:

$$\psi=\frac{\psi_0}{1+\theta^2},\quad \psi_0=\frac{-A}{t},\quad \theta=\frac{x-x_0}{\Delta},$$

$$x_0=A\ln(-t),\quad \Delta=-t>0, \tag{9}$$

with an amplitude that increases with time in the interval $-\infty < t < 0$. A general method of deducing new equations containing the Hilbert operator from known integrable ones, such as the Korteweg-de Vries, sine-Gordon, Kadomtsev-Petviashvili equations, and also from ordinary differential equations, was proposed in Ref. 12. Unfortunately, none of these methods is suitable for the solution of the FF equation (2), but a solution that increases with time can be sought in the form of an expansion in power of the amplitude by assuming

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$$\psi=\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3+\varepsilon^4\varphi_4+\ldots,$$

$$\varphi_n=\sum_{k=0}^{max} P_n^{(n-2k)}\sin[(n-2k)X]. \qquad (10)$$

Here $\varepsilon=\psi_0 e^T$, and P_n^m are polynomials that depend on the dimensionless time $T=\gamma t$, with $P_1^1=1$. Substitution of (10) in (2) leads to the recurrence relation

$$\sum_{k=1}^{n}[(n-k)P_n^k+\dot{P}_n^k]\sin kX=-\sum_{k=1}^{n-1}\varphi_{n-k}\varphi_{kx}', \qquad (11)$$

from which, in particular, we find that $P_n^n=(-nT/2)^{n-1}/n!$, but the derivation of a general equation for P_n^m is an extremely complicated problem. We present therefore only the first few terms of the series:

$$\varphi_1=\sin X,\quad \varphi_2=-\frac{T}{2}\sin 2X,$$

$$\varphi_3=\frac{3}{8}T^2\sin 3X+\frac{1-2T}{16}\sin X, \qquad (12)$$

and also

$$\varphi_4=P_4^4\sin 4X+P_4^2\sin 2X,$$

$$\varphi_5=P_5^5\sin 5X+P_5^3\sin 3X+P_5^1\sin X, \qquad (13)$$

where the polynomials are equal to

$$P_4^4=-\frac{1}{3}T^3,\quad P_4^2=(6T^2-4T+1)2^{-5},\quad P_5^5=\frac{125}{384}T^4, \qquad (14)$$

$$P_5^3=(-64T^3+12T^2+48T-33)2^{-8},$$

$$P_5^1=(-24T^3+74T^2-73T+105/4)2^{-10}. \qquad (15)$$

Note, however, that Eq. (2) allows correctly only for quadratic corrections.

3. We now derive Eq. (2). To this end we note that the flows outside the front are adiabatic, so that the density perturbations are connected with the pressure perturbations by the relations $\delta\rho=c_s^{-2}\delta p$, where c_s is the speed of sound, and if slow combustion with essentially subsonic velocities $v^0\ll c_s$ is considered we can, following Landau,[1] neglect completely the density perturbations, assuming the latter to be constant and unperturbed. The equations of motion of the gas take then the form

$$\operatorname{div}\mathbf{v}=0,\quad \rho[\mathbf{v}_t'+(\mathbf{v}\nabla)\mathbf{v}]=-\nabla p \qquad (16)$$

and the flow ahead of the flame front can be regarded as potential, putting $\mathbf{v}_1=\mathbf{v}_1^0+\boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_1=\nabla\varphi_1$ and $\nabla^2\varphi_1=0$. Behind the front, on the other hand, we assume that $\mathbf{v}_2=\mathbf{v}_2^0+\boldsymbol{\xi}_2+\boldsymbol{\eta}$, where $\boldsymbol{\xi}_2=\nabla\varphi_2$ and $\nabla^2\varphi_2=0$, and $\boldsymbol{\eta}$ is an "entropy wave" that satisfies the nonlinear equation

$$\operatorname{div}\boldsymbol{\eta}=0,\quad \boldsymbol{\eta}_t'+((\mathbf{v}_2^0+\boldsymbol{\xi}_2)\nabla)\boldsymbol{\eta}+(\boldsymbol{\eta}\nabla)(\boldsymbol{\xi}_2+\boldsymbol{\eta})=0, \qquad (17)$$

whereas for the potential terms we have the Bernoulli equations

$$\Delta\varphi_j=0,\quad \varphi_{jt}'+v_j^0\xi_j+{}^1/_2(\xi_j)^2=-\delta p_j/\rho_j. \qquad (18)$$

