# Nondissipative gravitational turbulence in an expanding universe

A. V. Gurevich and K. P. Zybin

*Lebedev Physics Institute, USSR Academy of Sciences*
(Submitted 6 April 1988)
Zh. Eksp. Teor. Fiz. **94**, 5–15 (October 1988)

We investigate the nonlinear stage in the development of Jeans instabilities in an expanding universe. We show that the relationship between density and velocity in a growing gravitational mode is such that in the nonlinear stage, the maximum initial density evolves into a self-confined nondissipative gravitational singularity (NGS). An NGS possesses perfectly well-defined scaling relations for the gas density and the field velocity and potential in the vicinity of a singularity. Well-developed gravitational turbulence consists of a hierarchical system of NGS on a range of scales, where the fields of large-scale singularities confine small-scale ones. Comparison of the theory with observational data provides a crude estimate of the original large-scale fluctuation spectrum.

The Friedmann model of a homogeneous and isotropic universe is valid only on scales of the order of the radius of the horizon. Small deviations from homogeneity tend to grow as a result of gravitational forces. On scales much smaller than the radius of the horizon, the dynamics of these perturbations is of crucial importance in the formation of galaxies, galactic clusters, superclusters, etc. It is normally assumed that the missing mass, being nondissipative (that is, interacting solely through gravitational forces), plays a major role in this process.[1,2]

Lifshits[3] has developed the linear gravitational theory that describes the growth of small fluctuations in an expanding universe, but to ascertain just what structures might evolve and continue to exist in the universe, it is necessary to study perturbation dynamics in the nonlinear stage. The latter is the subject of the present paper. We naturally limit our attention to scales much smaller than the radius of the horizon; the velocity of matter is much less than the speed of light on such scales so our treatment can remain within the framework of Newtonian mechanics.

We previously considered the nonlinear stage of gravitational instability[4] in a noncosmological setting, assuming certain special initial conditions: initial matter-density perturbations $\delta\rho$ were taken to be at rest ($\mathbf{v} = 0$) and to exhibit nonlinear characteristics ($\delta\rho \sim \rho$). We then showed that every maximum of the initial density undergoes three-dimensional compression and subsequent mixing, resulting in the formation of a nondissipative gravitational singularity (NGS), a self-confined clump of matter having at its center $r \to 0$ a singularity of density $\rho$, speed $v^2$, and field potential $\psi$:

$$\rho \sim r^{-24/13}, \quad v^2 \sim \psi - \psi_m \sim r^{2/13}, \tag{1}$$

where $\psi_m = \psi(0)$ is the value of the potential $\psi$ at the singular point $r = 0$. Nondissipative gravitational turbulence develops when there is a random, irregular distribution of initial density $\rho(\mathbf{r})$, and consists of a hierarchical structure containing superposed moving NGS on various scales, with smaller-scale singularities being confined by the gravitational field of larger-scale singularities. The scaling relations (1) hold in every such NGS, both for the gas density and for the number density of confined small-scale NGS.

A different, more general cosmological approach was taken by Zel'dovich[5] and further developed by him and by Arnol'd, Shandarin, Doroshkevich, and others.[1,6] In Lagrangian form, they considered nonlinear perturbations of arbitrary form in an expanding universe, and found that the main (i.e., the most likely) linear structure is a planar singularity (a Zel'dovich pancake). In such a singularity $\rho \sim x^{-2/3}$ and $v^2 \sim x^{2/3} \gg \psi - \psi_m$, so it is not capable of self-confinement—that is, it cannot form a stationary structure.

Note that discussions based on the general properties of Lagrangian singularities do not take account of the specifics involved in the growth of small fluctuations subject to gravitational instability. The point here is that prior to the nonlinear stage, only perturbations of a particular kind will grow out of arbitrary small initial perturbations, and consequently a special kind of singularity might also emerge. This is exactly what proved to be crucial to an understanding of the nature of singularities of nonlinear structures that might arise in an expanding universe.

In Sec. 1 of the present paper, we first discuss the growth of small fluctuations, given arbitrary initial density and velocity perturbations. In so doing, we distinguish between two modes[1,2]—growing and damped. We show that the initial conditions required for an investigation of the nonlinear dynamics of the growing mode comprise a well-defined scalar combination of density and velocity. As a result, we obtain for the growing mode a nonlinear solution that is in fact identical with the NGS (1).

In Sec. 2, we examine the effect of the damped mode on the growing mode during the nonlinear stage of development of an instability. We show that it has a significant influence only in a very small neighborhood of the singularity. We then go on to discuss the relationship between or results and Lagrangian singularities.

