Theory of attenuation of Rayleigh surface acoustic waves on a free randomly rough surface of a solid

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The problem of attenuation of Rayleigh surface acoustic waves on a free weakly rough boundary of an elastic isotropic solid is solved using the Born approximation of perturbation theory and the Green function technique developed by Maradudin and Mills [Ann. Phys. (N.Y.) **100**, 262 (1976)]. The errors in the paper of Maradudin and Mills are identified and the reasons for them are elucidated. The conditions are formulated under which the Green function method gives physically correct results for the problems dealing with the scattering of elastic waves on a weakly rough boundary. It is shown that in the limit $\bar{\lambda} \ll a$ ($\bar{\lambda}$ is the wavelength of the incident Rayleigh wave and a is the correlation radius of the roughness) the reciprocal of the attenuation length tends to a constant value on increase in the frequency. It is shown that the results obtained in the present study are in agreement with those reported by Eguiluz and Maradudin [Phys. Rev. B **28**, 728 (1983)] and it is pointed out that Eguiluz and Maradudin draw an incorrect conclusion that decay of a Rayleigh wave into volume waves predominates at any wavelength. It is also shown that in the range $\bar{\lambda} \ll a$ the attenuation is governed by the scattering into secondary Rayleigh waves, whereas the decay into volume waves is exponentially weak, in good agreement with the results of Urazakov and Fal'kovskiĭ [Sov. Phys. JETP **36**, 1214 (1973)]. A strong dependence of the attenuation coefficient on the properties of the medium is deduced from numerical calculations.

1. INTRODUCTION

The present paper reports a theoretical investigation of the attenuation of Rayleigh surface acoustic waves on a free randomly rough boundary of an elastic isotropic solid. Detailed investigations are of considerable interest because they can provide the information on surface properties of solids and also because of various potential technical applications.

This problem has been solved theoretically before in the case of weakly rough surfaces by a variety of methods.[1–7] For example, in the cornerstone paper[1] it was shown for the first time that propagation of a Rayleigh wave along a free weakly rough surface $\delta/\bar{\lambda} \ll 1$ (δ is the rms amplitude of the roughness and $\lambda = 2\pi\bar{\lambda}$ is the Rayleigh wavelength) results in its attenuation because of multiple scattering into volume (bulk) and secondary Rayleigh waves of the same frequency but with a different direction of the wave vector. The dispersion law of Rayleigh surface waves was found in an integral form. Since the integral in question could not be calculated explicitly, Urazakov and Fal'kovskiĭ,[1] estimated the order of magnitude of this integral and obtained the following results for the attenuation. In the long-wavelength approximation $\bar{\lambda} \gg a$ (a is the correlation radius of the surface roughness) the surface and volume attenuations are of the same order of magnitude, whereas in the short-wavelength limit $\bar{\lambda} \ll a$ the surface attenuation predominates.

Urazakov[2] calculated the energy flux in a wave averaged over the surface roughness. He demonstrated that in the limit of high values of a the attenuation is governed by the transport time which increases for high values of a. In contrast to Refs. 1 and 2, Maradudin and Mills[3] found that for any value of $a/\bar{\lambda}$ the surface attenuation is at least an order of magnitude higher than the volume process. The attenuation of Rayleigh surface waves was studied by Krylov and Lyamov[4] in the $\bar{\lambda} \gg a$ approximation. The scattering into the bulk of a medium was ignored following the conclusions of Ref. 3. However, it was pointed out in Refs. 5 and 6 that the results of Ref. 3 are in error. It was also stated by Eguiluz and Maradudin in Ref. 6 that the attenuation of Rayleigh waves on a weakly rough surface is governed primarily by the scattering into volume acoustic waves. On the other hand, it was shown in Ref. 7 that in the short-wavelength limit the attenuation is entirely due to the scattering into secondary Rayleigh waves, in agreement with the estimates obtained in Ref. 1. However, the frequency dependences of the attenuation coefficients given in Refs. 1 and 7 are fundamentally different.