These equations must be supplemented with boundary conditions at $z=a(t,x)$:

$$\mathbf{n}(\mathbf{v}_j-\mathbf{v}_f)_a=v_j^0,\quad \mathbf{t}(\mathbf{v}_1-\mathbf{v}_2)_a=0,\quad (\delta p_1-\delta p_2)_a=0, \qquad (19)$$

which mean invariance of the normal and continuity the tangential velocity components, and also equality of the pressures on the front. In the four boundary conditions (19), $\mathbf{n}=(n_x,n_z)=(-\sin\chi,\cos\chi)$ denotes the normal to the front, $\mathbf{t}=(t_x,t_z)=(\cos\chi,\sin\chi)$ is a unit vector tangential to the front, and $\mathbf{v}_f=\mathbf{n}a_t'\cos\chi$ is the velocity of the front itself, with $\tan\chi=a_x'$.

The foregoing relations constitute the exact nonlinear formulation of our essentially subsonic problem, but the actual calculation can be performed only by retaining small terms of order not higher than quadratic. We note in this connection that it follows from the condition $\nabla^2\varphi_{1,2}=0$ that

$$\varphi_{1,2}(t,x,z)=\int\varphi_{1,2}(t,k)\exp(\pm|k|z+ikx)\,dk, \qquad (20)$$

and if we put for brevity $\alpha=(\xi_{1z})_0$ and $\beta=(\xi_{2z})_0$, we obtain, by expanding in terms of the small amplitude $a\ll\lambda$, the relations

$$(\xi_{1x})_0=\hat{H}\alpha,\quad (\xi_{2x})_0=-\hat{H}\beta,\quad (\xi_{1z})_a=\alpha-a\hat{H}\alpha_x',$$
$$(\xi_{2z})_a=\beta+a\hat{H}\beta_x',\quad (\xi_{1x})_a=\hat{H}\alpha+a\alpha_x',\quad (\xi_{2x})_a=-\hat{H}\beta+a\beta_x'. \qquad (21)$$

To shorten the subsequent calculations we separate right away the growing branch of the perturbations, using for the linear factors in the quadratic corrections the linear equations

$$a_\tau'=-\nu\hat{H}a_x',\quad a_x'=\nu^{-1}\hat{H}a_\tau', \qquad (22)$$

where $\tau=v_1^0\mu t$. The first three boundary conditions (19) yield then

$$\alpha/v_1^0\mu=a_\tau'+\nu(aa_x')_x'+(a_x')^2/2\mu,$$
$$(\eta_z)_0=\alpha-\beta-v_1^0(\mu-1)[aa_{xx}''+(a_x')^2/2], \qquad (23)$$
$$(\eta_x)_0=\hat{H}(\alpha+\beta)-v_1^0(\mu-1)[a_x'+(1+\mu\nu)aa_{x\tau}'']$$

and substitution of the entropy components in (17) yields an expression for the quantity $G=(\beta_\tau'+\hat{H}\beta_x')/v_1^0\mu$, which can conveniently be written in the form

$$G=\frac{\alpha_\tau'-\hat{H}\alpha_x'}{v_1^0\mu}+\left(1-\frac{1}{\mu}\right)a_{xx}''+\left[\frac{\mu+\lambda^2}{2\nu}\sigma+\frac{1-\mu}{2\mu}(a_x')^2+\frac{1-\mu}{\nu}aa_{\tau\tau}''\right]_\tau', \qquad (24)$$

where we put for brevity $\sigma=(a_\tau')^2+(\nu a_x')^2$, and also $\lambda=(1+\mu\nu)/(1-\nu)$. On the other hand, from the last condition $(\delta p_1-\delta p_2)_a=0$ of (19), we obtain for the same quantity G the expression

$$G=\frac{1}{v_1^0\mu}(\hat{H}\alpha_x'-\mu\alpha_\tau')+\hat{H}\left(\frac{\mu-\lambda^2}{2}\sigma+2\frac{1+\mu\nu}{\nu}aa_{\tau\tau}''\right)_x', \qquad (25)$$