In Sec. 3, we study the process whereby a hierarchical structure of nondissipative turbulence is produced; we show that the scaling laws (1) also hold in an expanding universe, and that they characterize both the matter density distribution and the distribution of smaller-scale confined NGS.

In a brief summary, we point out that the observational data not only confirm that a hierarchical structure consistent with the scaling relations (1) exists, but they also enable us to investigate the initial growth stage of gravitational perturbations for the largest-scale inhomogeneities. By comparing theory with the latest observations, we can roughly approximate the initial spectrum of large-scale fluctuations.

 0038-5646/88/101957-06$04.00  

## 1. DYNAMICS OF THE GROWING MODE

We consider the stage in the evolution of the universe when its expansion dynamics are governed by cold nondissipative matter. We assume small velocity and density fluctuations $\mathbf{v}_i$ and $\delta\rho_i$ at some initial time $t_i$, and investigate their subsequent evolution.

The equations describing the motion of a cold nondissipative gas in an expanding reference frame take the form[2]

$$\frac{\partial}{\partial t}\mathbf{v}+a^{-1}(\mathbf{v}\nabla)\mathbf{v}+\frac{\dot a}{a}\mathbf{v}+a^{-1}\nabla\varphi=0,$$
$$\frac{\partial}{\partial t}\delta+a^{-1}\nabla\{(1+\delta)\mathbf{v}\}=0,\quad \Delta\varphi=a^2\delta\rho_0(t). \tag{2}$$

Here $\mathbf{v}$ is the particle velocity, $\delta=[\rho(\mathbf{x},t)-\rho_0(t)]/\rho_0(t)$ is the deviation of the gas density from the mean background density $\rho_0(t)$, and $a(t)$ is a scale factor. In (2), as in Ref. 4, we use a system of units in which $4\pi G=1$.

In view of the smallness of the initial perturbations, we first investigate their growth in the linear stage. Equations (2) then take the form

$$\frac{\partial}{\partial t}\mathbf{v}+\frac{\dot a}{a}\mathbf{v}+a^{-1}\nabla\varphi=0,$$
$$\frac{\partial}{\partial t}\delta+a^{-1}\nabla\mathbf{v}=0,\quad \Delta\varphi=a^2\delta\rho_0(t). \tag{3}$$

The solution of (3) has been analyzed repeatedly (for example, see Ref. 2), and is of the form

$$\delta=\delta_i\{D_1(t)\dot D_2(i)-D_2(t)\dot D_1(i)\}/E$$
$$+\nabla\mathbf{v}_i\{D_1(t)D_2(i)-D_2(t)D_1(i)\}/a_iE, \tag{4}$$
$$\mathbf{v}=\frac{a(t)}{4\pi E}\int d^3x'\,\delta_i(\mathbf{x}')\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\{\dot D_1(t)\dot D_2(i)-\dot D_2(t)\dot D_1(i)\}$$
$$+\mathbf{v}_i^d\frac{a(t)}{a_iE}\{\dot D_2(t)D_1(i)-\dot D_1(t)D_2(i)\}+\mathbf{v}_i^r a_i/a(t).$$

Here $a_i$, $\delta_i$, and $\mathbf{v}_i$ are quantities that are prescribed at some initial time $t_i$, $\mathbf{v}_i^d$ and $\mathbf{v}_i^r$ are the irrotational and rotational components of velocity, the constant $E$ is defined by

$$E=D_1(i)\dot D_2(i)-D_2(i)\dot D_1(i),$$

and $D_1(t)$ and $D_2(t)$ are the growing and damped solutions, respectively, of the differential equation

$$\frac{\partial^2}{\partial t^2}D+2\frac{\dot a(t)}{a(t)}\frac{\partial}{\partial t}D=\rho_0(t)D.$$

In the remainder of this section we will consider only the unstable (growing) mode. This implies that we must choose the initial conditions for the velocity and density so that there is no damped component in Eq. (4). This can always be accomplished if at the initial time $t_i$ we impose the following conditions:

$$\delta_i(\mathbf{x})-\frac{\nabla\mathbf{v}_i(\mathbf{x})}{a_i}\frac{D_2(i)}{\dot D_2(i)}=0,\quad \mathbf{v}^r=0. \tag{5}$$

Of course, the relations (5) significantly restrict the class of initial functions; the effects of all other initial conditions will be the subject of Sec. 2.

Substituting (5) into (4), we can readily show that at the initial time $t_i$, the growing mode is completely determined by a single scalar function, for which we can use the initial potential $\varphi_i(\mathbf{x})$, whereupon

$$\mathbf{v}_i=-\frac{\nabla\varphi_i(\mathbf{x})}{a_i\rho_0(i)}\frac{\dot D_1(i)}{D_1(i)},\quad \delta_i(\mathbf{x})=\frac{\Delta\varphi_i}{a_i^2\rho_0(i)}. \tag{5a}$$

Notice that one might also choose some arbitrary density $\delta_i(\mathbf{x})$ for the initial conditions here. Equation (2) then implies that the potential $\varphi_i(\mathbf{x})$ would be determined to within some function $S(\mathbf{x})$ satisfying the equation $\Delta S=0$.

