Torons and breaking of chiral symmetry in supersymmetric gluodynamics

A. R. Zhitnitskiĭ

Nuclear Physics Institute, Siberian division, Academy of Sciences of the USSR
(Submitted 30 May 1988)
Zh. Eksp. Teor. Fiz. **95**, 24–34 (January 1989)

A novel class of self-dual solutions in $SU(2)$ gauge theories is considered. The solution has half-integral topological charge $Q = 1/2$, an action $S = 8\pi^2 Q/g^2$, and is characterized by a conic singularity. The contribution of the corresponding fluctuations to the gluino condensate $\langle\bar\psi\psi\rangle$ in supersymmetric gluodynamics is calculated. The result turns out to be finite indicating spontaneous breaking of chiral symmetry. The relation to analogous calculations in the supersymmetric $O(3)$ σ-model is discussed. Various ways of describing fluctuations with fractional Q, such as analytic continuation to a complex space containing several Riemann sheets or passing to a description in terms of orbifolds, are considered.

1. INTRODUCTION

The aim of the present work is the description of fluctuations with fractional topological charge Q in gauge theories and an analysis of its physical consequences on the example of 4-dimensional supersymmetric gluodynamics with gauge group $SU(2)$.

The analogous problem for the 2-dimensional supersymmetric $O(3)$ σ-model was considered in Ref. 1. It was shown in that paper that fluctuations with half-integral topological charge give a finite contribution to the fermion condensate $\langle\bar\psi\psi\rangle$ and thus ensure spontaneous breaking of chiral symmetry in the model. It will be seen below that an analogous effect arises in supersymmetric gluodynamics as well. Namely, it will be shown that fluctuations with $Q = 1/2$ make a nonzero contribution to the gluino condensate $\langle\bar\psi\psi\rangle\neq 0$, and that use of the quasi-classical approximation is justified parametrically.

Before describing in detail the $Q = 1/2$ fluctuation we recall that the integral nature of Q for instantons[2,3] is connected with the compactification of the physical space to a sphere, i.e., with the identification of all infinitely distant points. A different choice of boundary conditions could result in fractional topological charges. In particular, in gluodynamics with $SU(2)$ gauge group the introduction of so-called twisted boundary conditions[4] makes it possible to obtain solutions of the classical equations—torons,[5] possessing $Q = 1/2$ and action $S = 4\pi^2/g^2$. The fundamental reason for the admissibility of such solutions is related to the nontriviality of the first homotopy group $\pi_1(SU(2)/Z_2) = Z_2$, which in turn is due to the existence of the elements of the center $Z_2 = \exp\{i\pi\sigma_3 k\}$, $k = 0$, 1, belonging to the group and leaving invariant the gluon fields in the adjoint representation:

$$A_\mu' = Z_2 A_\mu Z_2^{-1} = A_\mu.$$

Thus in fact the group is $SU(2)/Z_2$ and the nontriviality $\pi_1(SU(2)/Z_2) = Z_2$ indicates the existence of new (non-instanton) solutions of the classical equations.

Beside twisted boundary conditions there exist other ways of describing fluctuations with fractional Q: analytic continuation to a complex space containing several Riemann sheets, or passage to a description in terms of orbifolds. It was the latter approach that was exploited in the analysis of the $O(3)$ σ-model.[1]

In this article the self-duality equations of the gauge theory will be formulated in terms analogous to the corresponding Cauchy-Riemann conditions for the $O(3)$ σ-model. It is in these terms that we are able to describe the self-dual solution with $Q = 1/2$ and to follow fully the logic of Ref. 1 in the calculation of the gluino condensate $\langle\bar\psi\psi\rangle$.

A few words about supersymmetric gluodynamics proper. There exist a number of serious arguments[6–11] to the effect that indeed in the model spontaneous breaking of discrete chiral symmetry occurs and a nonzero condensate $\langle\bar\psi\psi\rangle\neq 0$ arises. We recall that the theory possesses naive $U(1)$ symmetry: $\psi^a \to \exp\{i\alpha\}\psi^a$, broken by the anomaly $\partial_\mu a_\mu \sim G\tilde G$. However in the process the discret symmetry $Z_2\times Z_2$ survives. The nonvanishing of $\langle\bar\psi\psi\rangle$ signals spontaneous breaking of this discrete symmetry down to Z_2, in accord with the Witten[6] index, which equals two.

Further, the arguments in Refs. 7 and 8 are based on calculating the instanton contribution to the correlator

$$\Pi = \langle\bar\psi\psi(x),\ \bar\psi\psi(0)\rangle = \text{const}.$$

Making use of clustering as $x\to\infty$ one might conclude that $\Pi = \langle\bar\psi\psi\rangle^2 \neq 0$. These arguments, however, do not constitute a proof since it was clarified[9] that theoretically uncontrollable contributions from large distances may in principle compensate the contribution from a small size instanton.

Let us note that although the instanton contribution to Π is nonzero, the corresponding contribution directly to $\langle\bar\psi\psi\rangle$ does vanish, and therefore a direct calculation of $\langle\bar\psi\psi\rangle$ using instantons is impossible. The reason for this is trivial and due to the existence of four gluino zero modes in accord with the index theorem. This results in the instanton solution changing the chiral charge by four units and the corresponding vacuum transition being necessarily accompanied by the creation of precisely two gluino pairs $\langle\bar\psi\psi,\bar\psi\psi\rangle$.

