Diffraction inversion of a wavefront

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A general formalism is proposed for the solution of the Maxwell equations with the aid of complete Green functions. This formalism is based on the fifth-parameter technique. A new effect is predicted: an object wave scattered by a smooth inhomogeneity forms a conjugate image additional to the main one. The two images are on opposite sides of the inhomogeneity and are simultaneously real or virtual. This effect is of wave origin: the intensity of the conjugate image vanishes in the limit $\lambda \to 0$.

The theory of propagation of waves relies mainly on approximate methods in spite of the fact that the exact solution of any system of linear equations can be represented in a series form.[1,2] This is because series are not very suitable for numerical calculations. Several asymptotic methods for solving the wave equations are known at present: these are integral equations,[3,4] geometric optics,[3,5–7] smooth perturbations,[7] short-wavelength asymptotes,[8,9] and a parabolic equation.[3,7,10] Use is also made of the methods developed in quantum field theory[11,12] and of variational principles.[13]

The most general approach to the solution of linear equations with variable coefficients involves the use of complete Green functions. At present the main method for the calculation of Green functions is perturbation theory. On the other hand a nonperturbative approach is used extensively in quantum theory and this is based on the fifth-parameter technique developed by Fock and Schwinger.[14–16] We shall use this method to solve the Maxwell equations.

COMPLETE GREEN FUNCTION FOR THE MAXWELL EQUATIONS

The vector representing the intensity of an electric field in an inhomogeneous medium exhibiting time dispersion satisfies the equation

$$(\mathrm{rot\ rot}\ \mathbf{E})_m+\partial_0^2\hat{\varepsilon}_{mn}(t,\ \mathbf{x})E_n=0. \tag{1}$$

Going over to a Fourier transformation, we find from Eq. (1) that

$$\Lambda_{ms}e_s(\omega,\ \mathbf{x})=0, \tag{2}$$

$$\Lambda_{ms}=-\varepsilon_{mjk}\varepsilon_{kls}\partial_{jl}+\omega^2\varepsilon_{ms}=\delta_{ms}\partial^2-\partial_{ms}+\omega^2\varepsilon_{ms}. \tag{3}$$

Here, $\varepsilon_{ms}(\omega,\mathbf{x})$ is the permittivity tensor.

We shall assume that the functions $\varepsilon_{ms}(\omega,\mathbf{x})$ decrease sufficiently rapidly outside a finite number of inhomogeneities located in a limited amount of space. If a source creating a field $\mathbf{e}^{(0)}(\mathbf{x})$ is specified outside this region, there should be a solution of Eq. (2) which differs in the asymptotic limit from $\mathbf{e}^{(0)}(\mathbf{x})$ by the presence of a diverging spherical wave. Our task will be to find the solution of Eq. (2) subject to these boundary conditions.

We shall introduce a complete Green function satisfying the equation

$$\Lambda_{ms}G_{sn}(\mathbf{x},\ \mathbf{x}')=-\delta_{mn}\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{4}$$

Then, the solution of Eq. (2) can be represented in the form[17]

$$e_m(\mathbf{x})=e_m^{(0)}(\mathbf{x})+\omega^2\int d^3x'\,G_{mk}(\mathbf{x},\mathbf{x}')\sigma_{ks}(\mathbf{x}')e_s^{(0)}(\mathbf{x}'), \tag{5}$$

where $\sigma_{ks}=\varepsilon_{ks}-\delta_{ks}$. Both Eqs. (2) and (4) cannot be integrated in a finite form. However, an approximate solution of Eq. (4) for the Green function has a number of important advantages. Firstly, the Green function determines the structure of the general solution (1) irrespective of the source or of the function $\mathbf{e}^{(0)}(\mathbf{x})$. [It should be noted that in most approximate methods a specific $\mathbf{e}^{(0)}(\mathbf{x})$ function is used as the first approximation.] Secondly, the main advantage of the adopted approach is the fact that the solution (4) can be obtained on the basis of a nonperturbative approximation based on the fifth-parameter technique.