Now consider one arbitrary maximum of the function $\delta_i(\mathbf{x})$. Near this maximum, the density (and therefore in accordance with (5a), the velocity) can always be represented in the form

$$\delta\rho_i=\delta\rho_0(1-\xi^2+\ldots),\quad \xi^2=\frac{x_1^2}{b_1^2}+\frac{x_2^2}{b_2^2}+\frac{x_3^2}{b_3^2}$$
$$\mathbf{v}_i=\frac{\mathbf{x}}{3}a_i\frac{\delta\rho_0}{\rho_0(i)}\frac{\dot D_1(i)}{D_1(i)}\left(1-\frac{3}{5}\xi^2+\ldots\right. . \tag{6}$$

Let us see how the density and velocity of a collisionless gas will vary in the neighborhood of this maximum, starting with the system of equations (2).

An analysis shows that using (6) to take the initial velocity into account simply leads to a renormalization of the coefficients in the solution (10), (13). Without loss of generality, therefore, we may assume that $\mathbf{v}_i=0$. It is most convenient to seek a solution in the frame of reference associated with the maximum density. In the usual manner, therefore, we may go from the expanding reference frame to the rest frame via the transformation[2]

$$\mathbf{r}=a(t)\mathbf{x},\quad \varphi=\psi+{}^1/_2a(t)\ddot a(t)x^2,$$
$$\mathbf{u}=\mathbf{v}+\mathbf{x}\dot a(t),\quad \rho=(1+\delta)\rho_0(t). \tag{7}$$

Substituting (7) into (2), we obtain

$$\frac{\partial}{\partial t}\rho+\nabla_r(\rho\mathbf{u})=0,$$
$$\frac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\nabla_r)\mathbf{u}+\nabla_r\psi=0,\quad \nabla_r^2\psi=\rho. \tag{8}$$

We supplement (8) with initial conditions derived from (5) and (6): at $t=t_i$,

$$\rho(t_i)=\rho_0(t_i)+\delta\rho_0(1-\xi^2),\quad \mathbf{u}(t_i,\mathbf{r})=\frac{\dot a_i}{a_i}\mathbf{r}, \tag{9}$$

where

$$\xi^2=\frac{x^2}{c_1^2}+\frac{y^2}{c_2^2}+\frac{z^2}{c_3^2},\quad c_k=b_ka_i,\quad k=1,2,3.$$

We have previously obtained[4] an exact solution of the system of equations (8) with initial conditions like (9):

$$\nabla\psi=\frac{\mathbf{r}}{\xi}B(\xi,t),\quad \mathbf{u}=\frac{\mathbf{r}}{\xi}u(\xi,t),\quad \rho=\rho(\xi,t),$$
$$\tau=(2Y)^{-1/2}\mathscr{H}^{-3/2}(Y)\left\{\operatorname{arctg}\mathscr{Z}+\frac{\mathscr{Z}}{1+\mathscr{Z}^2}\right\}+\mathscr{M}(Y), \tag{10}$$
$$\xi=1/\mathscr{H}(Y)(1+\mathscr{Z}^2)\qquad \mathscr{Z}=-u/(2Y\mathscr{H})^{1/2}.$$

Here $Y=\xi^2(\partial B/\partial\xi)$, the functions $\mathscr{M}(Y)$ and $\mathscr{H}(Y)$ are determined by the initial conditions (9), and the time $\tau=t-t_i$ is conveniently measured from the initial time $t_i$.

Below it will also be convenient to measure $\xi$ in units of $((1+\delta_i)/\delta_i)^{1/2}$, $Y$ in units of $Y_0=\rho_{10}R_0^3/3$, the velocity $u(\xi,t)$ in units of $R_0^{-1}(3/\rho_{10})^{1/2}$, and time $\tau$ in units of $(3/\rho_{10})^{1/2}$. Here $R_0=r_0((1+\delta_i)/\delta_i)^{1/2}$, and $r_0$ is the characteristic scale length of the density maximum:

$$r_0=(c_1^2+c_2^2+c_3^2)^{1/2}, \tag{11}$$

$\rho_{10}=\rho_0(i)+\delta\rho_0(i)$. At $\tau=0$, the initial conditions (9) then take the form

$$Y=\xi^3-{}^3/_5\xi^5, \quad u=\lambda\xi, \tag{12}$$

where $\lambda=(-\dot{a}_i^2/\ddot{a}_i a_i(1+\delta_i))^{1/2}$. In deriving this equation, we made use of the dynamical relation[2] between $\rho_0(t)$ and $a(t)$:

$$\ddot{a}(t)=-{}^1/_3 a(t)\rho_0(t).$$

We emphasize that since we have $\rho_0(t)>0$ and $\delta_i \ll 1$, the expression whose square root is being taken is always positive definite.