We note also that a consistent calculation of $\langle\bar\psi\psi\rangle$ was carried out in Ref. 10. However, the corresponding very indirect calculation cannot answer the question about the fluctuations that determine this condensate.

And, lastly, a final argument which we shall consider in detail, connected with direct evaluation[11] of $\langle\bar\psi\psi\rangle$ on the basis of the toron solution of Ref. 5. In this case the topological charge Q equals 1/2 and the number of gluino zero modes equals two (and not four as for the instanton with $Q = 1$) in accord with the index and the expression for the anomaly. Therefore the toronic vacuum transition changes the chiral

 0038-5646/89/010014-06$04.00

charge by two units and is accompanied by the production of the pair $\bar{\psi}\psi$, as is verified by the explicit calculation in Ref. 11.

We note that the toron solution[5] is defined in a box of size L and uniformly "smeared out" over the entire volume: $G_{\mu\nu} \sim 1/L^2$. However the size of $\langle\bar{\psi}\psi\rangle$ is finite as $L\to\infty$. It is clear that the calculation in Ref. 11 is only of heuristic value since literally the standard quasi-classical approximation used in that paper is not valid in the strong coupling regime, $g(L\to\infty)\to\infty$. In addition the solution of Ref. 5 exists only for definite ratios of the lengths of the sides of the box.

In this paper we propose a different formulation of the toron solution[1] influenced by the corresponding analysis[1] of the $O(3)$ σ-model. In place of the field $G_{\mu\nu}\sim 1/L^2$ "smeared out" over the entire volume we obtain in this case, as in Ref. 1, a solution defined in the infinite space R^4 with $G_{\mu\nu}{}^2$ concentrated in a small neighborhood $\Delta\to 0$. In consequence the quasi-classical evaluation of $\langle\bar{\psi}\psi\rangle$ discussed below becomes fully justified in view of asymptotic freedom: $g(\Delta)\to 0$. We note that, just as in Ref. 11, the value of $\langle\bar{\psi}\psi\rangle$ is independent of the regulator Δ and determined by the only dimensional parameter Λ of the theory; in the process the correct renormalization theory relation is automatically reproduced.

It is clear from what has been said above that the here-obtained assertion about $\langle\bar{\psi}\psi\rangle\neq 0$ is not new. The purpose of this paper is different, namely to find a self-dual solution with $Q=1/2$ and to describe methods for using this solution on the example of $SU(2)$ supersymmetric gluodynamics. It is to be hoped that analogous solutions exist in a broad class of theories—whenever the group π_1 is nontrivial. In such cases there should exist point singularities with nontrivial values for the Wilson loop, see Ref. 1.

Finally, a last remark about the parameter $\Delta\to 0$ figuring in the definition of the classical self-dual solution both in this paper and in Ref. 1. As will be seen below, the solution is determined as the limit for $\Delta\to 0$, with Δ set equal to zero only at the end of the calculations. Were we to set $\Delta=0$ at the beginning we would obtain the trivial result $G_{\mu\nu}=0$. Such a situation, connected with the passage to the limit, is apparently common in the description of solutions with fractional Q. For example, to 't Hooft's toron solution[5] corresponds the field strength $G_{\mu\nu}=1/L^2$, which equals zero for $L\to\infty$, yet the condensate $\langle\bar{\psi}\psi\rangle$ turns out to be finite.[11]

The geometric interpretation of the parameter $\Delta\to 0$ is discussed in detail in the Appendix of Ref. 1 and is connected with the regularization (referred to in the literature as blowing up) of the fixed points of the orbifold—a singular manifold of a special kind. Since the introduction of the parameter Δ is not new in principle in comparison with Ref. 1, we shall not discuss it in detail but refer the reader to the above-mentioned paper.

The plan of this paper is as follows. In Sec. 2 we obtain the solution of duality equations, which is expressed in terms of an arbitrary analytic function. This makes it possible to follow the logic of Ref. 1, redefine the theory on two Riemann sheets and obtain a self-dual solution with $Q=1/2$. In Sec. 3 we formulate the criteria for the selection of modes that must be taken into account in calculating the corresponding functional integral. In Sec. 4 as an application of above-mentioned ideas the toron measure is calculated in supersymmetric gluodynamics and it is shown that the condensate $\langle\bar{\psi}\psi\rangle$ appears due to fluctuations with $Q=1/2$.