We shall write down the inverse of the operator Λ in the form of an integral with respect of the parameter τ:

$$\Lambda^{-1}=-\frac{i}{2\omega}\int_0^\infty d\tau\exp\left[i\frac{\tau}{2\omega}(\Lambda+i0)\right]. \tag{6}$$

Using Eq. (6), we obtain the following solution of Eq. (4):

$$G_{mn}(\mathbf{x},\mathbf{x}')=\frac{i}{2\omega}\int_0^\infty d\tau\,U_{mn}(\mathbf{x},\mathbf{x}',\tau), \tag{7}$$

$$U(\mathbf{x},\mathbf{x}',\tau)=\exp\left[i\frac{\tau}{2\omega}(\Lambda+i0)\right]\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{8}$$

There are several methods of calculating the function $U(\mathbf{x}, \mathbf{x}', \tau)$.

A. Let us assume that $\varphi_n(\mathbf{x})$ is any complete system of orthonormalized functions. We then have

$$U(\mathbf{x},\mathbf{x}',\tau)=\sum_n\varphi_n^*(\mathbf{x}')\exp\left[i\frac{\tau}{2\omega}(\Lambda+i0)\right]\varphi_n(\mathbf{x}).$$

We shall find initially the Green function for homogeneous space ($\varepsilon_{ms}=\delta_{ms}$), assuming that $\varphi_\mathbf{k}(\mathbf{x})=(2\pi)^{-3/2}\times\exp(i\mathbf{kx})$:

$$U_{mn}(\mathbf{x},\mathbf{x}',\tau)=\int\frac{d^3k}{(2\pi)^3}Q_{mn}(\mathbf{k},\tau)e^{i\mathbf{k}\boldsymbol{\xi}},\quad \boldsymbol{\xi}=\mathbf{x}-\mathbf{x}',$$

$$Q=\exp\left(i\frac{\tau}{2\omega}A\right),\quad A_{mn}=(\omega^2-k^2)\delta_{mn}+k_mk_n. \tag{9}$$

Allowing for the rules of calculation of a function of matrices, we obtain

$$Q_{mn}=\sum_\mu P_{mn}^{(\mu)}\exp\left(i\frac{\tau}{2\omega}\lambda_\mu\right).$$

Here, the projection operator is $P_{mn}^{(\mu)} = u_m^{(\mu)}(u_n^{(\mu)})^*$; $u_m^{(\mu)}$ and λ_μ are the eigenvectors and eigenvalues of the equation $A_{ms}u_s = \lambda u_m : \lambda_{1,2} = \omega^2 - k^2, \lambda_3 = \omega^2; \mathbf{u}^{(1)} = \mathbf{e}_\theta, \mathbf{u}^{(2)} = \mathbf{e}_\varphi$, $\mathbf{u}^{(3)} = \mathbf{k}/k$ are the unit vectors of a spherical coordinate system in the momentum space. Obviously, we have $Q_{mn}(\mathbf{k},0) = \delta_{mn}$. Since

$$\sum_\mu \lambda_\mu^{-1} P_{mn}^{(\mu)} = \left(\delta_{mn} - \frac{k_m k_n}{\omega^2}\right)(\omega^2 - k^2 + i0)^{-1},$$

we find from Eqs. (7) and (9) that[18]

$$G_{mn}(\mathbf{x},\mathbf{x}') = \left(\delta_{mn} - \omega^{-2}\frac{\partial^2}{\partial x_m \partial x_n'}\right) G^{(0)}(\xi),$$

$$G^{(0)}(\xi) = \frac{e^{i\omega|\xi|}}{4\pi|\xi|}. \tag{10}$$

B. We shall represent $G(\mathbf{x}, \mathbf{x}')$ as a matrix element of the operator $\hat{G} = -\Lambda^{-1}$ in the coordinate representation: $G(\mathbf{x},\mathbf{x}') = \langle \mathbf{x}'|\hat{G}|\mathbf{x}\rangle$. The vectors $|\mathbf{x}\rangle$ are normalized by the condition $\langle \mathbf{x}|\mathbf{x}'\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$. Equation (4) represents a matrix element of the equation $\hat{\Lambda}\hat{G} = -I$, where the operators $\hat{x}_\alpha$ and $\hat{p}_\alpha = -i\partial_\alpha$ obey the usual commutation relations. Consequently, the quantity Λ in Eq. (8) can be regarded as a shift operator in the variable τ. The operator $\hat{U} = \exp(i\tau\Lambda/2\omega)$ describes evolution of a certain system with the Hamiltonian $H = -\Lambda/2\omega$. Equation $i\partial_\tau \hat{U} = \hat{H}\hat{U}$ is formally identical with the equation for the S matrix. Its solution can be found by the field-theory methods.