The functions $\mathscr{H}(Y)$ and $\mathscr{M}(Y)$ which satisfy the initial conditions (12) are of the form

$$\mathscr{H}(Y)=\alpha Y^{-1/3}+\beta Y^{1/3}, \quad \mathscr{M}(Y)=-\sigma_0+\sigma_1 Y^{2/3},$$

where

$$\alpha=1-\lambda^2/2, \quad \beta=-{}^1/_5(1+\lambda^2),$$

$$\sigma_0=\alpha^{-3/2}2^{-1/2}K_0, \quad \sigma_1=2^{-1/2}\alpha^{-3/2}\left\{\frac{3\beta K_0}{2\alpha}+\frac{\lambda}{(2\alpha)^{1/2}}\left(\frac{1}{5}-\frac{\beta}{2\alpha}\right)K_1\right\},$$

$$K_0=-\operatorname{arctg}(\lambda/(2\alpha)^{1/2})-\frac{\lambda/(2\alpha)^{1/2}}{1+\lambda^2/2\alpha}, \quad K_1=2/(1+\lambda^2/2\alpha)^2. \tag{13}$$

Proceeding to the linear case in the solution (10), (13), we obtain a well-known result[1-3] for $\Omega=1$:

$$\delta(\mathbf{x},t)=\tilde{\delta}_i(\mathbf{x})t^{2/3}. \tag{14}$$

Let us first consider the case in which $a>0$. An investigation of the solution (10)–(13) shows that the gas is compressed under the influence of gravity, and at time

$$\tau_0=\frac{\pi}{2\sqrt{2}}\alpha^{-3/2}-\sigma_0$$

a singularity appears: the density at the center goes to infinity. Equations (10) and (13) imply that at $\tau=\tau_0$, all quantities in the neighborhood of the singular point $\xi=0$ behave in the following way:

$$\rho=\frac{9}{7}\left(\frac{2}{9}Q^{-2}\right)^{3/7}\xi^{-12/7}, \quad \psi=\psi_m+\frac{7}{2}\left(\frac{2}{9}Q^{-2}\right)^{3/7}\xi^{2/7},$$

$$u^2=2\left(\frac{2}{9}Q^{-2}\right)^{3/7}\xi^{2/7}, \quad Q=\frac{3}{2}\frac{\beta}{\alpha}\frac{\pi}{2\sqrt{2}}\alpha^{-3/2}-\sigma_1. \tag{15}$$

These relations are analogous to Eq. (20) of our previous paper.[4] But we stress once again that the central singularity (15) differs fundamentally from planar "pancake" singularities in that it is a capture singularity, $v^2\sim\psi-\psi_m$ at every point. The subsequent mixing of this singularity in multiple-current hydrodynamics is described in Ref. 4 in identical fashion, and leads to the emergence of the NGS (1).

So far, we have examined the nonlinear dynamics of the growing mode of a gravitational instability for the case $a>0$. For $a<0$, there will be no singularity at the origin, and a kinematic current reversal takes place at a point a certain distance away from the density maximum, with the particle kinetic energy in the vicinity of the singularity being much higher than the potential energy. Outside the neighborhood of the density maximum, Lagrangian singularities of general form will emerge; these are classified in Ref. 6.

The sign of $\alpha$ depends on the cosmological model; according to (13).

$$\alpha=1-\frac{3}{2}\left(\frac{\dot{a}_i}{a_i}\right)^2\Big/\rho_0(i)(1+\delta_i). \tag{16}$$

Equation (16) may conveniently be expressed in terms of the parameter $\Omega=\frac{2}{3}\rho_0(t)/(\dot{a}/a)^2$, which determines the rate of expansion of the universe. Thus, we find from (16) that

$$\alpha=1-\Omega_i^{-1}(1+\delta_i)^{-1}.$$

This makes it apparent that the region in which an NGS will occur ($\alpha>0$) depends heavily not just on the initial values $\delta_i$, but also on how close the parameter $\Omega$ is to unity. Therefore, an analysis of the actual distribution of matter in the universe might perhaps enable one to improve the value of $\Omega$. Taking $\Omega_i=1$, we find

$$\alpha=\frac{\delta_i}{1+\delta_i}, \quad \tau_0=\frac{\pi}{2^{1/2}}\alpha^{-3/2}-\frac{2^{1/2}}{3}\alpha^{-3/2}\delta_i^{3/2}. \tag{17}$$

