Superthermal particles and electron thermal conductivity

S. I. Krasheninnikov

I. V. Kurchatov Institute of Atomic Energy, Moscow
(Submitted 26 July 1988)
Zh. Eksp. Teor. Fiz. **94**, 166–171 (December 1988)

A class of solutions which can be represented in terms of self-similar variables is found for the collisional kinetic equation for electrons which describes energy transport. These solutions correspond to a constant value of the ratio of the electron mean free path to the length scale of the variation in the average electron energy, γ. An analytic solution is derived for the self-similar kinetic equation in the case $\gamma \ll 1$. In large systems in which the electron temperature varies comparatively rapidly with distance, for $d \ln T/d \ln x > 2/7$, most of the energy flux is carried at low temperatures by superthermal particles at arbitrarily small values of γ.

1. INTRODUCTION

Research on the electron thermal conductivity in a fully ionized plasma has shown that the theory of Ref. 1 fails at small values of the parameter $\gamma = \lambda/L \gtrsim 10^{-2}$, where λ is the mean free path, and L the length scale of the variations in the electron temperature.[2–5] Such values of γ are extremely typical in the plasma near the wall of a tokamak,[6,7] in space plasmas,[8] in plasma produced in the interaction of intense energy fluxes with matter,[9] etc.

One possible reason for the discrepancy between the results of Ref. 1 and experimental data is that the component of the heat flux carried by superthermal electrons,[2–5] whose mean free path increases with increasing energy, is not described adequately. The effect of superthermal electrons on the electron thermal conductivity has been studied numerically[3–5] and also by various approximate analytic methods (Refs. 10 and 11, for example). The primary complicating factor in the problem of determining the electron thermal conductivity is the multidimensional nature of the electrons. Even in the very simple case in which the electron temperature varies only as a function of a single spatial coordinate, the number of independent variables is three: this spatial coordinate plus the longitudinal and transverse velocities.

In the present paper we show that there exists a class of solutions of the collisional kinetic equation for electrons which can be represented in terms of self-similar variables (Sec. 2), which reduce the dimensionality of the problem. In Sec. 3 we find a solution of the self-similar kinetic equation for small values of the parameter γ. In Sec. 4 we calculate the heat fluxes carried by the superthermal particles, discuss the applicability of the results, and compare the results with the results of other studies.

2. SELF-SIMILAR VARIABLES

We assume that the ions are at rest, and we consider the steady-state kinetic equation for the electron distribution function, which varies as a function of x:

$$V\mu\frac{\partial f}{\partial x}-\frac{eE(x)}{m}\frac{\partial f}{\partial V_x}=\frac{2\pi e^4}{m^2}\Lambda\left[\hat{C}(\mathbf{V},f)+\frac{Z_i n(x)}{V^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}\right], \quad (1)$$

$$\hat{C}(\mathbf{V},f)=\frac{\partial}{\partial V_\beta}\int\left[f(\mathbf{V})\frac{\partial f(\mathbf{V}')}{\partial V_\alpha'}-f(\mathbf{V}')\frac{\partial f(\mathbf{V})}{\partial V_\alpha}\right]U_{\alpha\beta}\,d\mathbf{V}'$$

$$U_{\alpha\beta}=(u^2\delta_{\alpha\beta}-u_\alpha u_\beta)/u^3,\quad u_\beta=V_\beta-V_\beta',$$

where $\mu=\cos\theta$, and θ is the angle between the particle velocity vector $\mathbf{V}$ and the x axis; $f(V,\mu,x)$ is the electron distribution function; e, m, and $n(x)$ are the charge, mass, and density of the electrons; Z_i is the effective charge of the ions; Λ is the Coulomb logarithm; $E(x)$ is the ambipolar electric field; and the operator $\hat{C}(\mathbf{V},f)$ describes the Coulomb interaction between electrons.

The sole dimensionless parameter which determines the solution of (1) is the ratio of the mean free path of the electrons with the average energy $T(x)$ to the length scale of the variations in $T(x)$:

$$\gamma=-T^2(d\ln T/dx)/(2\pi e^4\Lambda n).$$

It is natural to assume that if the parameter γ does not depend on x then Eq. (1) will have a solution in terms of self-similar variables. Setting $\gamma=\text{const}$, and writing the distribution function $f(\mathbf{V},x)$ in the form

$$f(\mathbf{V},x)=NF(\mathbf{v})/[T(x)]^\alpha,\quad \mathbf{v}=\mathbf{V}(m/2T(x))^{1/2} \quad (2)$$

where N is a normalization factor, $\int F(\mathbf{v})d\mathbf{v}=1$, and α is an adjustable parameter, we find from (1) an integrodifferential equation for $F(\mathbf{v})$:

$$\gamma v\mu\left(\alpha F+\frac{v}{2}\frac{\partial F}{\partial v}\right)-\frac{\gamma_E}{2}\frac{\partial F}{\partial v_x}=\frac{1}{4}\left[\hat{C}(\mathbf{v},F)+\frac{z_i}{v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial F}{\partial\mu}\right]. \quad (3)$$