C. The transformation function $U_{mn}(\mathbf{x},\mathbf{x}',\tau)$ satisfies a Schrödinger-type equation

$$i\partial_\tau U_{mn} = -\Lambda_{ms}U_{sn}/2\omega \tag{11}$$

subject to the initial condition

$$U_{mn}(\mathbf{x},\ \mathbf{x}',\ 0) = \delta_{mn}\delta^{(3)}(\mathbf{x}-\mathbf{x}').$$

The solution of Eq. (11) can be obtained by methods developed by Fock and Schwinger.[14,15,18,19]

GREEN FUNCTION IN THE WKB APPROXIMATION

We shall seek a particular solution of Eq. (11) in the form

$$U_{mn} = L_{mn}e^{iS}.$$

Separating the real and imaginary parts, we obtain

$$[2\omega\delta_{mn}\partial_\tau S + \varepsilon_{mjk}\varepsilon_{kls}(\partial_{jl} - \partial_j S\partial_l S) - \omega^2\varepsilon_{ms}]L_{sn} = 0, \tag{12}$$

$$2\omega\partial_\tau L_{mn} = \varepsilon_{mjk}\varepsilon_{kls}[\partial_j(L_{sn}\partial_l S) + \partial_j S(\partial_l L_{sn})]. \tag{13}$$

Assuming that $L = aK$, $S = \omega\psi$, $\mathbf{p} = \nabla\psi$, we shall write down the zeroth-approximation equations:

$$2\partial_\tau\psi K_{mn} = (-p^2\delta_{ms} + M_{ms})K_{sn}, \tag{14}$$

$$2\partial_\tau(aK_{mn}) = \varepsilon_{mjk}\varepsilon_{kls}[\partial_j(aK_{sn}p_l) + p_j\partial_l(aK_{sn})], \tag{15}$$

where $M_{ms} = \varepsilon_{ms} + p_m p_s$. The solution of Eq. (14) can be represented by an expansion in normal modes. With this in mind, assuming that $\mathbf{p} = \mathbf{n}$, we shall find the eigenvalues and the eigenvectors of the equation

$$(\varepsilon_{ms} + n_m n_s)u_s = n^2 u_m. \tag{16}$$

The equation $\det|M - n^2 I| = 0$ yields the refractive indices n_μ for normal modes. The eigenvectors $u_m^{(\mu)}$ satisfy the conditions

$$\varepsilon_{mn}(u_m^{(\mu)})^* u_n^{(\nu)} + p_m p_n (u_m^{(\mu)})^* u_n^{(\nu)} = n_\mu^2\delta_{\mu\nu}, \tag{17}$$

$$(u_m^{(\mu)})^* u_m^{(\nu)} = \delta_{\mu\nu}.$$

Consequently, the matrix obeys

$$K_{mn} = u_m^{(\mu)}(u_n^{(\mu)})^* \equiv E_{mn}^{(\mu)}(\mathbf{x},\mathbf{x}').$$

Using Eq. (17), we find from Eq. (14) the eikonal equation for a normal wave:

$$\partial_\tau\psi_\mu + {}^1/_2(\nabla\psi_\mu)^2 - {}^1/_2 n_\mu^2(\nabla\psi/|\nabla\psi|,\ \mathbf{x}) = 0. \tag{18}$$

We shall now represent a in the form $a = A\exp(i\alpha)$. We shall multiply Eq. (15) by $a^*(E_{mn}^{(\mu)})^*$ and add it to its complex-conjugate. This yields the energy transport equation (from which the index μ is omitted)

$$\partial_\tau A^2 + \partial_j\sigma_j A^2 = 0, \tag{19}$$

$$\sigma_j = {}^1/_2\varepsilon_{jmk}E_{mn}^* B_{kn} + \text{c.c.},\quad B_{kn} = \varepsilon_{kls}p_l E_{sn}.$$

In terms of the eigenvectors, we have

$$\boldsymbol{\sigma} = {}^1/_2([\mathbf{u}^*\mathbf{v}] + \text{c.c.}),\quad \mathbf{v} = [\mathbf{pu}].$$