## 2. THE EFFECT OF DAMPED MODES ON THE FORMATION OF A NONDISSIPATIVE GRAVITATIONAL SINGULARITY

In treating the nonlinear theory of a growing perturbation mode, we have completely avoided any consideration of the damped mode, thereby considerably restricting the class of initial conditions. Indeed, arbitrary initial conditions are determined by four scalar functions ($\delta_i(\mathbf{x})$; $\mathbf{v}_i(\mathbf{x})$), while there is but one for the growing mode. Since the separation into growing and damped modes is only possible in the linear stage, however, it is necessary to consider the effects on (15) of all sorts of perturbations that we have not into account. Particle velocities in the damped mode take the form[2]

$$\mathbf{v}=-\dot{D}_2(t)a(t)\left\{\dot{D}_1(i)\int d^3x'\,\delta_i(\mathbf{x}')\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}-D_1(i)\frac{v_i^d}{a(t)}\right\}\Big/E+\mathbf{v}_i^r\frac{a_i}{a(t)}. \tag{18}$$

Let us first confine our attention to irrotational motion; i.e., we assume that $\mathbf{v}_i^r=0$. We shall be interested in the solution of (18) in a small neighborhood of the density maximum discussed earlier. In general, then, the velocity $\mathbf{v}$ can be expanded in a Taylor series. To first order, we obtain

$$u_\alpha=U_{\alpha\beta}x_\beta.$$

The velocity $u_\alpha$ and position $x_\beta$ here are made dimensionless in the same way as in (12).

Equation (18) only holds as long as $\delta<1$. For $\delta\geqslant 1$, it is necessary to consider the compressed solution described by (10). We denote by $\varepsilon$ the characteristic magnitude of the tensor $U_{\alpha\beta}$ at the instant when $\delta=1$; it depends on the rate of expansion of the universe and the fluctuation amplitude $\delta_i$. For $\Omega = 1$, we have

$$\varepsilon \approx \delta_i^{3/2} \ll 1. \tag{19}$$

An analysis of (10) shows that without loss of generality, we may consider values of the velocity field for the damped mode having $\nabla \cdot \mathbf{v} = 0$. If we then diagonalize the matrix $U_{\alpha\beta}$ (bearing in mind that $\nabla \times \mathbf{v} = 0$), we obtain

$$u_x = \frac{\varepsilon}{3} x, \quad u_y = -\frac{2}{3} \varepsilon y, \quad u_z = \frac{\varepsilon}{3} z. \tag{20}$$

In writing out (20), we have chosen a direction along the $y$-axis for the velocity gradient near the maximum-density point $r = 0$.

Let us now investigate how the choice of the initial velocity (20) at time $t = t^*(\delta = 1)$ affects the compression of a clump induced by a growing mode. Since we are interested in the neighborhood of $\xi = 0$, we look at the solution (10) expanded about this point during the compression stage, obtaining

$$\mathbf{v} = -{}^2/_3 \mathbf{r}(1-\tau)^{-1}, \quad \rho = {}^2/_3 (1-\tau)^{-2}. \tag{21}$$

Here $r$ has been normalized to $r_0$, and $\tau$ to $\tau_0$. Subtracting (21) from the complete hydrodynamic system (8) and calling the differences $\delta \mathbf{v}$, $\delta\rho$, and $\delta\psi$, we find

$$\begin{gathered}\frac{\partial}{\partial\tau}\delta\mathbf{v} - \frac{2}{3(1-\tau)}(\mathbf{r}\nabla_r)\delta\mathbf{v} + \frac{2}{3(1-\tau)}\delta\mathbf{v} + \nabla_r\delta\psi + (\delta\mathbf{v}\nabla_r)\delta\mathbf{v} = 0, \\ \frac{\partial}{\partial\tau}\delta\rho + \frac{2}{3(1-\tau)^2}(\nabla_r\delta\mathbf{v}) - \frac{2}{3(1-\tau)}\nabla_r(\mathbf{r}\delta\rho) + \nabla_r(\delta\rho\delta\mathbf{v}) = 0, \\ \Delta\delta\psi = \delta\rho.\end{gathered} \tag{22}$$

The initial conditions for (22) are as given in (20). We seek a solution of (22) in the form

$$\delta v_k = r_k h_k \eta^4, \quad \delta\rho = q(\eta)\eta^4, \quad \nabla_r\delta\psi = {}^1/_3 \mathbf{r}\delta\rho, \tag{23}$$

where $\eta = (1-\tau)^{-1/3}$; there is no summation over the subscript $k$.

