Quantum fluctuations and coherent effects in compact tunnel junctions

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The quantum properties of compact Josephson tunnel junctions are investigated for the case where there are two types of quasiparticle dissipation caused by discrete electron tunnelling directly through the junction and by current flow through a shunting resistance. It is demonstrated that quasiparticle and Cooper pair tunnelling causes quantum diffusion of the quasicharge and increases the conductivity of the system in the case of weak currents. The influence of quasiparticle dissipation on the Coulomb tunnelling blockade and coherent voltage oscillations is investigated. The I-V characteristics is found for various combinations of the system parameters and the fixed voltage "steps" are investigated. The frequency of voltage oscillations is twice the frequency of the Bloch oscillations for moderate values of the quasiparticle conductivity $1/R_T$ of the tunnel junction.[1] When Cooper pair tunnelling exists, such oscillations may occur with both large and small values of R_T. It is demonstrated that various types of dissipative phase transitions exist due to linear and "cosine" energy dissipation, and the phase diagram of the system is investigated at various Josephson energy levels.

1. INTRODUCTION

Research on macroscopic quantum phenomena in compact Josephson weak links has been the subject of extensive interest in recent years. One of the most interesting cases from the physical viewpoint is the situation where the "quantumness" of the Josephson phase difference φ is most strongly manifested. This is the study of the quantum behavior of Josephson junctions at very low temperatures and very low external current values (or in the absence of such current). It has been determined that quantum fluctuations of φ are significant when dissipation is small and, consequently, the "classical" theory of Josephson effects is not applicable. It turned out that in this case the most important aspects were qualitatively new effects associated with the discrete nature of electron and Cooper pair tunnelling. It has been shown that coherent oscillations in the voltage V may occur when an external current flows through compact tunnel junctions.[1–3] So-called Bloch voltage oscillations[1] are caused by the change in charge across the junction from successive Cooper pair tunnelling. The frequency of such oscillations is $\omega_E = \pi I/e$, where I is the external current through the junction and e is the electron charge. Such oscillations represent a "dual" effect to the well-known Josephson current oscillation effect at the prescribed voltage.

Coherent voltage oscillations in "ideal" tunnel junctions shunted by an ohmic resistance were considered in Ref. 1. Quasiparticle tunnelling is not significant for such systems in the temperature and frequency range of interest to us (substantially below the superconducting gap), and the problem is reduced to describing quantum particle motion in the "wash-board" potential when linear (ohmic) dissipation of particle energy is present. Such a system also has many other interesting quantum effects (in addition to those noted above) investigated in Refs. 4–14.

From the physical viewpoint a very interesting case is the situation where quasiparticle (one-electron) tunnelling becomes significant and largely determines the low-temperature properties of compact tunnel junctions. The first important theoretical results relating to such a situation were obtained in Refs. 2, 3, 14. Specifically it was demonstrated that coherent voltage oscillations of frequency $\omega_s = 2\omega_B$ are also related to single electron jumps across the barrier. It is important to emphasize that such single-electron oscillations can, unlike Bloch oscillations, exist independently of superconductivity.

The purpose of the present article is to provide a detailed description of the quantum dynamics of tunnel junctions in which discrete quasiparticle and Coulomb pair tunnelling occurs simultaneously as well as a "continuous" charge transfer through the shunt. Quantum fluctuations are analyzed below and it is demonstrated that such fluctuations can partially eliminate the Coulomb tunnelling blockade in the case of weak currents (see Ref. 3); moreover, coherent voltage oscillations are investigated and the I-V characteristics of tunnel junctions are determined for different Josephson energy values and a random ratio between the quasiparticle conductivities of the shunt and the tunnel junction.

2. GENERAL RELATIONS; QUANTUM FLUCTUATIONS IN THE CASE OF WEAK CURRENTS

We will begin with the effective action for the tunnel junction–normal shunt system which takes the simplest form in the Mössbauer representation:

$$S=\int_0^{1/T} d\tau\left\{\frac{C}{2}\left(\frac{\dot\varphi}{2e}\right)^2+U(\varphi)+\int_0^{1/T} d\tau'\left[\frac{\alpha_s}{8\pi^2}\left(\frac{\varphi(\tau)-\varphi(\tau')}{\sin(\tau-\tau')}\right)^2\right.\right.$$
$$\left.\left.+G_\tau(\tau-\tau')\left(1-\cos\left(\frac{\varphi(\tau)-\varphi(\tau')}{2}\right)\right)\right]\right\}, \qquad (1)$$

where T is temperature, C is the capacitance of the junction, $\alpha_s = R_Q/R_s$, $R_Q = \pi/2e^2 \approx 6.5\ k\Omega$, R_s is the shunt resistance. The expression for the effective action (1) derives directly from microscopic theory,[16–19] which states that in the adiabatic approximation we have

$$U(\varphi) = -E_J\cos\varphi - I\varphi/2e, \qquad (2)$$

$$G_\tau(\tau) = \frac{\alpha_\tau}{4}\frac{\pi^2T^2}{\sin^2(\pi\tau T)}, \qquad (3)$$

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where $\alpha_T = 4R_Q/\pi^2 R_T$, $\tau \gtrsim \omega_c^{-1}$, ω_c is the characteristic cut-off frequency. Expression (3) is obtained assuming nonzero conductivity of the tunnel junction $1/R_T$ in a frequency and temperature range substantially below the superconducting gap. Such a situation occurs either in conjunction with inelastic processes that "smear" the gap feature in the state density near the tunnel junction or when an additional normal junction is connected in parallel. Aside from the "gapless" contribution of (3), G_T also contains a term proportional to ω^2 at low frequencies. This term yields the familiar capacitance renormalization effect[17] and will be included in the quantity C henceforth.

It is convenient to introduce, in addition to the phase difference, one additional variable q called the quasicharge[1,11]. The quantities φ and q are analogs of the particle position and quasimomentum in the periodic potential, respectively[1]. Reference 11 has demonstrated that in the absence of quasiparticle tunnelling ($\alpha_T = 0$) the density matrix in the quasiparticle representation $\rho_{qq'}$ is discrete when

$$\max(T,\ I/e\alpha_s) \ll 1/\tau_s, \quad t \gg \tau_s = R_s C, \tag{4}$$

where t is time. In this case the quantity q is an even multiple of the charge of the Cooper pair $2e$. A direct generalization of calculations[11] to the case $\alpha_T \neq 0$ in the conditions of (4) yields

$$q = ke + Q_0, \quad Q_0 = I\tau_s, \quad k = 0,\ \pm 1, \ldots \tag{5}$$

and makes it possible to find expressions for the diagonal elements of the density matrix $\rho_{qq}(t)$[20]. The physical meaning of the quantities ρ_{qq} is rather simple: These determine the probability that k electrons will have tunnelled through the barrier by time t, i.e., that the quasicharge (although not the charge Q_0) will have changed by ke. Theory[11,20] therefore makes it possible to describe comprehensively quantum diffusion in the q-space attributable to Cooper pair and quasiparticle tunnelling. Such diffusion effectively reduces the charge across the junction and consequently must be taken into account in determining the I-V characteristic of the system in the weak current range.