It follows from this brief review that the various investigations cited above were carried out by different methods and yielded contradictory results. Our aim will therefore be to identify the reason for the published contradictory conclusions about the nature of the attenuation of Rayleigh surface waves on a free weakly rough surface and to obtain quantitative expressions for the attenuation coefficients.

2. FORMULATION OF THE PROBLEM. EQUATIONS OF MOTION OF AN ELASTIC MEDIUM BOUNDED BY A FREE ROUGH SURFACE

We shall assume that a plane monochromatic Rayleigh surface wave is incident on a randomly rough part of a boundary separating an elastic isotropic solid from vacuum, described by a random function $x_3 = f(x_1,x_2)$. The solid occupies the half-space $x_3 > 0$ and the rough region is a rectangle of linear dimensions L_1 and L_2 along the x_1 and x_2 axes, respectively. It is assumed that roughness is weak: $\delta/\bar{\lambda} \ll 1$ ($\delta^2 = \langle f^2(x_1,x_2)\rangle$, where the angular brackets denote averaging over an ensemble of all possible realizations of the roughness profile). We now have to find the field of displacements in scattered volume (longitudinal and transverse) and secondary Rayleigh waves, the energy fluxes of the scat-

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tered waves, and the attenuation length of the incident Rayleigh surface acoustic wave as a function of the parameters δ and a of the roughness microprofile, of the wavelength λ of the incident wave, and of the Poisson ratio σ of the medium.

We shall solve this problem in the first Born approximation of perturbation theory using the Green function method applied in Ref. 3 specifically to the problem of scattering of elastic waves by a free weakly rough surface of an isotropic solid. However, as pointed out in the Introduction, the results of Maradudin and Mills[3] are in error. Therefore, we have to determine why the use of this method gives incorrect results. We recall that the Green function method applied in Ref. 3 is based on the statement[8] that surface states can be investigated in the theory of elasticity by two equivalent methods. If we consider a semi-infinite solid with a free flat surface, we can solve the equations of motion of an elastic homogeneous medium occupying the half-space $x_3 > 0$,

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = C_{\alpha\beta\mu\nu} \frac{\partial^2 u_\mu}{\partial x_\beta \partial x_\nu}, \quad (1)$$

subject to the boundary conditions on the flat free surface $x_3 = 0$,

$$C_{\alpha 3\mu\nu} \frac{\partial u_\mu}{\partial x_\nu} = 0, \quad (2)$$

where $u_\alpha(\mathbf{x},t)$ is the αth Cartesian component of the displacement field, ρ is the density of the medium (mass per unit volume), $C_{\alpha\beta\mu\nu}$ are the elastic moduli of the medium, and $\mathbf{x} = (x_1,x_2,x_3)$; alternately, we can solve the equations of motion for an infinite inhomogeneous space

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta} C_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial u_\mu}{\partial x_\nu} + C_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial^2 u_\mu}{\partial x_\beta \partial x_\nu}, \quad (3)$$

where the elastic moduli of the medium $C_{\alpha\beta\mu\nu}(\mathbf{x})$ are functions of the spatial coordinates. Following Ref. 8, the authors of Ref. 9 propose the use of coordinate-dependent elastic moduli in the form

$$C_{\alpha\beta\mu\nu}(\mathbf{x}) = C_{\alpha\beta\mu\nu}\theta(x_3), \quad \theta(x_3) = \begin{cases} 1, & x_3 \geqslant 0 \\ 0, & x_3 < 0 \end{cases}, \quad (4)$$

where $\theta(x_3)$ is the Heaviside step function, so that Eq. (4) can be used to rewrite Eq. (3) in the form

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \delta(x_3) C_{\alpha 3\mu\nu} \frac{\partial u_\mu}{\partial x_\nu} + \theta(x_3) C_{\alpha\beta\mu\nu} \frac{\partial^2 u_\mu}{\partial x_\beta \partial x_\nu}, \quad (5)$$

where—as in Eq. (3)—ρ represents the ordinary coordinate-independent density of the medium.