Substituting (23) into (22), we obtain

$$\begin{gathered}\frac{1}{3}\frac{\partial}{\partial\eta} h_k + h_k^2 + \frac{1}{3} q\eta^{-4} = 0, \\ \frac{1}{3}\frac{\partial}{\partial\eta} q - \frac{2}{3}\frac{q}{\eta} + q\sum_k h_k + \frac{2}{3}\eta^2 \sum_k h_k = 0.\end{gathered} \tag{24}$$

Since according to (19) we have $\varepsilon \ll 1$, it suffices to examine the linear solution of the system (24). Discarding nonlinear terms and inserting the initial conditions (20), we have

$$\begin{gathered}\delta\rho = 0, \quad \delta v_k = g_k r_k (1-\tau)^{-4/3}, \\ g_1 = \varepsilon/3, \quad g_2 = -{}^2/_3\varepsilon, \quad g_3 = \varepsilon/3.\end{gathered} \tag{25}$$

We see, then, from (25) that to first order the density does not increase, $q = 0$, but the velocity grows more rapidly than that of the main flow (21), significantly exceeding the latter near the singularity. Let us consider, therefore, the nonlinear solution of the system (24), assuming $\eta \to \infty$. From the first of Eqs. (24), we obtain asymptotically

$$\frac{1}{3}\frac{\partial h_k}{\partial\eta} + h_k^2 = 0. \tag{26}$$

Equation (26) describes the kinematic motion of particles, and it has a solution of the form

$$h_k(\eta) = h_k(0)/[1 + 3h_k(0)(\eta - 1)]. \tag{27}$$

What is important for our purposes is that perturbations in $y$ grow more rapidly than in the other two coordinates. The solution (27) becomes singular at $\eta = 1 + \varepsilon/2$ (as can be seen from the initial conditions (20)). From (26), this singularity is due to a nonlinear kinematic breaking, and is one of the possible Langrangian singularities.[6]

The statement made in Ref. 6, to the effect that under general initial conditions the singularities that occur in the system (8) are Lagrangian, thus turns out to be correct. This means that the central singularity (15) that we are considering gets smeared out over some small neighborhood of $\xi = 0$ as a result of mixing due to the advent of a Lagrangian singularity which itself results from the presence of a damped mode. Estimates indicate, however, that this smeared-out region is typically extremely small—of order $\varepsilon^2$ (i.e., of order $\delta_i^3$), so this process has no significance under real physical conditions.

To conclude this section, we note that even when small rotational velocities $\mathbf{v}_i^r$ are taken into account, one simply obtains an entirely analogous small smoothing region for the central singularity (15).

## 3. FORMATION OF HIERARCHICAL STRUCTURE

In the preceeding sections we considered the dynamics of a single initial density maximum. In reality, there will be a spectrum of initial fluctuations. We shall now assume, as usual,[1,2] that the spectrum has a maximum at some scale length $L = L_m$. Accordingly, the initial values of the effective-density maxima are greatest at $L = L_m$:

$$\bar{\delta}_i(L_m) > \bar{\delta}_i(L) \quad \text{for} \quad L \neq L_m. \tag{28}$$

This same scale length $L_m$ will then prove to be special, and in a time [see (14)]

$$t_m = t_i(\bar{\delta}_i(L_m))^{-3/2} \tag{29}$$

a compression of the corresponding maxima will take place, leading to NGS with a basic scale length of $L_m$.

Hierarchical structure—NGS on a variety of scales $L$—will develop differently, depending on the value of $L$. For large scales, $L > L_m$, it evolves in the manner indicated in Ref. 4, a result obtained by averaging. Following this same approach, it can readily be shown that the time for formation of an NGS of size $L > L_m$ is

$$t_L = t_m(\bar{\delta}_i(L_m)/\bar{\delta}_i(L))^{3/2}. \tag{30}$$

By virtue of (28), we always have $t_L \gg t_m$. Hierarchical structure on scales larger than $L_m$ will thus develop consistently in a time given by (30), and on those scales for which it has managed to evolve by some time $t$, it will possess[4] the scaling relations (1) both for the gas density and the number density of smaller-scale NGS trapped within.