Here the quantity $\gamma_E = eET/(2\pi e^4\Lambda n)$ is found from the condition that the particle flux is ambipolar; in the problem as formulated here, this condition reduces to the equation $\int F(\mathbf{v})v_x d\mathbf{v}=0$.

The assumption that the parameter $\gamma\sim T^2\times|d\ln T/dx|n^{-1}$ is constant, combined with the self-similar nature of distribution function (2), leads to the following functional dependences $T(x)$ and $n(x)$:

$$T^{(\alpha-1/2)}\frac{dT}{dx}=\text{const},\qquad n\sim T^{(3/2-\alpha)}. \quad (4)$$

We now consider the expression for the energy flux density corresponding to (2):

$$q(x)=T(x)^{(3-\alpha)}N\left(\frac{2}{m}\right)^2 Q,\quad Q=\int F(\mathbf{v})v^2v_x\,d\mathbf{v}. \quad (5)$$

It is not difficult to see that the x dependence drops out of (5) in the case $\alpha = 3$. In other cases we have $dq/dx \neq 0$; i.e., the energy balance equation contains either time-dependent terms or energy sources or sinks which were not considered in (1) and (3). In the case $\gamma \ll 1$, however, the variation in $T(x)$ has little effect on the distribution function at thermal velocities, $V \lesssim (2T/m)^{1/2}$, and the function f is approximately a local Maxwellian function. We can thus assume that Eq. (3) in the case $\alpha \neq 3$, $\gamma \ll 1$ describes the effect of superthermal particles on $q(x)$ either for a slow variation of the function $f(\mathbf{V},x)$ in the thermal-velocity region or if there are energy sources or sinks in this region. The case $\alpha = 3$, we might note, also corresponds to energy flux conservation in the case of a classical[1] temperature dependence of the electron thermal conductivity $\varkappa_e$: $\varkappa_e \propto T^{5/2}$ [see (4)].

3. SOLUTION OF THE SELF-SIMILAR KINETIC EQUATION IN THE CASE $\gamma \ll 1$

Structurally, the self-similar kinetic equation (3) is approximately the same as the equation describing electron runaway in a static electric field.[12] Accordingly, as in Refs. 12–14, we seek solutions of (3) in different characteristic regions of the variable v, which we will subsequently join. We set $z_i = 1$ at this point.

I. The low-energy region, $\xi = v^2 \lesssim \gamma^{-1/3}$

In this case, a solution of (3) can be found[1] by expanding the function $F(\mathbf{v})$ in Legendre polynominals $P_i(\mu)$:

$$F(\mathbf{v}) = \sum_{i=0}^{\infty} F_i(\xi) P_i(\mu). \tag{6}$$

We find the asymptotic expression for $F_i(\xi)$ at $1 \ll \xi \lesssim \gamma^{-1/3}$ from (3) by using the well-known approximation for the collision term $C(\mathbf{v},F)$ at large values of $v (v \gg 1)$:

$$\gamma\left[\left(\alpha F + \xi\frac{\partial F}{\partial \xi}\right)\mu - \delta\left(\mu\frac{\partial F}{\partial \xi} + \frac{1-\mu^2}{2\xi}\frac{\partial F}{\partial \mu}\right)\right] = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(F + \frac{\partial F}{\partial \xi}\right) + \frac{1}{2\xi^2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial F}{\partial \mu}, \tag{7}$$

where $\delta = \gamma_E/\gamma \sim 1$. Writing $F_i(\xi)$ in the form

$$F_i(\xi) = \pi^{-3/2} e^{-\xi} \sum_{k=0}^{\infty} (2\gamma\xi^3)^{2k+i} a_k^i, \tag{8}$$

where $a_0^0 = 1$, we find the following recurrence relation from (7):

$$a_k^i = \left(\frac{i}{2i-1} a_k^{i-1} + \frac{i+1}{2i+3} a_{k-1}^{i+1}\right) \Big/ [i(i+1) + 6(2k+i)]. \tag{9}$$

Substituting (6), (8), and (9) into (5), we easily see that the term in the integrand which is linear in γ reaches a maximum at $\xi = \xi_m = 5$. Noting that the expansion in (8) is carried out in $\gamma\xi^3 \lesssim 1$, we see that the inequality $\gamma\xi_m^3 \lesssim 1$ leads to a limitation on the linear approximation[1] at a level $\gamma \lesssim 10^{-2}$.