It can be demonstrated (see Ref. 20) that the quasicharge mobility $\mu_q = 2e\langle \dot{q} \rangle / I$ in the conditions of (4) is related to the mobility in the φ-space $\mu_\varphi = 2e\langle \dot{\varphi} \rangle / I$ by the relation

$$\mu_\varphi = 4e^2 R_s (1 - \mu_q/2e). \tag{6}$$

This formula was obtained previously in Ref. 11 in the particular case $\alpha_T = 0$. If we take account of the obvious relation for the quantum averages $\langle V \rangle = (2e)^{-1} \cdot \langle \dot{\varphi} \rangle$ the mobility μ_φ determines the effective resistance of the system: $\mu_\varphi = 4e^2 R_{\mathrm{EFF}}$. Ignoring quantum tunnelling of the quasicharge ($\mu_q = 0$), we find $R_{\mathrm{EFF}} = R_s$, which corresponds to the conditions of the Coulomb tunnelling blockade.[3] The jump of electrons across the tunnel barrier with small T and I is not energy efficient, so charge transfer will run only through the shunting resistance R_s. The quantum fluctuations in the tunnel junction–shunt system, i.e., the virtual jumps of the quasiparticles and Cooper pairs will serve to reduce the effective resistance R_{EFF} compared to R_s. In other words the charge across the junction taking into account quantum corrections is equal to $\langle Q \rangle = Q_0 \cdot (1 - \mu_q/2e) < Q_0$, i.e., the Coulomb blockade is incomplete.

As noted above quasicharge diffusion is caused by a combination of single electron and Cooper pair tunnelling. However in virtually every case only one of these mechanisms is significant. Tunnelling of single electrons is insignificant for small α_T and large E_J, so it is possible to set $\alpha_T = 0$ in calculating μ_q. We considered this case in Ref. 11. In the opposite case of large α_T (small E_J) Cooper pair tunnelling makes no noticeable contribution to quasicharge dynamics, and diffusion in the q-space is determined by quasiparticle tunnelling. The expressions for the diagonal elements of the density matrix ρ_{qq} are represented as a series in powers of α_T in the limit $E_J \to 0$ in the conditions of (4). Calculating the coefficients of this series is largely analogous to the calculation carried out in Ref. 11 in which the quantity E_J^2 was the expansion parameter. The only important difference in this case is that expression (1) contains the nonlocal kernel $G_T(\tau - \tau')$ of (3) which is equivalent to the existence of an additional "attraction" between the trajectories corresponding to two successive jumps of single electrons across the barrier. If we take this into account, summation of the corresponding series for ρ_{qq} is easily carried out.

We will not present the final result here due to its unwieldiness. We will simply provide expressions for the quasicharge mobility that are easily solved by means of our expressions for $\rho_{qq}(t)$. For $E_J = 0$ and condition (4) we have

$$\mu_q = \begin{cases} \dfrac{2eR_s}{R_T \Gamma(2 + 1/2\alpha_s)} (IeR_s\tau_s)^{1/2\alpha_s}, & I \gg Te\alpha_s \\ \dfrac{\pi^{1/2} eR_s \Gamma(1 + 1/4\alpha_s)}{R_T \Gamma(3/2 + 1/4\alpha_s)} (\pi T \tau_s)^{1/2\alpha_s}, & I \ll Te\alpha_s \end{cases}, \tag{7}$$

where $\Gamma(x)$ is the Euler gamma-function.[21] The results in (7) were obtained for $\alpha_T \ll 1$, although with an arbitrary ratio between R_s and R_T.

The quantitative conditions allowing determination of the most effective quasicharge transfer mechanism are easily determined by direct comparison of (7) to the expressions for μ_q found in Ref. 11. The condition limiting the applicability of (7) for $T \ll I/e\alpha_s$ takes the form

$$I \gg e\tau_s [(E_J\tau_s)^2/\alpha_T\gamma]^{2\alpha_s/(4\alpha_s - 3)}, \quad \alpha_s > 3/4,$$
$$I \ll e\tau_s [(E_J\tau_s)^2/\alpha_T\gamma]^{2\alpha_s/(4\alpha_s - 3)}, \quad \alpha_s < 3/4,$$
$$\gamma = \Gamma(2/\alpha_s)\, 2^{2 + 1/2\alpha_s} \Gamma(2 + 1/2\alpha_s).$$

In the opposite limiting case quasicharge diffusion is caused by Cooper pair tunnelling.[11] The results (7) were obtained in the limit of small quantum effects, i.e., $\mu_q \ll e$. These formulae, however, allow estimation of the parameter values at which the Coulomb blockade will be broken down completely by quantum fluctuations. It is necessary to set $\mu_q \sim e$ for such an estimate. In the particular case $T \to 0$, $E_J \to 0$ and $\alpha_s \lesssim 1$ the Coulomb tunnelling blockade ceases to exist when $\alpha_T \gtrsim 1$.

We also note that results (6), (7) given here cannot be obtained by means of the approach in Ref. 22. The perturbation theory developed in Ref. 22 is in fact effective only at rather large external current values and when $E_J = 0$, and does not allow description of the I-V characteristic of the system in the presence of the Coulomb blockade or coherent voltage oscillations.

3. COHERENT VOLTAGE OSCILLATIONS IN THE CASE OF SMALL E_J and α_T

In the range of weak currents $I/e \ll 1/\tau_s$ considered in the preceding section the quantum effects served to cause an effective drop in the junction resistance due to quantum tunnelling in the quasicharge space. For large external current levels $I/e \gtrsim 1/\tau_s$ quantum diffusion no longer dominates but classical motion of the quasicharge through the Brillouin zone comes into play instead. Such motion is accompanied by coherent voltage oscillations across the junction[1-3] that, as noted above, are caused by tunnelling jumps of single electrons and Cooper pairs across the barrier. Both of these charge transfer mechanisms are "responsible" for the coherent voltage oscillations at moderate values of I. It is important to emphasize that such oscillations do not reduce to a simple superposition of the Bloch and one-electron oscillations, since there is a strong correlation between quasiparticle and Cooper pair tunnelling.

In this section we will consider the case of a small tunnelling probability (small α_T and E_J) and a very large shunt resistance, since the times of interest to us satisfy the condition $t \ll \tau_s$. In this case the term with α_s in (1) can be dropped. With a fixed current the charge across the capacitor C, ignoring electron tunnelling, will have a linear time dependence. In this regard the statistical sum of the system (as is the case with any other equilibrium quantity) is not, strictly speaking, physically meaningful. Therefore in the spirit of Ref. 23 we will consider the statistical sum in the charge representation and will calculate the imaginary part of this quantity with a fixed charge value, and will then describe the properties of the system in real time at a prescribed external current that adiabatically changes the charge across the junction.

The statistical sum with a fixed Q_0 is equal to the matrix element

$$Z_{Q_0} = \langle Q_0 | \exp(-\hat{H}/T) | Q_0 \rangle, \tag{8}$$

where $\hat{H}$ is the Hamiltonian of the system to which the expression for the effective action (1)–(3) corresponds. Expanding the Hamiltonian in powers of E_J and α_T and representing Z_{Q_0} as a functional integral, we obtain (see also Ref. 15)

$$Z_{Q_0} = \int_{Q(0)=Q_0}^{Q(1/T)=Q_0} DQ(\tau) \exp\left(-\int_0^{1/T} d\tau \frac{Q^2(\tau)}{2C}\right)$$
$$\times \left\{ \sum_{n=0}^{\infty} \sum_{\{\nu_i\}}{}' \frac{(E_J/2)^{2n}}{2n!} \int \prod_i^{2n} d\tau_i \right\}$$
$$\times \left\{ \sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_j^{m} d\tau_j' \, d\tau_j'' G_\tau(\tau_j' - \tau_j'') \right\}$$
$$\times \delta\left(Q(\tau) - Q_0 - 2e \sum_{i=1}^{2n} \nu_i \theta(\tau - \tau_i) - e \sum_{j=1}^{m} [\theta(\tau - \tau_j') - \theta(\tau - \tau_j'')] \right). \tag{9}$$

Here and henceforth Σ' implies summation over the neutral charge configurations $\{\nu_i\}$, $\nu_i = \pm 1$, and $\theta(x)$ is the Heaviside function.