Following Ref. 8, the authors of Ref. 9 state also that one can show that the solution of Eqs. (5) or (3) is equivalent to the solution of the system (1) satisfying the boundary condition (2) on the $x_3 = 0$ plane. However, this is not quite correct. The two approaches under discussion are equivalent if in Eqs. (3) and (5) the density of the medium is a function of the spatial coordinates together with the elastic moduli and in the case of a homogeneous elastic medium occupying the half-space $x_3 > 0$ is

$$\rho(\mathbf{x}) = \rho\theta(x_3). \quad (6)$$

In the opposite case, as we go from a semi-infinite homogeneous medium, bounded by a flat free surface $x_3 = 0$, to an infinite inhomogeneous space, we obtain a physically meaningless result $\rho(x_3 < 0) \neq 0$.

If we use Eq. (6), we find that the equations of motion in Eq. (5) become

$$\theta(x_3)\left[\rho \frac{\partial^2 u_\alpha}{\partial t^2} - C_{\alpha\beta\mu\nu} \frac{\partial^2 u_\mu}{\partial x_\beta \partial x_\nu}\right] - \delta(x_3) C_{\alpha 3\mu\nu} \frac{\partial u_\mu}{\partial x_\nu} = 0, \quad (7)$$

where $\theta'(x_3) = \delta(x_3)$ and the equivalence of the two approaches becomes obvious. Therefore, the system of equations of motion (3) can be written in the form

$$\rho(\mathbf{x}) \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta} C_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial u_\mu}{\partial x_\nu} + C_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial^2 u_\mu}{\partial x_\beta \partial x_\nu}. \quad (8)$$

We must stress that in Eq. (8) the density $\rho(x)$ is a function of the spatial coordinates, like the elastic moduli $C_{\alpha\beta\mu\nu}(\mathbf{x})$. We note that if this fact is ignored in the case of a flat free surface, the calculations still give the correct results (see, for example, Ref. 10). However, the situation changes fundamentally in the case of a nonflat boundary. For example, if the boundary is rough and is given by the equation $x_3 = f(x_1,x_2)$, then Eqs. (4) and (6) become

$$C_{\alpha\beta\mu\nu}(\mathbf{x}) = C_{\alpha\beta\mu\nu}\theta[x_3 - f(x_1, x_2)], \quad (9)$$

$$\rho(\mathbf{x}) = \rho\theta[x_3 - f(x_1, x_2)]. \quad (10)$$

In the case of a weakly rough surface considered in Ref. 3 and here we can expand $C_{\alpha\beta\mu\nu}(\mathbf{x})$ and $\rho(x)$ as functional Taylor series:

$$\begin{aligned} C_{\alpha\beta\mu\nu}(\mathbf{x}) &\approx C_{\alpha\beta\mu\nu}\theta(x_3) - f(x_1,x_2) C_{\alpha\beta\mu\nu}\delta(x_3) \\ &= C^{(0)}_{\alpha\beta\mu\nu}(\mathbf{x}) + C^{(1)}_{\alpha\beta\mu\nu}(\mathbf{x}), \\ \rho(\mathbf{x}) &\approx \rho\theta(x_3) - f(x_1, x_2)\rho\delta(x_3) = \rho^{(0)}(\mathbf{x}) + \rho^{(1)}(\mathbf{x}). \end{aligned} \quad (11)$$

Terms of higher order of smallness in f will be ignored. It follows from the system (11) that the expansion for $\rho(\mathbf{x})$ contains not only a term proportional to $\theta(x_3)$, but also a term proportional to $\delta(x_3)$, which alters fundamentally all the subsequent calculations.

This is precisely the point ignored by Maradudin and Mills in Ref. 3, where it is assumed that only the elastic moduli of the medium described by Eq. (9) depend on the spatial coordinates, whereas the density ρ is constant in all space. We shall show later that this error led eventually the authors of Ref. 3 to incorrect results.