II. The region $\gamma^{-1/3} \lesssim \xi \lesssim \gamma^{-1/2}$

We introduce the variable $z = \xi\gamma^{1/2}$, and we write the function $F(\mathbf{v})$ in the form $F(\mathbf{v}) = \pi^{-3/2}\exp(-\varphi(z,\mu))$. Expanding $\varphi(z,\mu)$ in a power series in $\gamma^{1/4}$,

$$\varphi = \frac{\varphi_0}{\gamma^{1/2}} + \frac{\varphi_1}{\gamma^{1/4}} + \varphi_2 + \ldots, \tag{10}$$

substituting (10) into (7), and carrying out the corresponding calculations, we find

$$\varphi_0 = z - z^3/3,$$

$$\varphi_1 = \int_0^z \left(\frac{x}{1-x^2}\right)^{1/2} dx + 4[z^3(1-z^2)]^{1/2}\left[1 - \left(\frac{1+\mu}{2}\right)^{1/2}\right],$$

$$\begin{aligned}\varphi_2 = &-\frac{1}{2}\left(\alpha - \frac{3}{4}\right)\ln(1-z^2) + \frac{\delta z^2}{2} - \frac{1}{4}\frac{z^2}{(1-z^2)} \\ &- \frac{1}{2}\ln\left[\frac{z^2}{(1-z^2)\gamma^{1/3}}\right] - \frac{1}{4}\ln\left(\frac{2}{1+\mu}\right) \\ &- (7-10z^2)\ln\left[\frac{2}{1+\left(\frac{1+\mu}{2}\right)^{1/2}}\right] \\ &+ \frac{2z^2}{(1-z^2)}(7-10z^2)\left[1 - \left(\frac{1+\mu}{2}\right)^{1/2}\right] \\ &- z^2(3-5z^2)(1-\mu)/(1-z^2) + C_1,\end{aligned} \tag{11}$$

where the normalization constant C_1 can be found by joining solutions (8) and (11).

III. The region $-1 < y\gamma^{1/6} < 1$, $y = (\xi^2\gamma - 1)/\gamma^{1/6}$.

We write the function φ as a series in $\gamma^{1/6}$:

$$\varphi(y,\mu) = \frac{\varphi^0}{\gamma^{1/6}} + \varphi^1 + \ldots + C_2. \tag{12}$$

Expanding φ^i in a Taylor series near $\mu = 1$,

$$\varphi^i(y,\mu) = \sum_{k=0}^{\infty} \varphi_k^i(y)\frac{(1-\mu)^k}{k!}, \tag{13}$$

and substituting (12) and (13) into (7), we find

$$\varphi_0^0 = -\frac{g}{2} - \frac{g^4}{4}, \quad \varphi_1^0 = g, \tag{14}$$

$$\varphi_0^1 = -\left(\alpha - \frac{7}{4}\right)\ln g + \frac{1}{2}\ln\left(\frac{2g^3+1}{g^3}\right) + \frac{g^3(1-g^3/2)}{6},$$

$$\varphi_1^1 = \frac{3}{4}(1-g^3) + \frac{1}{4}\frac{g^3}{(1+2g^3)} + \left[\alpha - \frac{7}{4} + \frac{3}{2}\frac{1}{(1+2g^3)}\right]\frac{1}{(1+2g^3)}.$$

The function $g(y)$, as in Ref. 11, is a positive solution of the algebraic equation

$$g^3 + yg - 1 = 0, \tag{15}$$

$$g = \begin{cases} 2(-y/3)^{1/2}\cos\left\{\frac{1}{3}\arccos\left[\left(-\frac{2^{2/3}y}{3}\right)^{-3/2}\right]\right\}, & y < -\frac{3}{2^{2/3}}, \\ \frac{1}{2^{1/3}}\left(\left\{\left[\left(\frac{2^{2/3}y}{3}\right)^3 + 1\right]^{1/2} + 1\right\}^{1/3} - \left\{\left[\left(\frac{2^{2/3}y}{3}\right)^3 + 1\right]^{1/2} - 1\right\}^{1/3}\right), & y > -\frac{3}{2^{2/3}}. \end{cases} \tag{16}$$