We shall now multiply Eq. (15) by $a^*(E_{mn}^{(\mu)})^*$ and subtract the complex-conjugate equation:

$$\partial_\tau\alpha + \sigma_m\partial_m\alpha = {}^1/_2\,\mathrm{Im}\,\varepsilon_{mjk}(E_{mn}^*\partial_j B_{kn} - B_{mn}^*\partial_j E_{kn}). \tag{20}$$

This yields an equation for the determination of the change in the phase of the amplitude of a wave propagating in a gyrotropic medium.[5]

The solution of Eq. (18), $\psi(\mathbf{x},\mathbf{x}',\tau)$, can be found using ray characteristics satisfying canonical equations

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{p} - \frac{1}{2}\frac{\partial n^2}{\partial\mathbf{p}},\quad \frac{d\mathbf{p}}{d\tau} = \frac{1}{2}\frac{\partial n^2}{\partial\mathbf{x}}$$

with a Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2}n^2\left(\frac{\mathbf{p}}{p},\mathbf{x}\right).$$

In optical terms, $\psi(\mathbf{x},\mathbf{x}',\tau)$ is a generating function of the canonical transformation $x_\alpha = x_\alpha(\mathbf{x}',\mathbf{p}',\tau)$, $p_\alpha = p_\alpha(\mathbf{x}',\mathbf{p}',\tau)$ to constant coordinates $\mathbf{x}' = \mathbf{x}(0)$ and momenta $\mathbf{p}' = \mathbf{p}(0)$ (Refs. 19 and 20). In the path of a ray the function $F(\mathbf{x},\mathbf{x}',\tau) = \psi(\mathbf{x}(\tau),\mathbf{x}',\tau)$ satisfies the equation

$$\frac{dF}{d\tau} = \frac{\partial\psi}{\partial\tau} + \partial_\alpha\psi\dot{x}_\alpha \equiv l(\mathbf{x}',\mathbf{p}',\tau).$$

Here, $l(\mathbf{x}',\mathbf{p}',\tau)$ is the value of the "Lagrangian"

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}n^2\left(\frac{\mathbf{p}}{p},\mathbf{x}\right) - \frac{1}{8}\frac{\partial n^2}{\partial\mathbf{p}}.$$

If using the equation $x_\alpha = x_\alpha(\mathbf{x}',\mathbf{p}',\tau)$ we deduce the value $p_\alpha' = \pi_\alpha(\mathbf{x},\mathbf{x}',\tau)$ corresponding to a ray passing through the points $\mathbf{x}'$ and $\mathbf{x}(\tau)$, we obtain the solution of Eq. (18):

$$\psi(\mathbf{x},\mathbf{x}',\tau) = \int_0^\tau d\tau'\, l(\mathbf{x}',\boldsymbol{\pi}(\mathbf{x},\mathbf{x}',\tau),\tau'). \tag{21}$$

Similarly[19] we obtain the solution of Eq. (19):

$$A(\mathbf{x},\ \mathbf{x}',\ \tau)=I^{1/2}(\mathbf{x}',\ \boldsymbol{\pi}(\mathbf{x},\ \mathbf{x}',\ \tau),\ \tau),$$
$$I(\mathbf{x}',\ \mathbf{p}',\ \tau)=c\det|\partial^2F/\partial x_\alpha\partial x_\beta'|. \tag{22}$$

The integration constant c is found from the condition

$$\lim_{\tau\to0}A(\mathbf{x},\mathbf{x}',\tau)=(2\pi i\tau/\omega)^{-3/2}.$$

After calculation of the zeroth approximation we shall seek the solution of Eqs. (12) and (13) as an expansion in reciprocals of ω:

$$U_{mn}(\mathbf{x},\mathbf{x}',\tau)=\sum_\mu E_{mn}^{(\mu)}\left(a_\mu+\frac{1}{\omega^2}b_\mu+\ldots\right)\times\exp\left[i\omega\psi_\mu+\frac{i}{\omega}\chi_\mu+\ldots\right].$$