As we know the rate of decay Γ of a state with a given Q_0 can be determined by the relation

$$\Gamma = 2\,\mathrm{Im}\, F, \tag{10}$$

where $F(Q_0) = -T \ln Z_{Q_0}$ is free energy. Initially let $E_J = 0$. Then the quantity $F(Q_0)$ is determined solely by the terms of the expansion of the statistical sum in powers of α_T. As a result we obtain

$$F = -\frac{\alpha_T}{4} \int_{\omega_c^{-1}}^{\infty} d\tau \frac{\pi^2 T^2}{\sin^2(\pi \tau T)} \exp\left[\frac{e}{C}\left(|Q_0| - \frac{e}{2}\right)\tau\right]. \tag{11}$$

The formal expression for $F(11)$ diverges when $|Q_0| > e/2$. However the imaginary part of the free energy is finite and can easily be calculated. The usual procedure for analytic continuation of the expression for $F(11)$ (see Refs. 24, 25) yields ($\tau_T = R_T C$)

$$\Gamma(Q_0) = \frac{1}{\tau_T} \frac{|Q_0| - e/2}{e} \left[1 - \exp\left(-\frac{e(|Q_0| - e/2)}{CT}\right)\right]^{-1}. \tag{12}$$

It is interesting to note that $\Gamma(Q_0)$ (12) is independent of the cut-off parameter ω_c, which drops out upon analytic continuation. For $E_J \neq 0$ the Coulomb pair tunnelling process also makes a contribution to the imaginary part of Z_{Q_0}. However for $T \ll E_J$ and $I \ll E_J^2/E_Q$ (where $E_Q = e^2/2c$) such a contribution is significant only near the points $q = ne$, $n = 0, \pm 1, \ldots$, and corresponds to the passage of a single Cooper pair through the tunnel junction at each of the times when the quasicharge $q(t)$ is in the narrow vicinity of the points ne. Each such tunnelling event is accompanied by a voltage jump whose magnitude is equal to the population probability of the corresponding state in the quasicharge space multiplied by $2e/C$. In these conditions the population probabilities w_A, w_B of the two branches of the lower Brillouin zone will be the only nonzero probabilities (see Fig. 1). For $T \ll E_J \ll E_Q$ parts of these branches with $E < e^2/8C$ are stable, while with $E > e^2/8C$ they are unstable and decay at a rate $\Gamma(Q_0)$ (12).

Taking this into account as well as the fact that for $(2n-1)e < q < (2n+1)e$, w_A determines the probability of the state with charge $Q_0 = q(t) - 2ne$, while for $2ne < q < 2(n+1)e$ it is w_B which determines the probability of state $Q_0 = q(t) - (2n+1)e$, it is easy to obtain the probability balance equations:

$$\begin{gathered} w_A + w_B = 1, \\ \dot{w}_A(q) = -\Gamma(q - 2ne) w_A(q), \quad e/2 + 2ne < q < 3e/2 + 2ne, \\ \dot{w}_B(q) = -\Gamma(q - (2n+1)e) w_B(q), \quad 2ne < q < 2ne + e/2, \\ 3e/2 + 2ne < q < 2(n+1)e, \end{gathered} \tag{13}$$

where Γ is determined by Eq. (12) in which it is necessary to set $T \to 0$.

With a constant external current the quasicharge is linearly dependent on time $(dq/dt) = I$) and Eqs. (13) are easily solved. This makes it possible to calculate the voltage averaged over the ensemble $\langle V(t) \rangle$ at any time. The function $\langle V(t) \rangle$ is a periodic function of time with the period $T_e = e/I$. We have

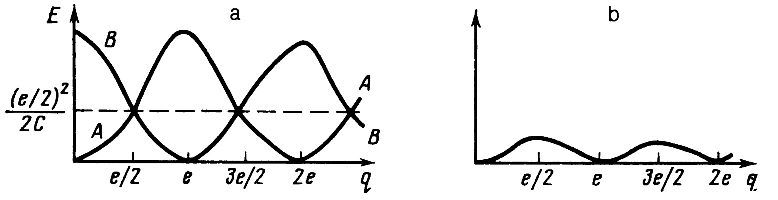

FIG. 1. Lower Brillouin zone $E(q)$ $(\alpha_s \to 0)$: a—Two branches A and B with the final population probability for $E_J \ll E_Q$, $\alpha_T \ll 1$; b—$\alpha_T > 1$.

$$\langle V(t)\rangle = It - e\left\{\frac{\exp[f_1(t)+f_2(t)-x\theta(It-e/2)]}{2\,\mathrm{ch}\,x} + 1 - \exp[f_2(t)]\right\},$$
$$f_1(t) = \frac{t(It-e)}{2e\tau_T}\theta\left(\frac{e}{2}-It\right),$$
$$f_2(t) = -\frac{(It-e/2)^2}{2Ie\tau_T}\theta\left(It-\frac{e}{2}\right), \qquad (14)$$

where $x = e/8I\tau_T$, $0 < t < T_e$, $\tau_T = R_T C$. For this system the I-V characteristic is easily determined after time averaging of (14), which yields

$$\langle \overline{V}\rangle = \frac{e}{8C\,\mathrm{ch}\,x}\left[\left(\frac{\pi}{x}\right)^{1/2}\Phi(x^{1/2})e^x - e^{-x}\left(\frac{1}{-x}\right)^{1/2}\gamma\left(\frac{1}{2},-x\right)\right]. \qquad (15)$$

Here Φ and γ are the probability integral and the incomplete gamma-function, respectively.[21] In the case of weak currents $(x \gg 1)$ the I-V characteristic is determined by one-electron oscillations[3]

$$\langle \overline{V}\rangle = \frac{e}{C}\left(\frac{\pi I\tau_T}{2e}\right)^{1/2}.$$

The I-V characteristic of the system was investigated numerically in Ref. 3 in the limit of strong current. Here we have found the analytic expression for the I-V characteristic with random x. In the particular case of strong currents $(x \ll 1)$ we have from (15)

$$\langle \overline{V}\rangle = e^2/24IC\tau_T. \qquad (16)$$

Such an I-V characteristic is common to Bloch oscillations.[1] It is, however, important to note that one-electron tunnelling is strongly manifested in the form of $\langle V(t)\rangle$. Consistent with (14) the frequency of oscillations $\langle V(t)\rangle$ for $x \ll 1$ (as well as for $x \gg 1$) is equal to the one-electron oscillation frequency $2\pi I/e$, unlike the case $\alpha_T = 0$, when the analogous frequency is equal to $\pi I/e$.[1] On the other hand for $x \ll 1$ coherent oscillations will exist only when $E_J \neq 0$, while the I-V characteristic of Eq. (16) is radically different from that found in Ref. 3 for the case $x \ll 1$, $E_J = 0$. Coherent voltage oscillations at strong currents $x \lesssim 1$ are therefore attributable to both tunnelling mechanisms (Cooper pair and quasiparticle tunnelling) and cannot be described by the simple "superposition" of Bloch and one-electron oscillations.