so that by equating (24) to (25) and substituting the value of α from (23) we obtain for the boundary $a(\tau,x)$ an equation that can expediently be written in the form

$$(1+\mu)a_{\tau\tau}''-2\hat{H}a_{x\tau}''+\left(1-\frac{1}{\mu}\right)a_{xx}''$$
$$=\hat{H}A_x'-\hat{L}[2\mu\nu(aa_x')_x'+B], \qquad (26)$$

where we have introduced for brevity the operator $\hat{L}$ and the notation

$$\hat{L}=\frac{\partial}{\partial\tau}+\nu\hat{H}\frac{\partial}{\partial x},\quad A=\mu\sigma+\left(1+\frac{\nu}{\mu}\right)(a_x')^2,$$

$$B=\frac{\mu+\lambda^2}{2\nu}\sigma+\left(\nu+\frac{1}{\mu}-\mu\nu\right)(a_x')^2. \tag{27}$$

4. The resultant equation (26) can be simplified. We note for this purpose that its left-hand side contains the operator of the usual Landau linear theory.[1] Separating therefore the growing branch of the perturbation, we can assume the left-hand side to be equal to

$$\left[(1+\mu)\frac{\partial}{\partial\tau}-\left(1+\mu+\frac{2}{\nu}\right)\nu\hat{H}\frac{\partial}{\partial x}\right]\hat{L}a\approx-2N\hat{H}\hat{L}a_x', \tag{28}$$

where $N=1+(1+\mu)\nu$. If we now introduce in place of $a(\tau,x)$ the function $b(\tau,x)=a+(\mu\nu/N)\hat{H}(aa_x')$, we obtain the equation

$$\hat{L}b_x'=-\frac{1}{2N}(A_x'+\hat{L}\hat{H}B)_{a\to b}, \tag{29}$$

where we can put $b_\tau'=-\nu\hat{H}b_x'$ in the right-hand quadratic terms, so that actually only the function b_x' is contained here. We can therefore replace further, to the same quadratic accuracy, b_x' by the new function $\Psi=b_x'+\hat{H}(B/2N)_{a\to b}$ and arrive at the final equation for the flame front:

$$\hat{L}\Psi=\Psi_\tau'+\nu\hat{H}\Psi_x'=-\frac{1}{2N}(A_x')_{a\to b,\,b_x'\to\Psi}$$
$$=-\frac{1}{2N}\left[\left(1+\frac{\nu}{\mu}+\mu\nu^2\right)\Psi^2+\mu\nu^2(\hat{H}\Psi)^2\right]'. \tag{30}$$

Introducing in place of τ, x, and Ψ the new dimensionless variables

$$X=kx,\quad k>0,\quad T=k\nu\tau=k\nu\mu v_1^0t,\quad \psi=\Psi/\nu,$$

we obtain the equation

$$\psi_T'+\hat{H}\psi_X'=-{}^1/_2[C_1\psi^2+C_2(\hat{H}\psi)^2]_X', \tag{31}$$

where the constants $C_{1,2}$ are equal to

$$C_1=\frac{1+\nu(\nu\mu+\mu^{-1})}{1+\nu(1+\mu)},\quad C_2=\frac{\nu^2\mu}{1+\nu(1+\mu)}. \tag{32}$$

For $C_1=C_2$ we would obtain an equation of type (7) with a solution of type (9), while at $C_1=-C_2$, using the identity $(\hat{H}\psi)^2-\psi^2=2\hat{H}(\psi\hat{H}\psi)$ and putting $\psi=\hat{H}\Phi_x'/C_2$, we would obtain the equation

$$\Phi_T'+\hat{H}\Phi_X'=\Phi_X'\hat{H}\Phi_X' \tag{33}$$

of type (6), which describes a tangential discontinuity with a fully integrable solution of type (8).

Unfortunately, the equation (31) for the flame front, with constants (32), is unsuitable for either case, and we were unable to solve Eq. (31) for arbitrary constants $C_{1,2}$.