The normalization constant C_2, found by joining solutions (10) and (12) in the region $-\gamma^{-1/6}<y<-\gamma^{1/12}$, where both expansions, (10) and (12), are valid, is

$$C_2=\frac{2}{3}\gamma^{-1/2}+\gamma^{-1/4}\frac{(2\pi)^{3/4}}{[\Gamma(1/4)]^2}-\frac{(\alpha-15/4)}{12}\ln\gamma+\frac{1}{8}+\frac{\delta}{2}-\frac{\ln 2}{2}+C_1, \qquad (17)$$

where $\Gamma(x)$ is the gamma function.

IV. The region $z=\xi\gamma^{1/2}\gg 1$

Here we seek the function $F(\mathbf{v})$ in the form

$$F(\mathbf{v})=\Phi_0(z,\mu)+\gamma^{1/2}\Phi_1(z,\mu). \qquad (18)$$

Substituting (18) into (7), we find an equation for Φ_0:

$$z^2\mu\left(\alpha\Phi_0+z\frac{\partial\Phi_0}{\partial z}\right)=z\frac{\partial\Phi}{\partial z}+\frac{1}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Phi_0}{\partial\mu}. \qquad (19)$$

In contrast with the runaway-electron problem,[12-14] Eq. (19) does not lead to the formation of a beam of particles, as can be seen without difficulty by going through the corresponding calculations. Since the higher-energy electrons are essentially insensitive to Coulomb collisions and thus to the values of $T(x)$ and $n(x)$, we would expect that the distribution function $f(\mathbf{v},x)$ would become independent of x at $z\gg 1$. In view of the self-similar nature of (2), we see that the only function Φ_0 which satisfies this condition is

$$\Phi_0(z,\mu)=C_3\tilde{\Phi}(\mu)/z^{\alpha}, \qquad (20)$$

where C_3 is a normalization constant, and the function $\tilde{\Phi}(\mu)$ describes the angular dependence of $F(\mathbf{v})$ at $z\gg 1$.

It is not difficult to show that (20) is the asymptotic solution of (19) for $z\gg 1$. It is not possible to determine the function $\tilde{\Phi}(\mu)$ without a numerical solution of (7). However, since the angular asymmetry of $F(\mathbf{v})$ in the region $z^2-1\lesssim 1$ [see (12) and (14)] is approximately unity, we can assume that the estimate $\int\tilde{\Phi}(\mu)\mu d\mu\sim 1$ holds for $z\gg 1$. The constant C_3 is determined to $O(1)$ by joining solutions (13) and (20):

$$C_3=\pi^{-3/2}\exp\left(-C_2+\frac{(\alpha-1/4)}{6}\ln\gamma\right). \qquad (21)$$

4. ENERGY FLUXES

The power-law dependence $F(\mathbf{v})$ in (20) in the case $\alpha\leqslant 3$ causes the integrand for the dimensionless energy flux density Q in (5) to diverge at an arbitrarily small value of γ. However the appearance of superthermal particles is related to the transport of these particles out of hotter regions (the effect of the electric field, $\sim dT/dx$, is unimportant here). In real, bounded systems, in which the maximum temperature is limited, the divergence can thus be removed through a cutoff of the integral Q in (5) at large values $v<v_{\max}$. At $\alpha=3$, the logarithmic divergence of Q which arises can be eliminated by imposing the restriction $v<c$, where c is the velocity of light. A more detailed discussion of relativistic effects goes beyond the scope of this paper and will be taken up in a separate paper.

However, even when we do incorporate a cutoff of the integral Q at large values of v, we find that the power-law tail of the distribution function can make a dominant contribution to the energy flux $\Delta Q(\mathrm{IV})$ under the conditions $\alpha<3$ and $\gamma\ll 1$, since this contribution contains a large numerical factor $\sim v_{\max}^{2(3-\alpha)}$:

$$\Delta Q(\mathrm{IV})\approx\frac{\pi C_3}{\gamma^{\alpha/2}}\frac{v_{max}^{2(3-\alpha)}}{(3-\alpha)}. \qquad (22)$$

For $\alpha>3$ we find a different situation. In this case the integral expression for Q converges as $v\to\infty$:

$$\Delta Q(\mathrm{IV})\approx\frac{\pi C_3}{\gamma^{3/2}(\alpha-3)}. \qquad (23)$$

We can determine the energy flux which is transported in the energy region $\gamma^{-1/3}\lesssim\xi\lesssim\gamma^{-1/2}$. Assuming $\gamma\ll 1$ we find from (5), (10), and (11), to $O(1)$,

$$\Delta Q(\mathrm{II})\approx\pi^{-1/2}\gamma^{-2/3}\exp(-A/\gamma^{1/3}), \qquad (24)$$

where the numerical factor A is on the order of unity.