If there are several rays connecting the points $\mathbf{x}$ and $\mathbf{x}'$, then Eq. (8) is a superposition of the function $U_{mn}(\mathbf{x},\mathbf{x}',\tau)$ for each of these rays. If we limit ourselves to this approximation, we obtain the Green function

$$G_{mn}(\mathbf{x},\mathbf{x}')=\sum_\mu E_{mn}^{(\mu)}(\mathbf{x},\mathbf{x}')\,g_\mu(\mathbf{x},\mathbf{x}'), \tag{23}$$

where

$$g_\mu(\mathbf{x},\mathbf{x}')=\frac{i}{2\omega}\int_0^\infty d\tau\,a_\mu(\mathbf{x},\mathbf{x}',\tau)\exp[i\omega\psi_\mu(\mathbf{x},\mathbf{x}',\tau)]. \tag{24}$$

Eikonal approximation. We can obtain the Green function in the eikonal approximation simply by confining ourselves to the approximate solution of Eq. (18) obtained by the Fock method in the form of an expansion in powers of τ:

$$\psi_\mu(\mathbf{x},\ \mathbf{x}',\ \tau)=\xi^2/2\tau+\tau\psi_\mu^{(1)}(\mathbf{x},\ \mathbf{x}')+\tau^3\psi_\mu^{(3)}(\mathbf{x},\ \mathbf{x}')+\ldots.$$

Equation (18) yields a system of recurrence equations:

$$(k+\xi_\alpha\partial_\alpha)\,\psi_\mu^{(k)}=f_\mu^{(k)},\quad f_\mu^{(1)}=\frac{1}{2}n_\mu^2\left(\frac{\boldsymbol{\xi}}{\xi},\mathbf{x}\right),$$
$$f_\mu^{(3)}=-\frac{1}{2}(\nabla\psi_\mu^{(1)})^2,\ldots,$$

the solution of which is

$$\psi_\mu^{(k)}(\mathbf{x},\mathbf{x}')=\int_0^1 ds\,s^{k-1}f_\mu^{(k)}(\mathbf{x}'+\boldsymbol{\xi}s). \tag{25}$$

We shall find the Green function for a gyrotropic medium representing a magnetically active cold plasma with a tensor

$$\varepsilon_{mn}=\varepsilon_\perp\delta_{mn}+(\varepsilon_\|-\varepsilon_\perp)\,b_mb_n+i\varkappa\varepsilon_{mnk}b_k,$$

where b is a unit vector parallel to the induction of a static homogeneous magnetic field.[21,22] In the eikonal approximation, we have

$$\mathbf{u}^{(1)}(\boldsymbol{\xi})=2^{-1/2}(\mathbf{e}_\theta+i\mathbf{e}_\varphi),\quad \mathbf{u}^{(2)}(\boldsymbol{\xi})=2^{-1/2}(\mathbf{e}_\theta-i\mathbf{e}_\varphi),$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit vectors perpendicular to the vector $\boldsymbol{\xi}$,

$$\psi_\mu^{(1)}(\mathbf{x},\mathbf{x}')=\frac{1}{2}\int_0^1 ds\,n_\mu^2\left(\frac{\boldsymbol{\xi}}{\xi},\mathbf{x}'+\boldsymbol{\xi}s\right),$$

$$n_\mu^2\approx\varepsilon_{mn}(u_m^{(\mu)})^*u_n^{(\mu)}$$
$$=\varepsilon_\perp+{}^1/_2(\varepsilon_\|-\varepsilon_\perp)\left[(\mathbf{b}\mathbf{e}_\theta)^2+(\mathbf{b}\mathbf{e}_\varphi)^2\right]+(-1)^\mu\varkappa\mathbf{b}\boldsymbol{\xi}/\xi.$$

The function $\psi(\mathbf{x},\mathbf{x}',\tau)$ is invariant under the substitution $\mathbf{x}\to\mathbf{x}',\mathbf{b}\to-\mathbf{b}$.

From Eqs. (20) and (22) we find that $\alpha=0$, $A=(2\pi i\tau/\omega)^{3/2}$. Consequently,

$$g_\mu(\mathbf{x},\mathbf{x}')=(4\pi\xi)^{-1}\exp(ik_\mu\xi),$$

$$k_\mu^2=\omega^2\int_0^1 ds\,n_\mu^2\left(\frac{\boldsymbol{\xi}}{\xi},\mathbf{x}'+\boldsymbol{\xi}s\right). \tag{26}$$

It should be noted that already in this approximation the Fourier transform $g_\mu(\mathbf{x},\mathbf{x}')$ contains a pole corresponding to an allowance for the multiple scattering effects. A similar result can be obtained by summing a Born series.[12,23] After substitution of Eqs. (26) and (23) into Eq. (5), we obtain a solution which is valid in the Born and eikonal approximations.