We confine ourselves therefore to the case of small $\nu\ll1$, when the densities $\rho_{1,2}$ are close enough. In practice it is just this case, called isothermal, which corresponds to the slow combustion regime. In this simplified variant one can put $C_1=1$ and $C_2=0$, and Eq. (31) takes the form (2)

$$\psi_T'+\psi\psi_X'+\hat{H}\psi_X'=0, \tag{34}$$

which was in fact considered by us above. We call this the nonlinear flame-front equation (NFFE). In view of its relative simplicity one can expect it to play a fundamental role also in certain other problems.

5. Let us examine this equation in greater detail and show that it can lead to a cellular structure of the flame. Note that at $\nu\ll1$ it can be approximately assumed that $\nu=(\rho_1-\rho_2)/2\rho_1>0$ and $b=a$, so that in terms of the initial variables t and x we have for the FF and NFFE:

$$\hat{\mathscr{L}}\psi=-\psi\psi_x',\quad \hat{\mathscr{L}}\psi=V_0^{-1}\psi_t'+\hat{H}\psi_x',\quad V_0=v_1^0\nu,\quad a_x'=\nu\psi, \tag{35}$$

which, as already noted, we were unfortunately unable to solve exactly. In practice, however, there is no need for this, for in their derivation we took into account correctly and fully only corrections quadratic in the amplitude, so that allowance for higher correction terms would be unjustified. To the same accuracy, for example, a solution periodic in the coordinate x for Eq. (10) leads to the following shape of the front:

$$a=\nu\int\psi\,dx=\frac{\nu}{k}\varepsilon\left(-\cos kx+\frac{\varepsilon}{4}T\cos 2kx+\ldots\right),\quad \varepsilon=\psi_0e^T. \tag{36}$$

As $T\to-\infty$ this perturbation is exponentially small, and the correction to the function $-\cos kx$ is maximal at $T=-1$; it is easily seen that this correction leads to a flattening of the crests of the function $-\cos kx$ and simultaneously to a deepening and narrowing of the troughs near its minima, so that qualitatively we get a cellular structure of the flame. Let us consider supplementary examples, putting

$$\psi=\psi_1+\psi_2+\ldots,\quad \hat{\mathscr{L}}\psi_1=0,\quad \hat{\mathscr{L}}\psi_2=-\psi_1\psi_{1x}'. \tag{37}$$

Landau[1] considered for the linear approximation $\hat{\mathscr{L}}\psi_1=0$ only a solution in the form $\psi_1\propto e^{\gamma t}\sin kx$, but here we can have also another simpler soliton-type solution with a Lorentz profile:

$$\psi_1=\psi_0(t)/(1+\theta^2),\quad \theta=x/\Delta,\quad \psi_0(t)=c/\Delta,$$
$$\Delta=-V_0t>0,\quad c=2a_0/\pi\nu, \tag{38}$$

and this solution corresponds to a steplike front shape:

$$a_1(t,x)=\frac{2}{\pi}a_0\,\mathrm{arctg}\frac{x}{\Delta(t)}, \tag{39}$$

which takes on values $\mp a_0$ far from a transition region having a width of order $\delta x\sim\Delta=-V_0t\geqslant0$. Out of a set of such steps it is easy to construct, by superposing linear solutions, a set of elementary platforms with gaps between them. Such solutions are apparently more typical of a flame than the Landau sinusoidal solutions.

Let us consider one such platform (in other words, "cell"), described in the linear approximation by the equations

$$a_1=\frac{a_0}{\pi}(\alpha_+-\alpha_-),\quad \alpha_\pm=\mathrm{arctg}\frac{x\pm x_0}{\Delta},\quad \Delta=-V_0t>0,$$
$$\psi_1=\frac{1}{\nu}a_{1x}'=AD_0,\quad A=\frac{a_0}{2\pi\nu},\quad D_0=\delta_+^{+}+\delta_+^{-}-\delta_-^{+}-\delta_-^{-}, \tag{40}$$

where $\delta_q^p=[\Delta+pi(x+qx_0)]^{-1}, p=\pm1, q=\pm1$. These functions δ_q^p, and also their powers (but not products) are solutions of the equation $\hat{\mathscr{L}}(\delta)^n = 0$, so that ψ_1^2 should be written in the form