We shall correct this error and use Eqs. (10) and (11) to represent the system of equations of motion (8) in the form

$$L^{(0)}_{\alpha\mu}(\mathbf{x}, t) u_\mu(\mathbf{x}, t) = -L^{(1)}_{\alpha\mu}(\mathbf{x}, t) u_\mu(\mathbf{x}, t), \quad (12)$$

where $L^{(0)}_{\alpha\mu}$ and $L^{(1)}_{\alpha\mu}$ are differential operators defined by

$$L^{(0)}_{\alpha\mu} = \frac{1}{\rho}\left[-\rho^{(0)}(\mathbf{x})\delta_{\alpha\mu}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial x_\beta} C^{(0)}_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial}{\partial x_\nu} + C^{(0)}_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial^2}{\partial x_\beta \partial x_\nu}\right], \quad (13)$$

$$L^{(1)}_{\alpha\mu} = \frac{1}{\rho}\left[-\rho^{(1)}(\mathbf{x})\delta_{\alpha\mu}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial x_\beta} C^{(1)}_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial}{\partial x_\nu} + C^{(1)}_{\alpha\beta\mu\nu}(\mathbf{x}) \frac{\partial^2}{\partial x_\beta \partial x_\nu}\right]. \quad (14)$$

Using a Green function $D_{\mu\beta}(\mathbf{x},\mathbf{x}';t-t')$, defined as the solution of inhomogeneous differential equations (subject to the boundary conditions for the free surface)

$$L_{\alpha\mu}^{(0)}(\mathbf{x},t)D_{\mu\beta}(\mathbf{x},\mathbf{x}';t-t')=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \quad (15)$$

we can reduce the system of differential equations (12) to a system of integral equations

$$u_\mu(\mathbf{x},t)=u_\mu^{(0)}(\mathbf{x},t)-\int d^3x'\int dt'\,D_{\mu\beta}(\mathbf{x},\mathbf{x}';t-t')\times L_{\beta\gamma}^{(1)}(\mathbf{x}',t')u_\gamma(\mathbf{x}',t'). \quad (16)$$

Here, $u_\mu^{(0)}(x,t)$ is the solution of the system of homogeneous equations

$$L_{\alpha\mu}^{(0)}(\mathbf{x},t)u_\mu^{(0)}(\mathbf{x},t)=0 \quad (17)$$

and it corresponds physically to a Rayleigh wave traveling along a free flat (not rough) surface $x_3=0$ of an isotropic solid.

Following Eq. (12) or (16), we shall find the displacement field from perturbation theory assuming that

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}^{(0)}(\mathbf{x},t)+\mathbf{u}^{(1)}(\mathbf{x},t)+\mathbf{u}^{(2)}(\mathbf{x},t)+\ldots; \quad (18)$$

the term of the second order of smallness in f amounting to $\mathbf{u}^{(2)}$ in Eq. (18) represents, over an ensemble of realizations of the function f, the attenuation of the average field of a Rayleigh wave traveling along a rough surface when this attenuation is due to the transfer of energy into scattered waves (see Refs. 1 and 6). In determination of the energy flux, which is a quadratic function of the field, this term as well as the term $\mathbf{u}^{(1)}$ make contributions of the second order of smallness in f, proportional to $\mathbf{u}^{(0)}\langle\mathbf{u}^{(2)}\rangle$ and equal in the absolute sense to the energy flux in the scattered wave $\mathbf{u}^{(1)}$, but with the opposite sign, i.e., this contribution represents the loss of energy from the original Rayleigh wave (this follows directly from the law of conservation of energy).

It is clear from the above discussion that in order to find the attenuation length of a Rayleigh wave traveling along a randomly rough surface we can use two physically equivalent approaches: 1) we can find the decreasing average field of the Rayleigh wave and use the dispersion law of this wave to calculate the attenuation length; 2) bearing in mind that the average field is attenuated because of the "transfer" of its energy into scattered waves, we can find the attenuation length directly using the energy flux in the scattered wave on the basis of the scattering theory. We shall adopt the second approach. Therefore, we shall be interested not in the average field of a Rayleigh wave traveling along a rough surface, but in the displacement field and the energy flux in the scattered waves. Consequently, we shall limit Eq. (18) to just the term of the first order of smallness in respect of f, which is $\mathbf{u}^{(1)}$, i.e., we shall adopt the Born approximation of perturbation theory for the displacement field and for the energy flux of the scattered waves.