2. SELF-DUAL SOLUTION WITH $Q=1/2$

We start from the axially-symmetric Witten ansatz[12] for the gauge fields $A_\mu^a(r,t)$:

$$A_0{}^a=A_0n^a,$$
$$A_i{}^a=\varepsilon^{iak}n^k\frac{1+\Phi_2}{r}+(\delta^{ai}-n^an^i)\frac{\Phi_1}{r}+n^an^iA_1, \qquad (1)$$
$$n^a=x^a/r,\ r=(x_ix_i)^{1/2}.$$

here A_0,A_1,Φ_1,Φ_2 are functions of r and t. Substituting the ansatz (1) in the expression for the field strength

$$G_{\mu\nu}{}^a=\partial_\mu A_\nu-\partial_\nu A_\mu-\varepsilon^{abc}A_\mu{}^bA_\nu{}^c,$$

we arrive at the following duality equations, written in two-dimensional notation:

$$D_\mu\Phi=i\varepsilon_{\mu\nu}D_\nu\Phi,\ (D_\mu\Phi\equiv\partial_\mu+iA_\mu)\Phi,$$
$$\Phi\equiv\Phi_1-i\Phi_2,\quad \mu=0,1,\quad \partial_\mu\partial_\mu\ln\frac{|\Phi|}{r}=\frac{|\Phi|^2}{r^2}. \qquad (2)$$

Here and below $\partial_\mu=(\partial_0,\partial_1)=(\partial/\partial t,\partial/\partial r)$ are the derivatives acting in the two-dimensional space (t,r). In this case, as in the analysis of two-dimensional theories, it is convenient to pass to complex notation[12]:

$$z=r+it,\ \partial=\partial/\partial z={}^1/_2(\partial_1-i\partial_0),$$
$$A=A_1-iA_0,\ \bar{z}=r-it,\ \bar{\partial}={}^1/_2(\partial_1+i\partial_0), \qquad (3)$$

and express the solution of the duality equations (2) in terms of an arbitrary analytic function $g(z)$[12]:

$$\Phi=\Phi_1-i\Phi_2=fe^\Psi,\quad A=A_1-iA_0=-2i\partial\Psi=-2i\partial\ln\Phi,$$
$$f=\frac{dg}{dz},\quad \Psi=\ln\left(\frac{z+\bar{z}}{1-\bar{g}g}\right),\quad g=g(z). \qquad (4)$$

In order that the solution be finite as $r\to 0$ it is necessary to impose the additional regularity requirement on Ψ, which in turn leads to the restriction on $g(z)$:

$$|g|=1 \text{ for } r=0,\ |g|<1 \text{ for } r>0. \qquad (5)$$

In this way expressions (4) for $\Phi=\Phi_1-i\Phi_2$ and $A=A_1-iA_0$ together with the additional condition (5) solve the posed problem. Namely, the self-duality conditions for gauge theories are formulated in the same terms of analyticity of $g(z)$ as in the $O(3)$ σ-model[3] with the additional requirement (5). The last circumstance is related to the fact that, in contrast to the two-dimensional σ-model, the theory is defined only on the half-plane $\mathrm{Re}\,z\geqslant 0$.

In order to find the analytic function $g(z)$ corresponding to $Q=1/2$ it is necessary to consider the explicit expression for the topological charge expressed in two-dimensional notation[12]:

$$Q=\frac{1}{2\pi}\int_0^\infty dr\int_{-\infty}^\infty dt\left[\frac{1}{2}\varepsilon_{\mu\nu}F_{\mu\nu}+i\varepsilon_{\mu\nu}\partial_\mu(\Phi D_\nu\Phi)\right]$$
$$=\frac{1}{2\pi i}\oint ds\frac{d}{ds}\ln f,\quad F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu. \qquad (6)$$

It is seen from expression (6) that the topological charge is determined by the phase acquired by the analytic function f on passing round the contour that encompasses the physical space $\mathrm{Re}\,z\geqslant 0$.

We are now prepared, following the logic of Ref. 1, to describe the self-dual solution defined on two Riemann sheets and possessing $Q = 1/2$. As for the $O(3)$ σ–model, the solution is defined with the help of the limit $\Delta \to 0$, corresponding to regularization of the fixed points of the orbifold (see the Appendix of Ref. 1). Keeping in mind what has been said above we shall write the analytic function $g(z)$, defined on two Riemann sheets, satisfying the finiteness requirement (5) and ensuring $Q = 1/2$ in the following form:

$$g(z)=\lim_{a+\bar{a}\to 0}\left(\frac{a-z}{\bar{a}+z}\right)^{3/2}=\left[\frac{\Delta-r+i(t_0-t)}{\Delta+r+i(t-t_0)}\right]^{3/2},$$
$$a=\Delta+it_0,\quad \mathrm{Re}\, a=\Delta>0, \tag{7}$$
$$f=\frac{dg}{dz}=-\frac{3}{2}\frac{a+\bar{a}}{(\bar{a}+z)^2}\left(\frac{a-z}{\bar{a}+z}\right)^{1/2}.$$

A characteristic feature of the solution (7), just as in the $O(3)$ σ–model, is the presence of the cut. Here $\mathrm{Im}\, a = t_0$ has the physical significance of localizing the pseudoparticle solution along the time axis.

In the following we set without loss of generality $t_0 = 0$. We note that the requirement $\mathrm{Re}\, a = \Delta > 0$ automatically ensures the regularity condition (5). Moreover, in going round the contour encompassing the physical space $\mathrm{Re}\, z \geqslant 0$ the function f acquires a phase equal to π, which in accordance with (6) means that $Q = 1/2$.[2] Let us note that by setting $\Delta = 0$ at the end of calculations we restore single-valuedness of the potential A_μ^a on one physical sheet; $G_{\mu\nu}^2$ on the other hand is single valued for any Δ.