Since we have $\Delta Q(\mathrm{IV})\propto C_3\propto\exp(-2/3\gamma^{1/2})$, in the case $\gamma\ll 1$ we have $\Delta Q(\mathrm{IV})<\Delta Q(\mathrm{II})$. Consequently, in the case $\alpha>3$ the power-law tail on the self-similar distribution function can contribute substantially to the net energy flux only if γ is sufficiently large.

We should point out that the analytic solution of the self-similar kinetic equation (Sec. 3) is of limited applicability even at comparatively small values of γ, since small terms of order $\gamma^{1/6}$ were taken into account in the derivation of that equation. The self-similar function itself, however, correctly describes the asymptotic behavior of the transition of the energy flux from a thermal-conductivity flux to a kinetic flux $\sim nT^{3/2}$ as γ increases from 0 to 1. Specifically, we find $Q\to O(1)$ as $\gamma\to 1$ (for the values of α for which the integral expression for Q converges). Hence, using (4), we find $q(x)\sim nT^{3/2}$.

Let us compare the results derived in this paper with the semiempirical expression for the energy flux density which is most frequently used, and which was found by analyzing the results of numerical calculations[5]:

$$q(x)=\frac{1}{2}\int\frac{dx' q_{SH}(x')}{\lambda(x')}\exp\left(-\left|\int_{x'}^{x}\frac{dl}{\lambda(l)}\right|\right), \qquad (25)$$

where $q_{SH}\sim T^{5/2}dT/dx$ is the expression for the energy flux density which follows from the theory of Ref. 1.

Substituting (4) with $1>\gamma|\alpha-3|$ into (25), we find

$$q(x)=\frac{q_{SH}(x)}{[1-\gamma^2(3-\alpha)^2]}. \qquad (26)$$

It is not difficult to see that expressions (25) and (26) give only a poor description of the distribution of the energy flux density in the class of functions $n(x)$ and $T(x)$ considered here. In particular, expressions (25) and (26) fail completely to differentiate between the fundamental differences in $q(x)$ at $\alpha<3$ and $\alpha>3$, which we discussed at the beginning of this section of the paper. Expression (25) apparently gives a fairly good description only of power-law profiles $T(x)$, which are the types ordinarily used in numerical calculations. A final conclusion, however, will require further study.

[1]L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).
[2]A. V. Gurevich and Ya. N. Pstomin, Zh. Eksp. Teor. Fiz. **77**, 933 (1979) [Sov. Phys. JETP **50**, 470 (1979)].
[3]A. R. Bell, R. G. Evans, and D. J. Nicholas, Phys. Rev. Lett. **46**, 243 (1981).
[4]R. J. Mason, Phys. Rev.Lett. **47**, 652 (1981).
[5]J. F. Luciani, P. Mora, and J. Virmont, Phys. Rev. Lett. **51**, 1664 (1983).
[6]A. V. Nedospasov, Usp. Fiz. Nauk **152**, 479 (1987) [Sov. Phys. Usp. **30**, 620 (1987)].
[7]S. I. Krasheninnikov, A. S. Kukushkin, and V. I. Pistunovich, Nucl. Fusion **27**, 1805 (1987).
[8]A. A. Galeev and A. M. Natanzon, Fiz. Plazmy **8**, 1151 (1982) [Sov. J. Plasma Phys. **8**, 650 (1982)].
[9]J. D. Hares *et al.*, Phys. Rev. Lett. **42**, 1216 (1979).
[10]M. H. Shirasian and L. C. Steinhauer, Phys. Fluids **24**, 843 (1981).
[11]Yu. L. Igitkhanov and P. N. Yushmanov, in: Proceedings of the Fourteenth European Conferences on Controlled Fusion and Plasma Physics, Madrid, Vol. 2, 1987, p. 686.
[12]A. V. Gurevich, Zh. Eksp. Teor. Fiz. **39**, 1296 (1960) [Sov. Phys. JETP **12**, 904 (1961)].
[13]A. N. Lebedev, Zh. Eksp. Teor. Fiz. **48**, 1392 (1965).
[14]J. W. Connor and R. J. Hastie, Nucl. Fusion **15**, 415 (1975).

Translated by Dave Parsons