Now assume that the external current flowing through the junction is a sum of time-dependent and a.c. components: $I(t) = I + I_1\cos\omega t$. It is easily determined that for certain ratios between the frequencies ω and $\omega_s = 2\pi I/e$ the alternating external current will cause voltage "steps" to appear in the I-V characteristic. We now consider the most interesting limiting cases of small $(x \gg 1)$ and large $(x \ll 1)$ values of the a.c. component of the external current I. In these cases the average value $\langle \overline{V}\rangle$ (15) is small compared to e/c, while the relation $\langle V(q(t))\rangle$ in the principal approximation have a "sawtooth" character. In the one-electron oscillation range $(x \gg 1)$ we have

$$\langle V(q)\rangle = \frac{e}{\pi C}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2\pi nq/e)}{n}. \qquad (17)$$

With an alternating external current the quasicharge value is determined by the obvious relation

$$q(t) = It + \frac{I_1}{\omega}\sin\omega t + q_0, \qquad -\frac{e}{2} < q_0 < \frac{e}{2}. \qquad (18)$$

Substituting (18) into (17) and after some simple transformations we find

$$\langle V(t)\rangle = \frac{e}{\pi C}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left\{J_0\left(\frac{2\pi nI_1}{e\omega}\right)\sin\left[\frac{2\pi n}{e}(It+q_0)\right] + \sum_{m=1}^{\infty}J_m\left(\frac{2\pi nI_1}{e\omega}\right)\left[(-1)^m\sin\left[\left(\frac{2\pi nI}{e}-m\omega\right)t+\frac{2\pi nq_0}{e}\right] + \sin\left[\left(\frac{2\pi nI}{e}+m\omega\right)t+\frac{2\pi nq_0}{e}\right]\right]\right\}, \qquad (19)$$

where J_m are the Bessel functions[21]. It is clear that at currents $I = me\omega/2\pi n$ (i.e., when $m/n = \omega_s/\omega$) a constant voltage component exists across the junction whose magnitude is determined from (19). Specifically for the amplitude of the step corresponding to the fundamental $\omega = \omega_s$ we obtain

$$V_1 = \frac{e}{\pi C}\sum_{m=1}^{\infty}\frac{J_m(m2\pi I_1/e\omega)}{m}. \qquad (20)$$

Hence $V_1 \propto (I/I_1)^{1/2}$ for $I_1/I \gg 1$ and $V_1 = I_1/\omega C$ when $I_1 \ll I$. In the latter case the amplitude of steps V_n for $n\omega_s = \omega$ are independent of n and are also equal to $V_n = I_1/\omega C$.

All these results remain valid in the range where the one-electron and the Bloch oscillations coexist $(x \ll 1)$. In this case we will also use Eq. (17), where it is necessary to carry out the substitution $q \to q \pm e/2$, to describe how the voltage $\langle V\rangle$ depends on quasicharge. Such a substitution will of course have no influence on the position or magnitude of the voltage steps which will still be determined by Eq. (19). The condition $n\omega_s = m\omega$ for such steps to appear differs from the analogous condition for the case of "pure" Bloch oscillations, $m\omega = n\omega_B \equiv n\omega_s/2$.[1] It is also important that the validity of the analysis carried out in this section is limited to the condition $\alpha_T \ll E_J^2/E_Q^2$ at moderate external current values $I \gtrsim e/\tau_T$, which makes it possible to neglect the Zener tunnelling process.

4. INSTANTON GAS APPROXIMATION

The other range of parameters in which it is possible to describe the quantum behavior of tunnel junctions is max $\{(E_J/E_Q)^{1/2}, \alpha_T\} \gg 1$. In this condition the φ-representation is more convenient for the statistical sum, and hence makes it possible to directly employ the saddle point method in calculating the functional integral for Z.

We will initially consider the case $(E_J/E_Q)^{1/2} \gg \max\{1, \alpha_s, \alpha_T\}$. Non-trivial quantum fluctuations (instantons) describing the tunnelling of phase φ between different minima of the potential (2) dominate at a sufficiently low temperature $T \ll \omega_0 = (8E_J E_Q)^{1/2}$ and $I/e \ll \omega_0$. The expression for Z is a functional integral of exp $(-S)$ over φ. Standard calculations of this integral using the well-known instanton solution $\tilde{\varphi}(\tau) = 4$ arctg $(\exp(-\omega_0 \tau))$ yield

$$Z = \sum_{n=0}^{\infty} \left(\frac{\Delta_1}{2}\right)^{2n} \int_0^{1/T} d\tau_{2n} \int_0^{\tau_{2n}-1/\omega_1} d\tau_{2n-1} \ldots \int_0^{\tau_2 - 1/\omega_1} d\tau_1 \sum_{\{\nu_i\}}{}' \exp\Big\{2 \times \sum_{\substack{i,j=1 \\ i>j}}^{2n} (\alpha_s \nu_i \nu_j + \alpha_T (-1)^{i-j}) \ln\left(\frac{\sin \pi T(\tau_i - \tau_j)}{\pi T \omega_1^{-1}}\right) + \frac{I\pi}{e} \sum_{i=1}^{2n} \nu_i \tau_i \Big\}. \tag{21}$$

Here $\Delta_1 = (32/\sqrt{\pi})(E_J)^{3/4}(E_Q/2)^{1/4} \exp[-(8E_J/E_Q)^{1/2}]$ is the peak-to-peak tunnelling amplitude between the nearest minima of potential (2), and $\omega_1 \sim \omega_0$ is the cutoff frequency.

Expression (21) differs slightly from the analogous expressions in Ref. 4,5 for the case of zero quasiparticle conductivity of the tunnel junction ($\alpha_T = 0$). The instantons experience logarithmic interaction for $\alpha_T \neq 0$ as is the case in Refs. 4, 5, although the sign of this interaction is determined not only by the signs of the topological charges v_i, but also by their relative position. For sufficiently small α_T this will not yield qualitatively new effects. For $\alpha_T > \alpha_s$ the sign of the interaction between any nearest charges v_i and v_{i+1} corresponds to attraction between them. When the interaction is sufficiently strong the instantons i and $i+1$ may "stick", corresponding to formation of an instanton with twice the topological charge. Its dimensions $\tau_2^* = |\tau_{i+1} - \tau_i|$ are determined by minimization of the instanton interaction:

$$\Delta S(\tau_2^*) = AE_J \tau_2^* e^{-\omega_0 \tau_2^*} + 2(\alpha_T - \alpha_s) \ln\left(\frac{\sin \pi T \tau_2^*}{\pi T \omega_1^{-1}}\right), \quad A \sim 1,$$

which yields

$$\tau_2^* = L/\omega_0, \quad L = \ln\left[(E_J/E_Q)^{1/2}/(\alpha_T - \alpha_s)\right].$$

For $\alpha_T - \alpha_s \gg L^{-1}$ fluctuations of τ_2^* are small and we find the following expression for the tunnelling amplitude between the k- and $k+2$-minima of potential (2)

$$\Delta_2 = \left(\frac{\pi}{\alpha_T - \alpha_s}\right)^{1/2} \frac{\Delta_1^2 L^{1/2 - 2\alpha_T}}{2\omega_0}, \quad \Delta_2 \ll \omega_0. \tag{22}$$