The resultant solution will be called a toron, just like the analogous solution in the $O(3)$ σ-model of Ref. 1 (see footnote 1). The important difference between the toron solution and solutions with integer Q is due to the presence of the cut. However physical gauge-invariant quantities are single valued and continuous across the cut, so that in essence the cut must be understood as the boundary of the physical space on which the values of all functions are defined. Consequently the solution is defined on a compact manifold with a boundary in contrast to, say, the instanton which is defined on S^4—a compact manifold without boundary. This fact may be realized[3] more explicitly by means of the conformal transformation which takes the cut z-plane into a disc of radius R, with the physical space understood as the limit $R \to \infty$. We note that such an interpretation is possible because of conformal symmetry of the Lagrangian. Solutions of classical equations, of course, "respect" this symmetry.

We return to relation (7) and describe the qualitative behavior of $G_{\mu\nu}^2$ as a function of r and t. Explicit calculations show that $G_{\mu\nu}^2$ is well-behaved at $r = 0$ and for $|z| \to \infty$. Namely

$$G_{\mu\nu}^2(r=0)\sim(1+t^2)^{-4},\quad G_{\mu\nu}^2(|z|\to\infty)\sim|z|^{-8}$$

(for technical details see Ref. 13). Here and in the following the coordinates are made dimensionless using Δ, so that in essence the described object has a small size Δ, degenerating in the limit $\Delta \to 0$ into a point defect. It is important to note, however, that our regularization of this point singularity is such that the self-duality equations are satisfied also for finite values of Δ.

Explicit calculations show further that in the neighborhood $z \to 1$ $G_{\mu\nu}^2$ has an integrable singularity:

$$G_{\mu\nu}^2(z\to 1)\sim[(z-1)(\bar{z}-1)]^{-1/2}. \tag{8}$$

This property will be important in the following section, in the analysis of zero modes in the toron field.

We note that although equality of (4) to the analytic function $g(z)$ (7) ensures solution of self-duality equations having $Q = 1/2$, the resultant expressions for R and Φ are too unwieldy for further analysis. Some simplification can be achieved by a convenient gauge choice. To this end we recall that the duality equations (2) have the $U(1)$-gauge freedom:

$$\Phi'=\Phi e^{i\alpha},\quad A_\mu'=A_\mu-\partial_\mu\alpha,$$

which may be expressed in terms of an arbitrary analytic function $h(z)$ as follows[12]:

$$f'(z)=f(z)h(z),\quad \Phi'=\Phi h(z)/|h(z)|=\Phi e^{i\alpha(z,\bar{z})},$$
$$\Psi'=\Psi-{}^1/_2\ln h\bar{h},\quad A'=A+i\partial\ln|h|^2. \tag{9}$$

One may choose $h(z)$ such that the field A_μ^a is represented in the form

$$A_\mu{}^a=-\bar{\eta}_{\mu\nu}{}^a\partial_\nu\ln P(r,t),$$
$$\bar{\eta}_{0i}{}^a=\delta^{ai},\quad \bar{\eta}_{i0}{}^a=-\delta^{ai},\quad \bar{\eta}_{ij}{}^a=-\varepsilon^{aij}, \tag{10}$$
$$\bar{\eta}_{\mu\nu}{}^a=-{}^1/_2\varepsilon_{\mu\nu\lambda\sigma}\bar{\eta}_{\lambda\sigma}{}^a,\quad \varepsilon_{0123}=1.$$

Here $P(r,t)$ depends on r,t; $\bar{\eta}^a_{\mu\nu}$ are 't Hooft's symbols.[4] If such a choice is possible then, as we show in Sec. 3, the problem of determining zero modes in the field of a toron is substantially simplified. In complex notation condition (10) is expressed as

$$(\Phi'-i)/r=A'=2i\partial\ln P(r,t). \tag{11}$$

One may explicitly verify that the choice of the gauge function $h(z)$ in the form

$$h(z)=[i+g(z)]^{-2} \tag{12}$$

exactly ensures that relations (10) and (11) are satisfied with the following expression for P:

$$P^{-1}=\frac{z+\bar{z}}{1-\bar{\sigma}\sigma}|i+g|^2=e^{\psi'}=e^{\psi}|i+g|^2. \tag{13}$$

In what follows we shall omit the prime, it being understood that all results refer to the gauge (10).

We note (see, e.g., the review in Ref. 14) that the duality condition for ansatz (10) has the form

$$P^{-1}\Box P=0. \tag{14}$$

One may verify by explicit calculation that our expression (13) indeed satisfies Eq. (14).