3. DISPLACEMENT VECTORS AND ENERGY FLUXES OF SCATTERED WAVES

The displacement field of the scattered waves considered in the Born approximation can be deduced from Eqs. (13), (14), and (16):

$$u_\mu^{(s)}(\mathbf{x},t)=u_\mu(\mathbf{x},t)-u_\mu^{(0)}(\mathbf{x},t)=\int d^2x_\parallel'\int dt'\,D_{\mu\beta}(\mathbf{x}_\parallel,\mathbf{x}_\parallel';t-t';x_3,x_3'=0)L_{\beta\gamma}^{(sh)}(\mathbf{x}_\parallel')u_\gamma^{(0)}(\mathbf{x}',t'), \quad (19)$$

where

$$L_{\beta\gamma}^{(sh)}(\mathbf{x}_\parallel')=\sum_{\alpha=1}^{2}\frac{1}{\rho}\frac{\partial f(x_1',x_2')}{\partial x_\alpha'}C_{\beta\alpha\gamma\nu}\frac{\partial}{\partial x_\nu'}\Big|_{x_3'=0}-\frac{1}{\rho}f(x_1',x_2')C_{\beta3\gamma\nu}\frac{\partial^2}{\partial x_3'\,\partial x_\nu'}\Big|_{x_3'=0},\quad \mathbf{x}_\parallel=(x_1,x_2,0). \quad (20)$$

The action of the operator (20) on a function $u_\gamma^{(0)}(\mathbf{x}',t')$ can be studied if after performing all the differentiations we assume that $x_3'=0$. It should also be noted that the operator (20) is independent of time because the surface roughness is stationary. Consequently, the spectral component of the scattered field [representing the Fourier transform $\mathbf{u}^{(s)}(x,t)$ with respect to time] is proportional to $\delta(\omega-\omega_0)$, i.e., the scattered wave has the same frequency as the incident wave.

Following the method described in detail in Refs. 3 and 11, we find that at large distances from a rough region of the surface the scattered field consists of three waves: volume longitudinal, volume transverse, and Rayleigh surface waves. The scattered volume waves are spherical and the scattered Rayleigh wave is cylindrical. The explicit form of the expressions for these scattered waves can be found in Ref. 11. These expressions can be used to calculate the energy fluxes of the scattered waves (see Ref. 11). In particular, the flux averaged with respect to time and over an ensemble of rough surfaces has the following form for a scattered Rayleigh wave:

$$\frac{dE_R}{dt}=|A|^2\rho\frac{P\omega^6}{8\pi R^2c_Rc_t^2\alpha^2\gamma^2}\int_0^{2\pi}d\varphi_s\langle|\hat{f}(\mathbf{k}_R-\mathbf{k}_\parallel^{(0)})|^2\rangle\times(1-\cos\varphi_s)^2[\cos\varphi_s+(1-c_R^2/2c_t^2)]^2, \quad (21)$$

where $\hat{f}(\mathbf{k}_\parallel)$ is the Fourier transform of $f(x_1,x_2)$; A is the complex amplitude of the incident Rayleigh wave; c_l, c_t, and c_R are the velocities of longitudinal, transverse, and Rayleigh acoustic waves, respectively; φ_s is the azimuthal angle $(0\leqslant\varphi_s\leqslant2\pi)$; $k_\parallel^{(0)}$ is the wave vector of the incident Rayleigh wave; $\mathbf{k}_R=(\omega/c_R)(\cos\varphi_s;\sin\varphi_s;0)$; P, R, α, and γ are the constants that depend only on σ. The correlation function of the surface roughness is of the Gaussian form:

$$W(|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|)=\langle f(\mathbf{x}_\parallel)f(\mathbf{x}_\parallel')\rangle=\delta^2\exp[-(\mathbf{x}_\parallel-\mathbf{x}_\parallel')^2/a^2]. \quad (22)$$

We note that in statistical averaging a distribution function is used for an infinite rough surface, but this function differs from zero only inside the region between L_1 and L_2. The application of this distribution function in the case of a rough region of finite dimensions naturally gives rise to an error. However, if the dimensions of the scattering rough area are considerably greater than the correlation radius of the roughness a, integration in the course of averaging can be carried out between infinite limits (see Ref. 12). The error inherent in this approach is estimated not to exceed 10^{-4}–10^{-5}.