We will show that such instanton configurations may be significant in certain conditions. We will consider the simplest such case: $+ + - -$. Its contribution is

$$\left(\frac{\Delta_2}{2}\right)^2 \int_0^{1/T} d\tau_2 \int_0^{\tau_2 - \tau_2^*} d\tau_1 \exp\left\{-8\alpha_s \ln\left[\frac{\sin \pi T(\tau_2 - \tau_1)}{\pi T \tau_2^*}\right] + \frac{2I\pi}{e}(\tau_2 - \tau_1)\right\}.$$

This expression coincides with the contribution of the configuration $+ -$ describing tunnelling between neighboring potential wells when the substitution $\Delta_2 \to \Delta_1$, $4\alpha_s \to \alpha_s + \alpha_T$ is made. We therefore have a situation that is qualitatively analogous to our case in Sec. 2 in investigating diffusion in the q-space. Here we also have what appears to be two mechanisms of wave packet motion (but in φ-space) attributable to jumps with a phase shift of 2π and 4π. The analysis of the general expression (21) therefore is substantially simplified, since only one of these mechanisms dominates in any case. With small T and I and moderate dissipation, Josephson "particle" motion manifests a band character. The width of the band is given by the larger of the two quantities

$$\Delta_{1r} \sim \Delta_1 \left(\frac{\Delta_1}{\omega_0}\right)^{(\alpha_s + \alpha_T)/(1 - \alpha_s - \alpha_T)}, \quad \alpha_s + \alpha_T < 1;$$
$$\Delta_{2r} \sim \Delta_2 \left(\frac{\Delta_2}{\omega_0}\right)^{4\alpha_s/(1 - 4\alpha_s)}, \quad \alpha_s < 1/4. \tag{23}$$

A direct comparison of the quantities in (23) demonstrates that for $\alpha_T > \alpha_s + {}^{1/2}$ the bandwidth is determined by tunnelling of φ between even (odd) minima of the potential (2) and is equal to Δ_{2r}. A phase transition occurs at the point $\alpha_s + \alpha_T = 1$ as $T \to 0$, $I \to 0$, causing total suppression of tunnel transitions between neighboring minima (see also Ref. 15), i.e., $\Delta_{1r} = 0$ when $\alpha_T + \alpha_s > 1$. The effective potential is 4π-periodic when the condition $\alpha_s < 1/4$ is satisfied simultaneously, and the phase in such a potential is delocalized. Complete phase localization ($\Delta_{1r} = \Delta_{2r} = 0$) occurs only for $\alpha_s + \alpha_T > 1, \alpha_s > 1/4$.

The instanton technique makes it possible to provide a quantitative solution of the problem in the limiting case

$$\alpha_T \gg \max\{(E_J/E_Q)^{1/2}, \alpha_s, 1\} \tag{24}$$

as well. In this case the primary role is played by instanton solutions describing tunnelling between potential minima separated by one. Neglecting the Coulomb and Josephson terms in the effective action (1)–(3) the exact solution appears[26] as $\tilde{\varphi}(\tau) = 4$ arctg$(\Omega\tau) + \varphi_0$, where Ω is the characteristic instanton frequency and φ_0 is random. Substituting this solution into (1)–(3), we find as $I \to 0$ and with sufficiently small α_s ($\varphi_0 \ll 1$)

$$S[\tilde{\varphi}] = \pi C\Omega/e^2 + 4\pi E_J/\Omega + \pi^2 \alpha_T/2 + E_J \varphi_0^2/2T. \tag{25}$$

The extreme values of φ_0 and $\Omega = \Omega_0$ are determined by minimization of (25), which yields $\varphi_0 = 0$ and $\Omega_0 = (8E_Q E_J)^{1/2}$. We note, however, that the introduction of the quantity Ω_0 makes sense only when the fluctuations of Ω are small compared to the extreme value, i.e., for $\partial^2 S(\Omega_0)/\partial \Omega_0^2 \gg \Omega_0^{-2}$, which is equivalent to the condition $E_J \gg E_Q$. In the opposite limiting case $E_J \ll E_Q$ fluctuations of the instanton frequency are significant and must be taken into account in determining the preexponential factor in the expression for the tunnelling amplitude $\Delta_2/2$.

This factor is determined by the eigenvalues of the operator $\hat{M} = \delta^2 S/\delta\varphi^2$ on the extremum trajectory. There are three eigenfunctions of this operator with zero eigenvalues as $E_J \to 0$, $C \to 0$:

$$\varphi_1(\tau) = J_1^{-1/2}\partial\tilde{\varphi}/\partial\tau, \quad \varphi_2(\tau) = J_2^{-1/2}, \quad \varphi_3(\tau) = J_3^{-1/2}\frac{\partial\tilde{\varphi}}{\partial\Omega}. \quad (26)$$

Here φ_1 and φ_2 correspond to the shift of the instanton along the τ and φ axes, while φ_3 corresponds to the change in the parameter Ω. Estimation of the normalizing factors in (26) yields $J_1 = 8\pi\Omega$, $J_2 = T^{-1}$, $J_3 = 8\pi/\Omega^3$. Identifying the contribution of these modes and going over to integration with respect to the collective variables τ, φ_0, and Ω, for Δ_2 we find

$$\Delta_2 = 2\int \frac{d\varphi_0\, d\Omega}{(2\pi)^{3/2}}(J_1J_2J_3K)^{1/2}\exp(-S[\tilde{\varphi}]), \quad (27)$$

where $K = \det \hat{M}^0/\det' \hat{M}$, $\hat{M} = \hat{M}^0 - \hat{M}^1$, and the operator $\hat{M}^0$ is calculated from the extremum $\tilde{\varphi}(\tau) = 0$, while the prime infers elimination of the eigenvalues corresponding to modes (26). The operator $\hat{M}^1$ in the expression for K is determined by the square-law form

$$(\varphi_1 \hat{M}^1 \varphi_1) = \int d\tau E_J(1-\cos\tilde{\varphi}(\tau))\varphi_1^2(\tau)$$
$$+\frac{\alpha_T}{8}\int d\tau\, d\tau'\left[1-\cos\left(\frac{\tilde{\varphi}(\tau)-\tilde{\varphi}(\tau')}{2}\right)\right]\left(\frac{\varphi_1(\tau)-\varphi_1(\tau')}{\tau-\tau'}\right)^2$$

The eigenfunctions and the eigenvalues of the operator $\hat{M}^0$ are easily found ($\omega_k = 2\pi Tk$, k is an integer):

$$\varphi_k^0(\tau) = (2T)^{1/2}\cos\omega_k\tau, \quad \lambda_k^0 = E_J + \pi\alpha_T|\omega_k|/8 + \omega_k^2/8E_Q, \quad (28)$$

and for the quantity K we obtain the expression

$$K = \lambda_1^0\lambda_2^0\lambda_3^0 \exp\left[\sum_{k=4}^{\infty}(\varphi_k^0\hat{M}^1\varphi_k^0)/\lambda_k^0 + o(\hat{M}^1)\right].$$