In conclusion of this section we list some properties of the functions P and Φ which are needed for further analysis. First we note that P is a regular function, single-valued on the cut and tending to a constant in the asymptotics $|z| \to \infty$. As regards Φ we have

$$[\Phi=\Phi_1-i\Phi_2](r=0)=i,\quad \Phi(|z|=\infty)=i,$$

which ensures regularity of the potential A_μ^a at $r = 0(1)$. Moreover, the imaginary part of Φ ($\mathrm{Im}\,\Phi = -\Phi_1$) is single-valued on the cut, while the real part ($\mathrm{Re}\,\Phi = \Phi_1$) experiences a jump—it changes sign keeping its magnitude unchanged. Consequently it follows that the expression

$$\bar{A}A=\left(\frac{\Phi-i}{r}\right)\left(\frac{\Phi+i}{r}\right)=\frac{1}{r^2}[\Phi_1^2+(1+\Phi_2)^2]$$

is single-valued, regular, and has no jumps across the cut.

The properties of $G^2_{\mu\nu}$, P, and Φ listed in this section play an exceptionally important role in the selection of the "correct" zero modes.

3. ZERO MODES IN SUPERSYMMETRIC GLUODYNAMICS

As is known, supersymmetric models are conveniently distinguished from ordinary ones in that only zero modes need be analyzed. Nonzero modes, as usual, cancel between bosons and fermions and give no contribution to the generating functional. This fact substantially simplifies the calculation of the toron measure in Sec. 4.

As regards zero modes, an explicit expression can usually be obtained for them in an arbitrary self-dual field without particular difficulties (in this case, however, this is not so, see below). Indeed, in the so-called background gauge $D^{ab}_\mu a^b_\mu=0$ the zero modes of a^b_μ can be explicitly expressed in terms of the classical field strength $G^a_{\mu\nu}$ (in this section μ, $\nu=0$, 1,2,3 are the usual Lorentz indices). To show this we recall[15] that the equation for zero modes a^a_μ in the background gauge has the form

$$\begin{gathered}[-(D^2)^{ac}g_{\mu\nu}+2\varepsilon^{abc}G_{\mu\nu}{}^b]a_\nu{}^c=0,\quad D_\mu{}^{ac}a_\mu{}^c=0,\\ D_\mu{}^{ac}=\delta^{ac}\partial_\mu-\varepsilon^{abc}A_\mu{}^b,\quad [D_\mu D_\mu]^{ac}=-\varepsilon^{abc}G_{\mu\nu}{}^b.\end{gathered}\tag{15}$$

It is easily verified that the four translational, one conformal and three gauge modes:

$$a_\mu{}^a(\lambda)\sim G_{\mu\lambda}{}^a,\ \lambda=0,\ 1,\ 2,\ 3;\ a_\mu{}^a\sim G_{\mu\nu}{}^a x_\nu;\ a_\mu{}^a(d)\sim G_{\mu\nu}{}^a\bar{\eta}_{\nu\lambda}{}^d x_\lambda,\tag{16}$$

expressed in terms of $G^a_{\mu\nu}$, satisfy relations (15). In particular, in the instanton field $G^a_{\mu\nu}\sim\eta^a_{\mu\nu}(1+x^2)-2$ the formulas (16) reproduce the well-known eight gluon modes[15]:

$$a_\mu{}^a(\lambda)\sim\eta_{\mu\lambda}{}^a(1+x^2)^{-2},\qquad a_\mu{}^a\sim\eta_{\mu\lambda}{}^a\frac{x_\lambda}{(1+x^2)^2},$$
$$a_\mu{}^a(d)\sim\eta_{\mu\nu}{}^a\bar{\eta}_{\nu\lambda}{}^d\frac{x_\lambda}{(1+x^2)^2}.$$

These modes are everywhere regular and normalizable, i.e. satisfy all necessary requirements.

In our case, for the toron having $Q=1/2$, the modes (16) continue to satisfy Eqs. (15), but they do not satisfy the regularity requirement and are therefore unacceptable. Indeed, since $G^2_{\mu\nu}$ (8) is singular for $z\to1$ the same singularity is present in the modes (16).

It is thus necessary to explicitly solve Eqs. (15). It is more convenient, however, to reduce the problem to a Dirac equation, i.e. to an equation of first order. Indeed, it can be shown[16] that to each fermion zero mode in the adjoint representation correspond precisely two vector zero modes (in the $D_\mu A^a_\mu=0$ gauge). In particular, the four gluino zero modes

$$\sigma_\mu{}^-D_\mu\psi=0,\ \sigma_\mu{}^\pm=(\pm i,\ \sigma),\ \psi_{1,\,2}{}^a\sim\sigma_\mu{}^+G_{\mu0}{}^a\varepsilon,\ \psi_{3,\,4}{}^a\sim\sigma_\mu{}^+G_{\mu\nu}{}^a x_\nu\varepsilon\tag{17}$$

in the instanton field correspond precisely to the eight gluon zero modes (16). Here ε is a constant two-component spinor.

In the toron case, although the modes (17) satisfy the Dirac equation, they are unacceptable for the same reason as the gluon modes discussed above (failure to satisfy the regularity requirement).

And so the problem is reduced to the search for regular solutions of the Dirac equation for a fermion in the adjoint representation in an external toron field. However, prior to any calculations, the number of such solutions can be predicted from the index theorem or from the form of the axial anomaly. Namely, there should be precisely two zero modes in accord with the fact that the toron vacuum transition changes the chiral charge by two units and must therefore be accompanied by the production of a $\bar{\psi}\psi$ pair.