On small scales $L \ll L_m$, hierarchical structure develops simultaneously with the process taking place at the fundamental scale. To demonstrate that this is so, we must examine the compression of one maximum at the fundamental scale $L_m$, assuming that at the initial time $t_i$ it contains small density fluctuations at other scales $L \ll L_m$. We can divide the development of these fluctuations into three stages. The first goes from the initial time $t_i$ to the time $t^*$ at which the processes taking place as the fundamental scale becomes nonlinear: $\tilde{\delta}(L_m, t^*) \sim 1$. During this time, according to (14), all inhomogeneities increase with time (we have in mind, of course, only the growing mode), and by time $t_0$, they are in an advanced linear stage:

$$\tilde{\delta}(L, t^*) \approx \tilde{\delta}_i(L)/\tilde{\delta}_i(L_m) \ll 1.$$

The second stage runs from the time when the fundamental scale experiences nonlinear compression until the formation of the central singularity (15), $t_m \approx t^* + t_g$, $t_g = t^* \delta_i^{3/2}$. Small-scale inhomogeneities are transported during this stage along with matter, so their number density increases in proportion to the density $\rho$. At the same time, they undergo compression, deformation, and growth; it can be shown, however, that this is a minor effect, since $t_g \ll t^*$. Except for those regions immediately adjacent to the singularity, inhomogeneities in this second stage remain linear.

Their rapid conversion into a nonlinear phase takes place in the third stage, when the first mixing caustic reaches the fundamental scale length. Let us consider this process in somewhat more detail.

If we formally continue the solution (10) beyond the formation of the central singularity, it will take the form shown in Fig. 1. The extrema depend on the problem parameters:

$$Y_1^{2/3} = \frac{3}{7}\frac{\tau_1}{Q_0}, \qquad Y_2^{2/3} = \frac{\tau_1}{Q_0}.$$

Here $\tau_1 = (\tau - \tau_0)/\tau_0$ is the time since reversal, and $Q_0 = 3\beta/2\alpha - \sigma_1\tau_0$. The caustic surface $Y(\xi_1, \xi_0)$. $\xi_0$ separates the multiple-current flow region from that of the hydrodynamic solution (15). Consider now the form of the solution near the caustic. Taking $\tau_1 \ll 1$, we obtain

$$Y = \left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{3/2} \pm 3\left(\frac{2}{7}\right)^{1/2}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{11/12}(Q_0\tau_0)^{-1/3}(\delta\xi)^{1/2},$$

$$u = \pm (Q_0\tau_0)^{-1/3}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{1/6}\left\{1 + \frac{3}{2}\left(\frac{2}{7}\right)^{1/2}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{-7/12} \times (Q_0\tau_0)^{-1/3}(\delta\xi)^{1/2}\right\}, \quad (31)$$

$$\rho = \frac{3}{8}\left(\frac{2}{7}\right)^{1/2}(Q_0\tau_0)^{-5/3}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{-17/12}(\delta\xi)^{-1/2},$$

$$\delta\xi = \xi_0(\tau_1) - \xi, \quad \xi_0 = 2(Q_0\tau_0)^{2/3}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{7/6}.$$

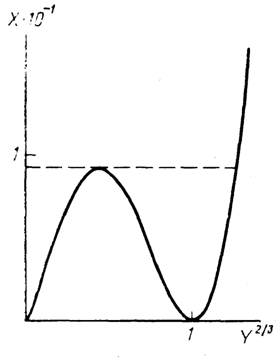


FIG. 1. Form of the solution (10) at time $\tau_1 = Q_0$ following the appearance of a singularity. The location of the caustic is shown by the dashed line. On the vertical axis we use the notation $2\xi^3/9Q_0^2 t_0^2 = X$.

It is clear from (31) that density and velocity singularities on the caustic are usually of the kinematic variety.

Let us now direct the $x$-axis along the normal to the caustic surface, introduce the variable $x = \xi - \xi_0(\tau_1)$, and consider one-dimensional perturbations, since one-dimensionality will have practically no effect on the way in which they grow. The equations for the perturbations (making the natural assumption that there are no mass sources) become

$$\frac{\partial}{\partial\tau_1}\delta v + t_0 u \frac{\partial}{\partial x}\delta v + t_0 \delta v \frac{\partial}{\partial x} u + t_0 \delta\varphi' = 0,$$
$$\frac{\partial}{\partial\tau_1}\delta\varphi' + t_0 u \frac{\partial}{\partial x}\delta\varphi' + t_0 \rho \delta v = 0, \quad (32)$$

where $u(x,t)$ and $\rho(x,t)$ are chosen to be consistent with (31). It is convenient to transport from the variables $x$, $\tau_1$ to $z$, $\varkappa$ in (32):

$$z = 3\left(\frac{2}{7}\right)^{1/3}(Q_0\tau_0)^{1/3}\left(\frac{3}{7}\frac{\tau_1}{Q_0}\right)^{7/12}, \qquad (-\varkappa)^{1/2} = (-x)^{1/2} + z.$$

From (31) and (32), we obtain

$$-z^{-5/7}\frac{\partial}{\partial z}\left\{z^{12/7}((-\varkappa)^{1/2} - z)\frac{\partial}{\partial z}\delta\varphi'\right\} + z\frac{\partial}{\partial z}\delta\varphi' + 2\delta\varphi' = 0. \quad (33)$$