Hence for $T = 0$ and the condition $E_J/\alpha_T \ll \Omega \ll E_Q\alpha_T$ we find

$$K^{1/2} = E_J^{3/2}\frac{2\pi\alpha_T E_Q}{\Omega}\exp\left[C_0 + \frac{E_J}{\Omega}\frac{16}{\pi\alpha_T}\ln\frac{\pi^2\alpha_T^2}{8E_JE_Q}\right], \quad (29)$$

where $C_0 = 0.577$. Using expression (29) and evaluating the integrals with respect to φ_0 and Ω in (27) we have

$$\Delta_2 = 2\alpha_T(8E_JE_Q)^{1/2}K_1[\pi(8E_J/E_Q)^{1/2}]\exp(-\pi^2\alpha_T/2 + C_0), \quad (30)$$

where $K_1(x)$ is the modified Bessel function. The crossover of the coefficient for $E_J \sim E_Q$ is related to the change in the type of Ω fluctuations. For $E_J \gg E_Q$ the value of Ω experiences Gaussian fluctuations with respect to $\Omega_0 = (8E_JE_Q)^{1/2}$, while the fluctuations of Ω are essentially non-Gaussian for $E_J \ll E_Q$ and are within an order of magnitude of $\Omega_0 \sim E_J$.

The difference between Eq. (30) and the result from Ref. 26 (after recovering the correct dimensionality) is due to the fact that the anharmonic effects considered above were neglected by Ref. 26 for $E_J \ll E_Q$, while in the case $E_J \gg E_Q$ this study evidently improperly accounted for the contribution of modes with frequencies $\omega \ll \Omega$. The instanton presence is not significant at times $\tau \sim \omega^{-1} \gg \Omega^{-1}$ and $\tau \ll \Omega^{-1}$, and the eigenvalues of the operator $\hat{M}$ in the principal approximation are identical to (28). On the other hand the eigenvalues of $\hat{M}$ (in the limit $E_J \to 0$, $E_Q \to \infty$ found exactly in Ref. 26) are also identical with the eigenvalues of $\hat{M}^0$ (28) for $E_J/\alpha_T \ll \omega \ll \alpha_T E_Q$. These inequalities heavily overlap for the instanton frequencies $\Omega \sim \Omega_0$ that are important here. This suggests that the eigenvalues λ_k^0 (28) in the principal approximation are the eigenvalues of the operator $\hat{M}$ with all values of ω_k and that perturbation theory in $\hat{M}^1$ can be used to calculate the corrections to λ_k^0. When the strong inequality $T \ll \Omega$ holds, such perturbation theory is valid for all frequencies ω_k. We also note that for $\alpha_T \sim (E_J/E_Q)^{1/2} \gg 1$ the expressions derived by various means for Δ_2 (22) and (30) transform into one another.

The interaction between the instantons considered here is inversely proportional to the square of the distance between them ($\propto \tau^{-2}$) (Ref. 26) as $I \to 0$, $\alpha_s \to 0$, and is small when $\Omega_0\tau \gg 1$. The average "distance" between the instantons is $\tau \sim \Delta_2^{-1}$ and thus for

$$\Delta_2 \ll \Omega_0 \quad (31)$$

the noninteracting instanton gas approximation holds. The range of applicability of expression (30) is limited by conditions (24),(31) for Δ_2.

An important consequence of the results obtained here is the possibility of establishing an interface $\alpha_T^*(E_J)$ between the ordered ($\langle\cos\varphi/2\rangle \neq 0$) and disordered ($\langle\cos\varphi/2\rangle = 0$) phases in the range of small E_J. We have $\alpha_T^* = 1 - \alpha_s$ for $\alpha_s < 1$ and as $E_J \to \infty$. For $E_J \ll E_Q$ the necessary condition for the existence of the ordered phase is that the Δ_2 bandwidth be small compared to E_J. In the opposite case $\Delta_2 > E_J$ the energy of the system may exceed the Josephson interaction E_J, so that transitions between the states φ and $\varphi + 2\pi$ become possible. In this situation the translational symmetry of the system in φ-space can be caused only by the symmetry of the Josephson term $\varphi \to \varphi + 2\pi$, i.e., the system has an equal probability of being in the states φ and $\varphi + 2\pi$, which corresponds to the disordered phase $\langle\cos\varphi/2\rangle = 0$. Thus, the interface $\alpha_T^*(E_J)$ is determined from the condition $\Delta_2 \sim E_J$ for $E_J \ll E_Q$ and $\alpha_T \gg 1$, which yields

$$E_J = BE_Q\alpha_T^* \exp\left(-\frac{\pi^2}{2}\alpha_T^*\right), \quad B \sim 1. \quad (32)$$

The existence of external current and ohmic dissipation causes additional instanton and anti-instanton interaction:

$$\Delta S = -\frac{2\pi I}{e}\tau + 8\alpha_s \ln\left[\frac{\sin(\pi T\tau)}{\pi T\omega_2^{-1}}\right], \quad \omega_2 \sim \Omega_0. \quad (33)$$

With small I, T and $\alpha_s < 1/4$ the Josephson "particle" experiences band motion, while the effective band width Δ_{2r} is determined by Eq. (23); it is necessary to substitute Δ_2 from (30) into this formula. In this case the quantity $\alpha_T^*(E_J)$ is also found from condition (32). The translational symmetry $\varphi \to \varphi + 4\pi$ is broken for $\alpha_s > 1/4$, $\alpha_T > \alpha_T^*$ and as $T \to 0$, and tunnelling between the even (or odd) minima is suppressed, and $\Delta_{2r} = 0$.

5. QUANTUM DIFFUSION OF PHASE AND VOLTAGE OSCILLATIONS

With sufficiently large values of T and (or) I diffusion-type motion replaces the band nature of charge carrier motion. The quantum diffusion of φ also exists at any small values of T and I in the rane $\alpha_s + \alpha_T > 1$, $\alpha_s > 1/4$. In the case of linear dissipation for $E_J \gg E_Q$ diffusive motion is caused by incoherent tunnelling between the nearest potential minima and has been investigated in considerable detail in Refs. 7, 9–11. The calculations have been carried out entirely analogously to the case in Refs. 7, 9–11 for $\alpha_T \neq 0$. As a result for $(E_J/E_Q)^{1/2} \gg \max\{1, \alpha_s + \alpha_T\}$ we have

$$\mu_1 = \begin{cases} \dfrac{\pi\Delta_1^2}{2I^2\Gamma(2(\alpha_s+\alpha_T))}\left(\dfrac{\pi I}{e\omega_1}\right)^{2(\alpha_s+\alpha_T)}, & I/e \gg T, \\ \dfrac{\pi^{3/2}\Delta_1^2\Gamma(\alpha_s+\alpha_T)}{4T^2\Gamma(\alpha_s+\alpha_T+1/2)}\left(\dfrac{\pi T}{\omega_1}\right)^{2(\alpha_s+\alpha_T)}, & I/e \ll T. \end{cases} \quad (34)$$