FORMATION OF DIRECT AND CONJUGATE IMAGES

We shall assume that the medium is isotropic. In this case, we have $n_\mu^2=n^2(\mathbf{x})$, and

$$\sum_\mu E_{mn}^{(\mu)}(\mathbf{x},\mathbf{x}')=\delta_{mn}-\xi_m\xi_n/\xi^2\equiv E_{mn}(\boldsymbol{\xi}/\xi).$$

The solution of Eq. (5) can be represented by

$$e_m(\mathbf{x})=e_m^{(0)}(\mathbf{x})+\omega^2\int d^3x'\,E_{ms}(\mathbf{x},\mathbf{x}')\,g(\mathbf{x},\mathbf{x}')\,\sigma(\mathbf{x}')\,e_s^{(0)}(\mathbf{x}'), \tag{27}$$

where $\sigma=n^2-1$. Let us assume that the function $\sigma(\mathbf{x})$ decreases rapidly outside a finite region of space of volume $\sim L^3$. We shall assume that the origin of the coordinate system is inside the scattering volume and we shall introduce a unit vector $\mathbf{n}$ directed toward the point of observation. We shall use

$$\xi_3=\mathbf{n}\boldsymbol{\xi},\quad \boldsymbol{\xi}_\perp=\boldsymbol{\xi}-\mathbf{n}\xi_3,\quad z''=\mathbf{n}\mathbf{x}'+\xi_3s.$$

Since the main contribution to the integral (27) comes from a region $|\boldsymbol{\xi}_\perp|\ll|\xi_3|$, the function (26) can be represented by

$$g(\mathbf{x},\mathbf{x}')=\frac{i}{2\omega}g^{(2)}\left(\boldsymbol{\xi}_\perp,\frac{|\xi_3|}{2\omega}\right)\exp\left[i\omega|\xi_3|+\frac{i}{2}\omega\,\mathrm{sign}\,\xi_3\int_{\mathbf{n}\mathbf{x}}^{\mathbf{n}\mathbf{x}}dz''\,\sigma(\mathbf{x}_\perp'+\mathbf{n}z'')\right],$$

$$g^{(2)}(a_n,\ c)=(4\pi ic)^{-1}\exp(ia_n^2/4c). \tag{28}$$

We shall assume that in a plane defined by $z=z_0<0$ there is a plane object outside an inhomogeneity. In the parabolic-equation approximation the wave field of this object in the region $z>z_0$ is given by

$$\theta(z-z_0)\,e_s^{(0)}(\mathbf{x})$$
$$=\int d^2x_{0n}\,g^{(2)}\left(x_n-x_{0n},\frac{z-z_0}{2\omega}\right)E_{sk}\left(\frac{\boldsymbol{\xi}_0}{\xi_0}\right)d_k(x_{0n})\,e^{i\omega(z-z_0)}. \tag{29}$$

Here, x_n and x_{0n} are two-dimensional vectors in the xy plane; $\boldsymbol{\xi}_0 = \mathbf{x} - \mathbf{x}_0$, $\mathbf{d}(x_{0n})$ is the density of the dipole moment of the radiation source. Substituting Eqs. (28) and (29) into Eq. (27), we obtain the field intensity at an arbitrary point $\mathbf{x}$.

We shall consider the field distribution in the region $z \gg L$. Using the relationship ($\boldsymbol{\xi}_0' = \mathbf{x}' - \mathbf{x}_0$)

$$\int d^2x_n'\, E_{ms}\left(\frac{\boldsymbol{\xi}}{\xi}\right) g^{(2)}\left(x_n - x_n', \frac{z-z'}{2\omega}\right) E_{sk}\left(\frac{\boldsymbol{\xi}_0'}{\xi_0'}\right)$$
$$\times g^{(2)}\left(x_n' - x_{0n}, \frac{z'-z_0}{2\omega}\right)$$
$$= E_{mk}\left(\frac{\boldsymbol{\xi}_0}{\xi_0}\right) g^{(2)}\left(x_n - x_{0n}, \frac{z-z_0}{2\omega}\right)$$