$$\begin{aligned}
&\psi_1^2 = A^2 D_0^2 = A^2 (D_1 + f_2 D_2 + f_3 D_3),\\
&D_1 = (\delta_+^{\ +})^2 + (\delta_+^{\ -})^2 + (\delta_-^{\ +})^2 + (\delta_-^{\ -})^2,\\
&D_2 = \delta_+^{\ +} + \delta_+^{\ -} + \delta_-^{\ +} + \delta_-^{\ -},\\
&f_2 = \frac{1}{\Delta} - \frac{\Delta}{\Delta^2 + x_0^2} = -\frac{\partial}{2\partial\Delta}\ln(1+\theta_0^2),\\
&D_3 = \delta_+^{\ +} - \delta_+^{\ -} + \delta_-^{\ -} - \delta_-^{\ +},\\
&f_3 = \frac{ix_0}{\Delta^2 + x_0^2} - \frac{i}{x_0} = -\frac{i\partial}{\partial\Delta}\left(\frac{1}{\theta_0} + \operatorname{arctg}\theta_0\right),
\end{aligned} \tag{41}$$

where $\hat{\mathscr{L}} D_{1,2,3} = 0$ and $\theta_0 = x_0/\Delta$, so that the functions $f_{2,3}(t)$ do not depend on the coordinate x. For the quadratic correction (37) we have therefore

$$\hat{\mathscr{L}}\psi_2 = -{}^1/_2(\psi_1^2)_x' = -{}^1/_2 A^2 (D_{1x}' + f_2 D_{2x}' + f_3 D_{3x}'), \tag{42}$$

and, recognizing that $\hat{\mathscr{L}}[f(t)D_j] = D_j \partial f/\partial t V_0 = -D_j \partial f/\partial\Delta$, we get

$$\begin{aligned}
&\psi_2 = {}^1/_2 A^2 [D_{1x}'\Delta - {}^1/_2 D_{2x}' \ln(1+\theta_0^2) - iD_{3x}'(1/\theta_0 + \operatorname{arc\,tg}\theta_0)],\\
&a_2 = \nu\int \psi_2\, dx = \frac{\nu}{2} A^2 \left[D_1\Delta - \frac{D_2}{2}\ln(1+\theta_0^2) - iD_3\left(\frac{1}{\theta_0} + \operatorname{arctg}\theta_0\right)\right].
\end{aligned} \tag{43}$$

In addition, the following expressions are valid:

$$\begin{aligned}
&\Delta^2 D_1 = (1+\cos 2\alpha_+)\cos 2\alpha_+ + (1+\cos 2\alpha_-)\cos 2\alpha_-,\\
&\Delta D_2 = 2 + \cos 2\alpha_+ + \cos 2\alpha_-, \quad i\Delta D_3 = \sin 2\alpha_+ - \sin 2\alpha_-,
\end{aligned} \tag{44}$$

so that we get ultimately for the flame front the equation

$$\begin{aligned}
a/a_0 = {}&(a_1 + a_2 + \ldots)/a_0 = \pi^{-1}(\alpha_+ - \alpha_-) - \tilde{A}\{(1+\theta_0 \operatorname{arctg}\theta_0)\\
&\times(\sin 2\alpha_+ - \sin 2\alpha_-) + \theta_0[(1 + {}^1/_2\cos 2\alpha_+ + {}^1/_2\cos 2\alpha_-)\\
&\times\ln(1+\theta_0^2) - (1+\cos 2\alpha_+)\cos 2\alpha_+ - (1+\cos 2\alpha_-)\cos 2\alpha_-]\},
\end{aligned} \tag{45}$$

where $\tilde{A} = a_0/8\pi^2\nu x_0$ is a constant. Figures 1 and 2 show separately the quantities a_1/a_0 and a_2/a_0 described by these equations, in the time interval $-\infty < t < 0$. At the instant $t = 0$ they acquire singularities that prevent consideration of the succeeding positive instants of time $t > 0$. Ths can be done, however, in the two-dimensional case.

6. Before proceeding to the two-dimensional case, we note that the one-dimensional equation (35) can be trans-

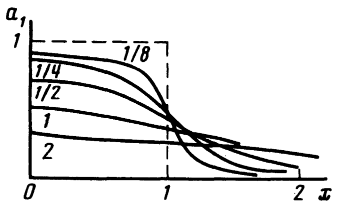

FIG. 1. Time evolution of the flame cell (4) with chosen parameters $x_0 = 1$ and $\tilde{A} = 0.1$. The numbers on the curves indicate the time in units of $\Delta = -V_0 t$.