As discussed above, in this case where the quasiparticle conductivity of the tunnel junction is nonzero there is one additional quantum diffusion mechanism of the Josephson phase related to direct tunnelling between the states φ and $\varphi + 4\pi$. In certain conditions the mobility μ_2 attributable to this mechanism exceeds μ_1. We will calculate μ_2 assuming that this mechanism dominates, and we will then determine the applicability conditions of this assumption based on a comparison of μ_1 and μ_2. For incoherent tunnelling between even (or odd) potential minima the probabilities of the Josephson "particle" being near such minima of $W_k(t)$ obey the simple equations

$$\dot{W}_k(t) = \Gamma_2^+ W_{k-2}(t) + \Gamma_2^- W_{k+2}(t) - \Gamma_2 W_k(t), \quad (35)$$

where $k = 2n$ (or $k = 2n + 1$), while Γ_2 are equal to

$$\Gamma_2^+ = \Gamma_2^- \exp\left(\frac{2\pi I}{eT}\right),$$

$$\Gamma_2 = \Gamma_2^+ + \Gamma_2^- = \frac{\Delta_2^2}{4\pi T}\left(\frac{2\pi T}{\omega_2}\right)^{8\alpha_s} \mathrm{ch}\left(\frac{\pi I}{eT}\right)\frac{|\Gamma(4\alpha_s + iI/eT)|^2}{\Gamma(8\alpha_s)}. \quad (36)$$

Here Δ_2 for $(E_J/E_Q)^{1/2} \gg \max\{1, \alpha_s + \alpha_T\}$ is determined by Eq. (22), while for $\alpha_T \gg \max\{E_J/E_Q)^{1/2}, \alpha_s, \ln(\alpha_T E_Q/E_J)\}$ it is determined by Eq. (30).

Equations analogous to (35) describing incoherent tunnelling between the nearest potential wells were initially postulated in Ref. 7 for the case of ohmic dissipation. A strict derivation of these equations by means of an analysis of the contributions of all instanton configurations and their applicability conditions were given in Ref. 11. The derivation of (35),(36) was entirely analogous to Ref. 11. The description of quantum diffusion of φ by means of (35),(36) is quite adequate for max $(T/I/e) \ll \omega_2$, while for $\alpha_s < 1/4$ the auxiliary condition $(T,I/e) \gg \alpha_s \Delta_{2r}$ must also be satisfied.

The averages $\langle\varphi(t)\rangle$ and $\langle\varphi^2(t)\rangle$ are easily calculated by means of relatins (35),(36). Specifically for the mobility μ_2 we have

$$\mu_2 = \frac{8\pi e \Gamma_2}{I}\,\mathrm{th}\,\frac{\pi I}{eT}, \quad (37)$$

which, subject to (36) yields

$$\mu_2 = \begin{cases} \dfrac{\pi\Delta_2^2}{2I^2\Gamma(8\alpha_s)}\left(\dfrac{2\pi I}{e\omega_2}\right)^{8\alpha_s}, & I/e \gg T, \\ \dfrac{\pi^{3/2}\Delta_2^2\Gamma(4\alpha_s)}{T^2\Gamma(4\alpha_s+1/2)}\left(\dfrac{\pi T}{\omega_2}\right)^{8\alpha_s}, & I/e \ll T. \end{cases} \quad (38)$$

We recall that the average voltage across the junction $\langle V\rangle = \langle\dot{\varphi}\rangle/2e$ in the diffusion approximation can also be determined by μ_1, so in the general case $\mu_\varphi = \max\{\mu_1, \mu_2\}$. Direct comparison of (34) and (38) reveals that in the case $(E_J/E_Q)^{1/2} \gg \max(1, \alpha_s, \alpha_T)$ and when the condition

$$\max(I/e, T) \ll \omega_0\left(\frac{\Delta_1}{\omega_0}\right)^{1/(\alpha_T - 3\alpha_s)} \varkappa(\alpha_T, \alpha_s) \quad (39)$$

is satisfied (the numerical factor $\varkappa$ is expressed through the combination of gamma-function) the second incoherent tunnelling mechanism under consideration dominates, i.e., $\mu_\varphi = \mu_2$, and, consequently, the effective potential in which the diffusion of φ occurs is a 4π-periodic potential. The effective potential in the opposite (quasicharge) space in this case is e-periodic, and the mobility of the quasicharge μ_q consistent with (6) and (38) is $\mu_q \approx 2e$ and this implies that the coherent voltage oscillations $\langle V(t)\rangle$ for $\alpha_s < 1/4$ and sufficiently small T have the frequency $\omega_s = 2\pi I/e$ (since the quasicharge "passes through" period e in q-space over time $T_e = e/I$).

In these conditions, as in the preceding case of small E_J and α_T, the Bloch and one-electron voltage oscillations coexist, while the I-V characteristic of the system $\overline{V} = \mu_2 I/4e^2$ is determined by the first relation of (38) for $\alpha_s < 1/4$. In the limit opposite (39), $\mu_\varphi = \mu_1$, i.e., incoherent tunnelling between the nearest minima of the 2π-periodic potential makes the primary contribution. This means that in this case the quasicharge "moving" with the external current "senses" solely the $2e$-periodic potential relief, while the coherent voltage oscillations have a frequency $\omega_B = \pi I/e$, i.e., they are purely Bloch oscillations. Such oscillations occur for $\alpha_s + \alpha_T < 1$, while the I-V characteristic of the system in such oscillatory conditions is determined by one of the expressions (34).

With large values of

$$\alpha_T \gg \max\left\{(E_J/E_Q)^{1/2}, \alpha_s, \ln\left(\frac{\alpha_T E_Q}{E_J}\right)\right\}$$

the I-V characteristic in the diffusion approximation is determined by $\mu_\varphi = \mu_2$. Specifically, for $I/e \gg T$ from (38) we have

$$\langle\overline{V}\rangle = \frac{\pi^2\Delta_2^2}{e\omega_2\Gamma(8\alpha_s)}\left(\frac{2\pi I}{e\omega_2}\right)^{8\alpha_s - 1}, \quad (40)$$

where Δ_2 is determined in (30). We note that expression (40) holds for virtually any (including small) values of E_J/E_Q at currents I less than and not too close to the Josephson critical current $2eE_J$. The classical dynamics of the Josephson phase become significant for $I \gtrsim 2eE_J$, so that in the case $I \gg eE_J$ and $\alpha_T \gg 1$ the I-V characteristic in the principal approximation is described by Ohm's Law. It is also necessary to take account of the auxiliary condition guaranteeing validity of (40) for $\alpha_T < 1/4$: $e\alpha_s\Delta_{2r} \ll I$. As a result we find that with small $E_J, \alpha_T \gg 1$ and $\alpha_s < 1/4$ the following condition must be satisfied in order to achieve quantum diffusion of φ and quantum voltage oscillations:

$$E_J \gg E_Q \alpha_T \alpha_s^{1-4\alpha_s} \exp\left(-\frac{\pi^2}{2}\alpha_T\right). \quad (41)$$

As before the Brillouin zone is e-periodic, while the frequency of oscillations $\langle V(t)\rangle$ is equal to ω_s.

A Coulomb tunnelling blockade can develop for small I and T. Specifically, for $\alpha > \alpha_T^*$ and $\alpha_s < 1/4$ it can occur when $I \ll e\alpha_s \Delta_{2r}$. It is easily demonstrated that the I-V characteristic of the system in such conditions is ohmic with an effective resistance $R_{\rm EFF} = R_s$. In the limit $\alpha_s \to 0$ and $\alpha_T \ll 1$ the initial section of the I-V characteristic with both small and large[27] values of E_J takes the form characteristic of one-electron oscillations.