and the identity

$$\sigma(\mathbf{x}')\exp[i\delta(\mathbf{x}', z)] = \frac{2i}{\omega}\frac{\partial}{\partial z'}\exp[i\delta(\mathbf{x}', z)],$$

$$\delta(\mathbf{x}', z) = \frac{\omega}{2}\int_{z'}^{z} dz''\,\sigma(x_n', z''),$$

we find that integration of Eq. (27) by parts demonstrates (as in the quantum scattering problem of Ref. 24) that the free field cancels out. We then obtain

$$e_m(\mathbf{x}) \approx E_{mk}(\mathbf{n}) \int d^2x_n'\, d^2x_{0n} g^{(2)}\left(x_n - x_n', \frac{z}{2\omega}\right) \exp\left[i\omega(z-z_0) + \frac{i\omega}{2}\int_{-\infty}^{z} dz''\,\sigma(x_n', z'')\right] g^{(2)}\left(x_n' - x_{0n}, \frac{-z_0}{2\omega}\right) E_{ks}(-\mathbf{n}_0)\, d_s(x_{0n}), \tag{30}$$

where $\mathbf{n}_0 = \mathbf{x}_0/x_0$. If the radius of the first Fresnel zone is considerably less than the transverse dimensions D of an inhomogeneity ($(\lambda|z_0|)^{1/2} \ll D$), we can ignore the diffraction effects limiting the attainable resolution of an image.[25] In this case the limits of integration with respect to x_n' and x_{0n} can be extended from $-\infty$ to ∞.

We shall assume that in the axial region we can represent $\sigma(\mathbf{x})$ by

$$\sigma(\mathbf{x}) = -\tfrac{1}{2}\gamma(x^2+y^2)h(z) + \ldots, \quad \gamma > 0, \tag{31}$$

where $h(z)$ is a function which decreases rapidly in the region $|z| \gg L$. Then, after integration with respect to x_n', we find from Eq. (30) that

$$e_m(\mathbf{x}) = E_{mk}(\mathbf{n}) \int d^2x_{0n} \frac{f_1(z)}{z_0} g^{(2)}\left(x_n - \frac{f_1}{z_0}x_{0n}, \frac{z-f_1}{2\omega}\right)$$
$$\times \exp\left[i\omega(z-z_0) - \frac{i\omega}{2F_1}\frac{f_1(z)}{z_0}x_{0n}^2\right] E_{ks}(-\mathbf{n}_0)\, d_s(x_{0n}). \tag{32}$$

Here,

$$\frac{1}{F_1(z)} = \frac{\gamma}{2}\int_{-\infty}^{z} dz''h(z''), \quad \frac{1}{f_1(z)} = \frac{1}{z_0} + \frac{1}{F_1(z)}. \tag{33}$$

We shall find the field in the $z = z_1$ plane, where z_1 is a root of the equation $z = f_1(z)$. Since $h(z)$ decreases rapidly, it follows that in the range $|z| \gg L$ the function $F_1(z)$ assumes the value $F_0 = F_1(\infty)$, which is equal to the focal length of a lens the role of which is played by the focusing inhomogeneity. Consequently, z_1 satisfies the lens equation: $z_1^{-1} - z_0^{-1} = F_0^{-1}$. Using the relationship

$$\lim_{z\to z_1} g^{(2)}\left(x_n - \frac{f_1(z)}{z_0}x_{0n}, \frac{z-f_1(z)}{2\omega}\right) = \delta^{(2)}\left(x_n - \frac{z_1}{z_0}x_{0n}\right),$$

we obtain the field intensity in the image plane $z = z_1$:

$$e_m(x_n, z_1) = \frac{z_0}{z_1} E_{mk}(\mathbf{n})\, d_k\left(\frac{z_0}{z_1}x_n\right)$$
$$\times \exp\left[i\omega(z-z_0) - \frac{i\omega}{2F_0}\frac{z_0}{z_1}x_n^2\right].$$

Consequently, the image is located at points with the coordinates $x_n = (z_1/z_0)x_{0n}$, where x_{0n} is a set of points belonging to a given object.