We return to Eq. (17), whose analysis is substantially simplified in the gauge where A^a_μ is expressed in the form (10). In that case the field is expressed through the function P, so that it is natural to search for the solution of (17) in the form

$$\psi^c=\bar{\eta}_{\lambda\mu}{}^c\sigma_\mu{}^+\partial_\lambda Pf(P)\varepsilon$$

with a constant spinor ε and arbitrary function $f(P)$. In that case Eq. (17) is written as

$$[\partial_\mu\delta^{ac}+\varepsilon^{abc}\bar{\eta}_{\mu\nu}{}^b\partial_\nu\ln P]\sigma_\mu{}^-\sigma_\lambda{}^+\bar{\eta}_{\lambda\sigma}{}^c(\partial_\sigma P)f(P)\varepsilon=0.\tag{18}$$

Using properties of the $\sigma^\pm_\mu$ matrices (17) and the relations for the $\bar{\eta}$ symbols:

$$\bar{\eta}_{\mu\nu}{}^a\bar{\eta}_{\mu\lambda}{}^b=\delta^{ab}g_{\nu\lambda}-\varepsilon^{abc}\bar{\eta}_{\nu\lambda}{}^c,$$

we arrive at the equation

$$\Box Pf+(\partial_\mu P)^2[df/dP+2f/P]=0.\tag{19}$$

Taking into account that P satisfies the equation $\Box P=0$ (14) we find for f the solution $f=P^{-2}$. In the end then we may write the two fermion zero modes as

$$\begin{gathered}\psi^a=-\bar{\eta}_{\lambda\sigma}{}^a\sigma_\lambda{}^+(\partial_\sigma\ln P)P^{-1}\varepsilon=\sigma_\lambda{}^+A_\lambda{}^aP^{-1}\varepsilon,\\ \bar{\psi}^a\psi^a=P^{-2}A_\lambda{}^aA_\lambda{}^a=3P^{-2}\bar{A}A=3P^{-2}\left|\frac{\Phi-i}{r}\right|^2.\end{gathered}\tag{20}$$

In the last stage we have again gone over to complex notation and expressed the gauge-invariant quantity $\bar{\psi}\psi$ in terms of the functions Φ and P (11).

The properties of the solution (20) will be discussed below, for now we note that the instanton field may also be expressed in the form (10) (the so-called singular gauge) with

$$P=\frac{x^2+1}{x^2},\qquad A_\lambda{}^a=\bar{\eta}_{\lambda\sigma}{}^a\frac{2x_\sigma}{x^2}(1+x^2)^{-1}.$$

In that case the mode (20) has the form

$$\psi^a\sim\sigma_\lambda{}^+\bar{\eta}_{\lambda\sigma}{}^a x_\sigma(1+x^2)^{-2}\varepsilon$$

and is not a new solution in comparison with the above-listed modes (17) (as can be seen from its regularity) and moreover, it can be shown (using the most general ansatz for the solution ψ^a), that no other regular solutions beside those given in (20) exist.

To demonstrate the regularity of the modes (20) it is sufficient to recall the properties of the functions Φ and P listed at the end of Sec. 2. It follows from these properties that $|\psi^a|^2$ is finite for $r\to 0$, $z\to 1$, single-valued on the cut, and tending rapidly to zero $|\psi|^2\to|z|^{-6}$ as $|z|\to\infty$, which ensures its normalizability:

$$\int \bar{\psi}^a\psi^a\,d^4x=1. \tag{21}$$

Thus the two fermion modes (20) satisfy all the requirements, while the fact that $|\psi|^2$ coincides on both sides of the cut indicates the existence of a gauge transformation "U" such that $U\psi$ is single valued [literally in the gauge (10) this condition is not fulfilled]. The explicit form of the matrix "U" is unimportant for our purposes and therefore we shall not discuss it in any further detail.

As was clarified above, the existence of precisely two fermion modes in the toron field was expected. Of interest was only the concrete realization of the general assertion of the index theorem.

We note that in the calculations of Ref. 11 also only two zero modes are kept, however the concrete way of their realization differs from ours. Namely, all the modes listed in (17) turn out to be regular, however only two of them—$\psi_{1,2}$—satisfy twisted boundary conditions, while the other two—$\psi_{3,4}$—are rejected for this reason.

As far as the number of gluon zero modes is concerned we have, as was clarified above, two gluon modes corresponding to each fermion mode ψ^a.[16] Introducing the solution (20) with $\varepsilon=\binom{1}{0}$ and $\varepsilon=\binom{0}{1}$ into Ref. 16 we obtain precisely four gluon zero modes satisfying the requirements of regularity and normalizability. Indeed, since $a^a_\mu a^a_\mu\sim|\psi^a|^2$ the fulfillment of above-mentioned requirements for $|\psi^a|^2$ ensures also their fulfillment for $(a^a_\mu)^2$. The procedure for deducing the zero modes a^a_μ from ψ^a is completely unambiguous, but since the resultant expressions are unwieldy we see no need for displaying them explicitly. What is important is that their number is precisely four in accord with the four collective coordinates describing the location of the toron (literally our solution describes a toron at the origin of the coordinates). The parameter Δ, present in the solution, is not a collective variable, as was explained in the Introduction, and plays the role of a regulator. At the end of the calculations it should be set equal to zero.