6. DISCUSSION OF RESULTS

Quantum effects therefore have a significant influence on the behavior of compact Josephson tunnel junctions at sufficiently low temperatures and produce a number of qualitatively new effects. An important role is played by dissipation which in our most general case is caused by both the nonzero quasiparticle conductivity of the tunnel junction and by the presence of a shunting resistance. In a weak external current and (or) temperature the quantum fluctuations in the tunnel junction-shunt system lead to virtual tunnelling of single electrons and Cooper pairs as well as the associated quantum diffusion in the quasicharge space. As a result for small α_T the Coulomb tunnelling blockade breaks down to some degree, while the charge and effective resistance drop below their classical values due to the quantum corrections. The Coulomb blockade breaks down entirely for $\alpha_T \gtrsim 1$ and $E_J = 0$. The Coulomb blockade exists with any α_T at nonzero (and even small) values of E_J and $\alpha_s \neq 0$, while the initial section of the I-V characteristic corresponds to the resistive state with $R_{\rm EFF} = R_s$.

Coherent voltage oscillations represent a very specific manifestation of the quantum nature of the Josephson phase difference. When quasiparticle and Cooper pair tunnelling exist simultaneously, these oscillations are substantially modified compared to the case of purely one-electron[2,3] or purely Bloch[1] oscillations. For moderately small values of α_T the frequency of the coherent voltage oscillations is twice the Bloch oscillation frequency. We emphasize that such oscillations should not be interpreted as purely one-electron oscillations (i.e., independent of Cooper pair tunnelling), in spite of the fact that their frequency coincides with the one-electron oscillation frequency ω_s.

The important role of Josephson tunnelling becomes particularly clear in the case $\alpha_T \gg 1$. In this case the Coulomb blockade and coherent oscillations are completely suppressed for $E_J = 0$, while the I-V characteristic is near-ohmic. The situation, however, changes radically with nonzero E_J: The coherent effects are recovered, and the I-V characteristic in the range of rather weak currents acquires its characteristic form (Fig. 2) and is substantially different from ohmic form. It is also important to call attention to the difference between dissipation attributable to quasiparticle tunnelling and ohmic dissipation: For $\alpha_s \gtrsim 1$ quantum coherent effects break down entirely, while for sufficiently small α_s these effects are not suppressed even for $\alpha_T \gg 1$.

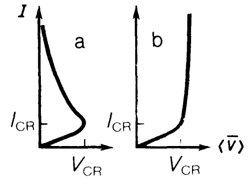

FIG. 2. Typical I-V characteristic for $\alpha_s < 1/4$ and $T \to 0$: a—For $\alpha_s < 1/8$, b—for $1/8 < \alpha_s < 1/4$. The initial segment of the characteristic $I \ll I_{\rm CR}$ corresponds to the Coulomb tunnelling blockade, while the region $I \gg I_{\rm CR}$ corresponds to coherent voltage oscillations. For $\Delta_{2r} \gtrsim \Delta_{1r}$ the crossover between these two conditions is characterized by the parameters $I_{\rm CR} \sim e\alpha_s \Delta_{2r}$, $V_{\rm CR} \sim \Delta_{2r}/e$.

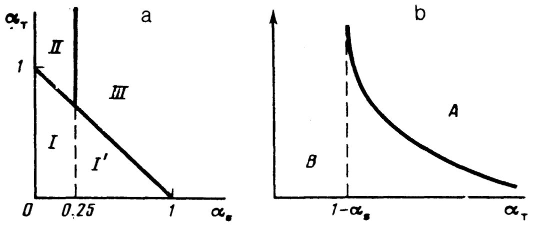

FIG. 3. a—Phase diagram for $T \to 0$, $I \to 0$ and $E_J/E_Q \gg 1$. Regions I and I′ correspond to the disordered phase $\langle \cos \varphi/2 \rangle = 0$, in regions II and III $\langle \cos \varphi/2 \rangle \approx 1$. The Josephson phase is localized in region III, where $\langle \varphi^2 \rangle \ll 1$. In regions I, I′ and II the macroscopic phase coherence is broken down by phase fluctuations and $\langle \varphi^2 \rangle \to \infty$. b—Phase diagram for $T \to 0$, $I \to 0$, and $\alpha_s < 1$. In phase A $\langle \cos \varphi/2 \rangle \neq 0$, in phase B $\langle \cos \varphi/2 \rangle = 0$. The interface between the phases A and B for $\alpha_T \gg 1$ is determined by relation (32). In phase for $\alpha_s < 1/4$, $\langle \varphi^2 \rangle \to \infty$, while for $1/4 < \alpha_s < 1$ the average value $\langle \varphi^2 \rangle$ is finite.

All these differences can be attributed to the physical differences between the three charge transport mechanisms under analysis: Nondissipative Cooper pair tunnelling, dissipative quasiparticle tunnelling and continuous normal electron transport through the ohmic shunt. These differences are manifested in particular in the different translational symmetry of the terms in action (1)–(3), which in turn influences the specific nature of the dissipative phase transitions in this system ($T \to 0$, $I \to 0$). For $E_J/E_Q \gg 1$ spontaneous translational symmetry breaking[2)] $\varphi + \varphi + 2\pi$ occurs at the point $\alpha_s + \alpha_T = 1$, so that for $\alpha_s + \alpha_T > 1$ the average $\langle \cos \varphi/2 \rangle$ becomes nonzero. The translational symmetry $\varphi \to \varphi + 4\pi$, which breaks down for $\alpha_s > 1/4$ (region III in Fig. 3, a) is implemented here for $\alpha_s < 1/4$ (region II in Fig. 3, a), and this symmetry breaking causes localization of φ. The phase transition at the point $\alpha_s = 1/4$ is in fact a dissipative Schmid phase transition[4–6] in the 4π-periodic potential and occurs only for $\alpha_s + \alpha_T > 1$. We note that the order parameter $\langle \cos \varphi/2 \rangle$ does not change in such a phase transition, while the phase localization and, consequently, the classical Josephson effect will occur only in region III. In regions I, I′ and II $\langle \varphi^2 \rangle \to \infty$ the quasicharge is localized, and the initial segment of the I-V characteristic corresponds to the resistive state.

For $E_J/E_Q \ll 1$ and $\alpha_s < 1$ the phase with broken symmetry $\varphi \to \varphi + 2\pi$ is implemented for $\alpha_T > \alpha_T^*(E_J)$, where α_T^* is determined by relation (32). As is the case with large E_J in this phase for $\alpha_s < 1/4$ the average value $\langle \varphi^2 \rangle$ diverges, while for $1 > \alpha_s > 1/4$ the phase φ is localized. The Josephson phase is localized for $\alpha_s > 1$ for any α_T, and the phase diagram in Fig. 3 b is meaningless.

The authors are grateful to K. K. Likharev for useful discussion of the results of this study.

[1)]We note that in investigating the quantum properties of the system the states φ and $\varphi + 2\pi$ should be considered nonequivalent states (see also Ref. 1). The validity of this statement is obvious for $I \neq 0$. In the case $I = 0$ these states are experimentally distinguishable at any level of weak interaction of the coordinate φ with the "external medium", a role that can be played by either the shunt or an external circuit. We also note that in virtually any experimental situation the charge of the superconducting electrons is not quantized in the scale e, which also suggests that the states φ and $\varphi + 2\pi$ are not identical.

[2)]Regarding translational symmetry we are in all cases referring to the symmetry of the effective action (or the effective Hamiltonian) but not of the quantum-mechanical state space of the entire system.

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