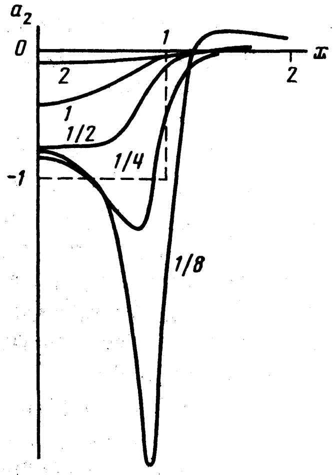

FIG. 2. Growth of the quadratic correction (43) that must be added to the linear solution (40) describing a single cell of the flame.

formed into

$$V_0^{-1}\frac{\partial a}{\partial t} - \int |k| a_k e^{ikx}\, dk = -\frac{1}{2\nu}\left(\frac{\partial a}{\partial x}\right)^2 + F(t); \tag{46}$$

it is assumed that

$$a(t,x) = \int a_k(t)\exp(ikx)\, dk.$$

It is easily seen that a two-dimensional generalization of this equation is

$$V_0^{-1}\frac{\partial a}{\partial t} - \int |\mathbf{k}| a_{\mathbf{k}} e^{i\mathbf{kr}}\, dk_x\, dk_y = -\frac{1}{2\nu}(\nabla a)^2 + F(t); \tag{47}$$

it is assumed that

$$a(t,x,y) = \int a_{\mathbf{k}}(t)\exp(ik_x x + ik_y y)\, dk_x\, dk_y.$$

We attempt next to solve this equation not by expansion in the amplitude, but by expansion in the eigenvectors of a two-dimensional periodic lattice. Such a procedure was used in Ref. 13 to construct a lattice made up of hexagons for parametric buildup of waves on the surface of an oscillating liquid.

Using the same procedure, we assume in our problem that

$$a(t,x,y) = A(t)R(x,y), \qquad R = \sum_1^3 c_j, \qquad c_j = \cos(|k|\mathbf{n}_j\mathbf{r}), \tag{48}$$

where $\mathbf{n}_j$ stands for three unit planar vectors equal respectively to

$$\begin{aligned}
&\mathbf{n}_1 = \mathbf{e}_y, \quad \mathbf{n}_2 = -\frac{3^{1/2}}{2}\mathbf{e}_x - \frac{1}{2}\mathbf{e}_y, \quad \mathbf{n}_3 = \frac{3^{1/2}}{2}\mathbf{e}_x - \frac{1}{2}\mathbf{e}_y,\\
&\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 = 0.
\end{aligned} \tag{49}$$

We obtain for such a sum of the three cosines the squared gradient

$$\begin{aligned}
(\nabla R)^2 = {}^1/_2 k^2 (R+Z), \quad Z = {}&3 - \cos 2\varphi_1 - \cos 2\varphi_2 - \cos 2\varphi_3\\
&-\cos(\varphi_1 - \varphi_2) - \cos(\varphi_2 - \varphi_3) - \cos(\varphi_2 - \varphi_3),
\end{aligned} \tag{50}$$

where $\varphi_j = |k|\mathbf{n}_j\cdot\mathbf{r}$. The term Z contains here only "nonresonant" spatial harmonics, and can therefore be approximately disregarded. We get then for the amplitude

$$A'(t)=\gamma A(1-|k|A/4\nu), \quad A(t)=(4\nu/|k|)/[1+e^{\gamma(t_0-t)}], \quad (51)$$

where t_0 is a certain integration constant. In the limit as the time $t\to-\infty$ this perturbation vanishes, $A\to 0$, and as $t\to+\infty$ the amplitude takes on a stationary value $A = 4\nu/|k|$, and this yields stabilized honeycomb-like hexagonal cells on the flame front. Equation (51) coincides with Zel'dovich's results[5] (see also Refs. 6 and 7) obtained from qualitative, essentially dimensional, considerations.

The theory developed here, in which full account is taken of all the quadratic corrections, permits thus a more rigorous explanation of the experimentally observed cellular structure of a flame front even without introducing dissipative terms with viscosity, heat conduction, or diffusion.

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