We note that in the calculations of Ref. 11 in supersymmetric gluodynamics with a 't Hooft[5] toron solution "smeared out" over all space, precisely four gluon zero modes occur just as in our case; the remaining modes, although satisfying the equations of motion, fail to satisfy twisted boundary conditions. The role of the regulator in Ref. 11 is played by the free parameter L (in place of our parameter Δ).

4. THE TORON MEASURE AND THE GLUINO CONDENSATE IN SUPERSYMMETRIC GLUODYNAMICS

As an application of the above constructed solution with $Q=1/2$ (Sec. 2) we calculate in this section the toron measure in supersymmetric gluodynamics and show how the $\langle\bar\psi\psi\rangle$ condensate arises as a consequence of this solution. As was already mentioned, the condributions of nonzero modes precisely cancel so that one must analyze only the zero modes given in Sec. 3.

As usual, in the evaluation of the generating functional Z in the quasi-classical approximation each bosonic zero mode gives rise to a factor $M_0S^{1/2}_{\rm cl}dx_0$. Here M_0 is the ultraviolet cutoff, and dx_0 the corresponding integral over the collective variable. Each fermionic zero mode is accompanied by the factor $d\varepsilon/M_0^{1/2}$, where $d\varepsilon$ is the factor connected with integration over the collective Grassman coordinate.

After the above has been taken into account the toron measure takes on the following form:

$$Z_{\rm toron}\sim\frac{M_0^4}{g^4}d^4x_0\frac{d^2\varepsilon}{M_0}e^{-4\pi^2/g^2}=\Lambda^3\,d^4x_0\,d^2\varepsilon,\qquad \frac{1}{g^4}M_0^3e^{-4\pi^2/g^2}=\Lambda^3. \tag{22}$$

Here the factor $g^{-4}M_0^4d^4x_0$ is due to the four bosonic zero modes while the factor $M_0^{-1}d^2\varepsilon$ is due to the two gluino zero modes (20) and, lastly, the factor $\exp\{-4\pi^2/g^2\}$ is the contribution of the classical toron action. As was to be expected the toron measure is independent of Δ, as is easily verified by a dimensionality check.

As in the case of instanton calculations[7–10] the expression for the toron measure is in a precisely renormalization-invariant form. It is easy to follow this phenomenon starting with the instanton calculations, Refs. 7–10:

$$Z_{\rm instant.}\sim\frac{M_0^4d^4x_0}{g^4}\frac{M_0^4d^4\rho}{g^4}\frac{d^2\varepsilon_1}{M_0}\frac{d^2\varepsilon_2}{M_0}\exp\left(-\frac{8\pi^2}{g^2}\right),$$
$$\Lambda^6=M_0^6g^{-8}\exp\left\{-\frac{8\pi^2}{g^2}\right\}. \tag{23}$$

In formula (23) the factor $M_0^8g^{-8}$ corresponds to the eight zero modes (16) in the instanton field, the factor $(d^2\varepsilon_1/M_0)(d^2\varepsilon_2/M_0)$ corresponds to the four gluino modes (17), while the standard factor $\exp\{-8\pi^2/g^2\}$ is connected with the classical instanton action. The renormalization invariance of the toron measure is now easy to understand: together with a decrease of the action by two the number of acceptable zero modes also decreased by precisely two, exactly reestablishing the correct renormalization group relation.

All is now ready for the calculation of the chiral condensate $\langle\bar\psi\psi\rangle$. Substituting in place of ψ its zero modes (20) and taking into account that integration over collective fermionic variables yields exactly unity ($\int\varepsilon^2d^2\varepsilon=1$) we verify that

$$\langle\bar\psi\psi\rangle\sim\Lambda^3\int d^4x_0\,\bar\psi_0^a(x-x_0)\psi_0^a(x-x_0)=\Lambda^3. \tag{24}$$

In the last step we used the value of the normalization integral (21).

Let us summarize. The instanton can ensure a nonzero value only for the $\langle\bar\psi\psi,\bar\psi\psi\rangle$ correlator,[7–10] in accord with the existence of four fermionic zero modes and integration over four Grassman variables $d^2\varepsilon_1d^2\varepsilon_2$ (23). The toron solution with $Q=1/2$ changes the chiral charge by two units and has two zero modes, so that the corresponding vacuum transition is necessarily accompanied by the production of the pair $\bar\psi\psi$, as was demonstrated in the explicit calculation (24).

The ideology of this calculation is analogous to that of Ref. 11. The difference has to do with the fact that in the calculations of Ref. 11 use is made of the standard quasi-classical approximation, which is not valid when the size L of the box is increased, since then the typical values of g that are important are large: $g(L\to\infty)\to\infty$.

In our case the toron solution, just like any other self-dual solution, exists for arbitrary values of g. However, keeping in mind that the object being considered is essentially pointlike (Sec. 2), it is clear that the characteristic scale is $x^2 \sim \Delta^2 \sim 0$. In that case the coupling constant $g(x \sim \Delta) \sim 0$ is small and the quasi-classical approximation is justified. This shows that the situation differs from the instanton calculations,[7,8] where the answer is determined by an integral over the instanton size $\int d\rho$, whereas only that part of the entire contribution that is due to a small-size instanton is justified and under control.