Equation (33) possesses two eigenfunctions. Let us write these out explicitly in the vicinity of the wavefront, i.e., for $z \to (-\varkappa)^{1/2}$. Making use of Eq. (33), we obtain the asymptotic expressions

$$\delta\varphi_1' = 1 + {}^1/_3(-\varkappa)^{-1/2}((-\varkappa)^{1/2} - z)^2 + \dots,$$
$$\delta\varphi_2' = |(-\varkappa^{1/2}) - z|^{-1} - {}^{12}/_7(-\varkappa)^{-1/2}\ln|(-\varkappa)^{1/2} - z| + \dots. \quad (34)$$

We see from (34), then, that the function $\delta\varphi_1'$ is regular near the caustic, while the function $\delta\varphi_2'$ has a singularity. If we analyze the derivation of (33), (34), it becomes clear that this singularity near the caustic is due to the fact that as $z \to (-\varkappa)^{1/2}$, the divergence of the velocity of the main flow tends to infinity, which in turn leads to a marked compression of the matter comprising it. When this occurs, the density perturbation $\delta\rho$ becomes larger than $\rho$, or in other words, the matter breaks down into small-scale clumps which then collapse. The characteristic scale size $L$ of the clumps increases with distance from the center of the main singularity as $(r/L_m)^{4/7}$, while the time at which they form is $t_L = t_m + (3/2)t_g(r/L_m)^{4/7}$. The number density of clumps is proportional to the matter density $\rho$.

Thus, hierarchical structure develops on scales $L \lesssim L_m$ at practically the same time that an NGS is produced with the fundamental scale size (17), while at scales $L > L_m$ it requires the longer time (30). The exact number of NGS formed at the various scales depends quite sensitively both on the spectrum and the degree of phase correlation among the initial fluctuations, and is not determined by the present theory. Furthermore, as can be seen directly from (1), the major contribution to the mass of an NGS does not come from the singularity itself, but is instead to be found in its outlying regions, which are in fact not governed by the physical laws operative inside the NGS, and where other nonlin-

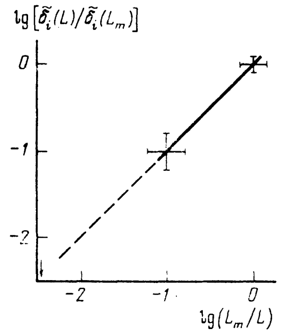


FIG. 2. Relative magnitude of initial perturbations $\tilde{\delta}_i(L)/\tilde{\delta}_i(L_m)$ as a function of $L_m/L$, where $L_m \approx$ 10–30 Mpc. The radius of the horizon corresponds to $L_m/L \approx (3\text{–}5)\cdot 10^{-3}$, and is indicated by the arrow.

ear structures may form. It must be assumed, however, that in a self-confined and thoroughly mixed state, the only place where there wil be a significant rise in density of nondissipative gas, i.e., a substantial increase in density $\rho \gg \rho_0$, will be within an NGS. It is also very important to note that when turbulence is well-developed, the scaling relations (1) hold at all scales, both for the gas density and for the number density of smaller-scale trapped singularities. Comparison of these theoretical results with the observational data on the distribution of galaxies in clusters, the distribution of clusters in superclusters, and the distribution of the missing mass in galaxies and clusters indicates that they are in fairly good agreement.[4] This affirms the crucial role played by nondissipative gravitational turbulence in the formation of the large-scale structure of the universe, where well-developed turbulence has been established on scales $L \lesssim$ 10–30 Mpc.

As an addendum to our previous brief analysis,[4] let us note that the scaling relations (1) are not observed on scales $L >$ 30–50 Mpc. This means that at these scales, turbulence has not yet, up to the present time $t$, become well-developed, i.e., $t < t_L$ (30). On larger scales, the nonlinear stage has not even been reached, so that effects related to the initial growth of perturbations may still be notable there. Indeed, recent observations[7,8] have divulged highly correlated mass motion on scales $L \sim 100$ Mpc with velocities $v \approx (4\text{–}8)\cdot 10^3$ km/sec. This is just the kind of divergent motion expected for a growing gravitational mode. The $L$-dependence of the relative magnitude of initial perturbations, using these data for a crude approximation, is shown Fig. 2. In this approximation, it is clear that the initial spectrum of fluctuations falls within a range of large scales proportional to $L^{-1}$. This is not inconsistent with previously discussed power-law spectra,[1,2] and lends credence to the idea that the primordial perturbations were adiabatic.

The authors thank V. L. Ginzburg, V. F. Mukhanov, and S. F. Shandarin for fruitful discussions.

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Translated by M. Damashek