We shall show now that in the region $z < 0$ the scattered wave also forms an image. If $z < 0$, we obtain from Eq. (27)

$$e_m(\mathbf{x}) = e_m^{(0)}(\mathbf{x}) + \exp(-i\omega z)\int d^3x'\, E_{mk}(\mathbf{n}) g^{(2)}\left(x_n - x_n', \frac{-z}{2\omega}\right)$$
$$\times \exp(i\omega z')\frac{\partial}{\partial z'}\exp[-i\delta(\mathbf{x}', z)]\, e_k^{(0)}(\mathbf{x}'). \tag{34}$$

Substituting Eq. (29) into Eq. (34), we reach the conclusion that a sharp image is observed in the $z = z_2$ plane defined by the condition

$$-\frac{1}{z} - \frac{1}{z_0} = \frac{1}{F_2(z)} \qquad \frac{1}{F_2(z)} = \frac{\gamma}{2}\int_{z}^{\infty} dz''h(z''). \tag{35}$$

Obviously, if $|z| \gg L$, the function $F_2(z)$ becomes $F_0 = F_2(-\infty)$. Integration with respect to transverse variables gives the field intensity in the $z = z_2$ plane:

$$e_m(x_n, z_2) = \frac{z_0}{z_2} E_{mk}(\mathbf{n}) E_{ks}(-\mathbf{n}_0)\, d_s\left(-\frac{z_0}{z_2}x_n\right)$$
$$\times R(\omega)\exp\left[-i\omega(z_2+z_0) - \frac{i\omega}{2F_0}\frac{z_0}{z_2}x_n^2\right], \tag{36}$$

$$R(\omega) = \frac{\gamma}{2}\int_{-\infty}^{\infty} dz\, h(z) F_2(z) e^{2i\omega z}. \tag{37}$$

Here, $R(\omega)$ is the reflection coefficient.

We shall assume that an object is located in a plane $z_0 = -c < 0$. If $c > F_0$, it then follows from Eqs. (32) and (33) that the image of this object is real and its coordinates are

$$z_1 = \frac{F_0 c}{c - F_0}, \quad x_n = -\frac{z_1}{c}x_{0n}.$$

The position of the image formed by the wave scattered in the negative direction of the z axis is described by Eqs. (35) and (36): $z_2 = -z_1$ and $x_n = -z_1 x_{0n}/c$, where z_2 is the root of the equation $-z_2^{-1} - z_0^{-1} = F_0^{-1}$. The position of this image is related by the inversion transformation $z \to -z$ to the main image. However, the illumination in this image, which is proportional to the square of the coefficient $|R(\omega)|^2$ is determined by the high-frequency spatial Fourier components of the effective longitudinal refractive index hF_2. We note that in the case of one-dimensional scattering the reflection coefficient is

$$r \sim \frac{\gamma}{2\omega} \int_{-\infty}^{\infty} dz\, h(z)\, e^{2i\omega z}$$

and in the case of the functions $h(z)$ satisfying the condition $|\lambda dh/dz| \ll |h|$ it is exponentially small.[26] The existence of a three-dimensional focusing medium has the effect that the integrand acquires a factor $F_2(z)$ which generates a pole singularity in the complex z plane. Consequently, the exponential smallness can transform into one of the power-law type.[26] By way of example, we shall assume that $h(z) = \cosh^{-2}(z/H)$. We then have

$$F_2(z) = \frac{2}{\gamma H}\left[1 - \mathrm{th}\,\frac{z}{H}\right]^{-1}.$$

If $\lambda \ll H$, the reflection coefficient and the refractive index are given by the Fresnel formulas.

We shall now make some comments.

1) If $c < F_0$, then the main and conjugate images are virtual.

2) The coefficient γ in Eq. (31) can be negative. In this case an inhomogeneity is similar to a diverging lens and both images are virtual.

3) In the case of real inhomogeneities the expansion of $\sigma(\mathbf{x})$ becomes

$$\sigma(\mathbf{x}) = \sigma(0) + x_n h_n(z) + {}^1/_2 x_m x_n h_{mn}(z) + \ldots.$$

In this case the image becomes deformed: it is rotated in the xy plane and shifted in the direction of the vector h.

4) Two direct and two conjugate images,formed by ordinary and extraordinary waves, appear in a gyrotropic medium. The area occupied by the image of a planar object consists of "polarization domains," which are regions in which the field orientation varies continuously.

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