Moreover, the solution of Ref. 5 on which the calculation in Ref. 11 is based exists only for definite ratios of the sides of the four-dimensional box. This casts doubt on the reality of the corresponding fluctuations, since a small change in the dimensions of the box should not lead to a change in the physical content of the theory.

A few words about the choice of the value $Q = 1/2$ as opposed to other fractional values of Q. The corresponding special feature manifests itself in that the quantity $\bar{\psi}^a \psi^a$ is required for all modes [in including the zero modes (20)] to be the same on both sides of the cut. This requirement is satisfied only for $Q = 1/2$. The situation just dscribed is analogous to the corresponding result of the $O(3)$ σ-model,[1] where the existence of translational zero modes satisfying all requirements unambiguously determined the acceptable value of Q to be 1/2. This fact was interpreted in Ref. 1 as indicating stability of the given solution and instability of solutions with other fractional topological charges.

We also note that, as in Ref. 1, the correct renormalization group dependence is reestablished only for $Q = 1/2$.

5. CONCLUSION

The main purpose of this work was the description of a self-dual solution with $Q = 1/2$ and an analysis of physical consequences connected with this solution on the example of supersymmetric gluodynamics. As was shown, the corresponding fluctuations ensure spontaneous breaking of discrete chiral symmetry and give nonzero contribution to the chiral condensate. Apparently this is a new independent contribution, which should be taken into account along with the instanton calculations of Refs. 7–10.

An alternative point of view is also possible, raised in Ref. 17, to the effect that the instanton is a superposition of two objects with half-integral topological charge. In Ref. 17 such an object with $Q = 1/2$ was the meron,[18] having infinite action. In some sense the above described solution (Sec. 2) is similar to a meron: they both have zero size (for $\Delta = 0$). There is also a difference: the toron has finite action, the meron—infinite. This interpretation is also favored by the instanton calculations of Ref. 9. As can be seen from that paper the evaluation of the various correlators reduces to integrals saturated by instantons of zero size [see formulas (32) and (33) of Ref. 9]. It is possible that this contribution can be interpreted as due to two torons (we recall that the toron size is zero for $\Delta = 0$). At least in the $O(3)$ σ-model the two-toron contribution indeed exists and gives a nonzero contribution along with the instanton to the correlator $\langle \bar{\psi}\psi, \bar{\psi}\psi \rangle$.

[1] We keep the term "toron", introduced in Ref. 5, for both the self-dual solution of the $O(3)$ σ-model[1] and for the gauge-theory solution constructed below. We thus emphasize the fact that this solution minimizes the action and has $Q = 1/2$, i.e., has all the properties associated with the toron of Ref. 5.

[2] We call attention to the fact that to the instanton corresponds $g(z) = \Pi_{i=1}^{2}(a_i - z)/(\bar{a}_i + z)$, while to empty space corresponds $g(z) = (a - z)/(\bar{a} + z)$,[12] so that expression (7) for $g(z)$ falls precisely between these solutions, as it should according to its sense.

[3] The author expresses gratitude to A. Morozov and A. Roslyĭ for clarification (with the $O(3)$ σ-model as the example) of this point of view with regard to the solution with fractional topological charge.

[4] The difference from the commonly used notation for $\bar{\eta}^a_{\mu\nu}$ is due to nonstandard expression for $G_{\mu\nu}$ [see (1)], containing a " − " sign in front of the term $\varepsilon_{abc} A_\mu$.

1. A. A. Zhitnitskiĭ, Zh. Eksp. Teor. Fiz. **94**, No. 6, 19 (1988) [Sov. Phys. JETP **67**, 1095 (1988)]; preprint 88-6 IYaF Novosibirsk.
2. A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
3. A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
4. G. 't Hooft, Nucl. Phys. **B138**, 1 (1978); **B153**, 141 (1979).
5. G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).
6. E. Witten, Nucl. Phys. **B202**, 253 (1982).
7. V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B229**, 407 (1983).
8. G. Rossi and G. Veneziano, Phys. Lett. **138B**, 195 (1984).
9. V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B260**, 157 (1985).
10. M. A. Shifman and A. I. Vainshtein, Nucl. Phys. **B296**, 445 (1988).
11. E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984).
12. E. Witten, Phys. Rev. Lett. **38**, 121 (1977).
13. A. A. Zhitnitskiĭ, Preprint IYaF-88-95. Novosibirsk, 1988.
14. A. Actor, Rev. Mod. Phys. **51**, 461 (1979).
15. G. 't Hooft, Phys. Rev. **D14**, 3432 (1976).
16. L. S. Brown, A. D. Carlitz, and C. Lee, Phys. Rev. **D16**, 417 (1977).
17. C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. **D17**, 2717 (1978).
18. V. de Alfaro, S. Fubini, and G. Furlan, Phys. Lett. **65B**, 1631 (1977).

Translated by Adam M. Bincer