In particular, it follows from Eq. (21) that the scattered field is absent for angles defined by the equations $\cos\varphi_s=1$, and $\cos\varphi_s = -(1 - c_R^2/2c_t^2)$. This is in agreement with the conclusions reached in Ref. 5, where the Born approximation is used in a study of the scattering of surface waves by three-dimensional inhomogeneities of the boundary and where it is shown that the results of Ref. 3 are in error. [In contrast to Ref. 5 and to Eq. (21) in the present paper, it follows from Eq. (4.21) of Ref. 3 that there are no angles for which the scattered field vanishes.]

4. ATTENUATION LENGTH OF A RAYLEIGH SURFACE ACOUSTIC WAVE

In view of the random nature of the roughness, the scattered longitudinal, transverse, and Rayleigh waves contribute additively to the total energy flux of the scattered field:

$$\frac{dE^{(T)}}{dt}=\frac{dE^{(l)}}{dt}+\frac{dE^{(t)}}{dt}+\frac{dE^{(R)}}{dt}. \tag{23}$$

On the other hand, it follows from the law of conservation of energy that the average energy flux in the scattered waves is

$$\frac{dE^{(T)}}{dt}=\frac{dE_0}{dt}-\frac{dE}{dt}\bigg|_{x_1=L_1}, \tag{24}$$

where dE_0/dt is the energy flux in the incident Rayleigh surface acoustic wave; dE/dt is the energy flux in the Rayleigh wave that has traveled a distance x_1 along the rough surface (see Ref. 6):

$$\frac{dE}{dt}(x_1)=\frac{dE_0}{dt}e^{-x_1/l}. \tag{25}$$

In the case of a weakly rough surface considered in the present study the influence of the roughness on the incident wave is not very great, i.e., $L_1 \ll l$. Since the energy fluxes of Eq. (23) are found in the present study to within the first nonvanishing term proportional to δ^2, we shall substitute Eq. (25) into Eq. (24) and expand Eq. (24) as a Taylor series in L_1/l to within the first nonvanishing term, which gives the following expression for the reciprocal of the attenuation length:

$$\frac{1}{l}=\frac{1}{L_1}\frac{(dE^{(T)}/dt)}{(dE_0/dt)}. \tag{26}$$

It follows from Eqs. (23) and (26) that

$$\frac{1}{l}=\frac{1}{l^{(l)}}+\frac{1}{l^{(t)}}+\frac{1}{l^{(R)}}, \tag{27}$$

where $l^{(l,t,R)}$ is the attenuation length of a Rayleigh surface acoustic wave scattered into volume longitudinal transverse waves and into Rayleigh surface waves, respectively. We shall now give the final expressions for the reciprocals of the attenuation lengths:

$$\frac{1}{l^{(l)}}=\frac{\delta^2a^2\omega^5}{4c_l^3c_t^2}\exp\left[-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\right]\int_0^{\pi/2}d\theta_s\Phi^{(l)}(z,\theta_s,\sigma)\times\exp\left[-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\frac{c_R^2}{c_l^2}\sin^2\theta_s\right], \tag{28}$$

$$\frac{1}{l^{(t)}}=\frac{\delta^2a^2\omega^5}{4c_t^5}\exp\left[-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\right]\int_0^{\pi/2}d\theta_s\,\Phi^{(t)}(z,\theta_s,\sigma)\times\exp\left[-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\frac{c_R^2}{c_t^2}\sin^2\theta_s\right], \tag{29}$$

$$\frac{1}{l^{(R)}}=\frac{\pi}{4R^2}\frac{\delta^2a^2\omega^5}{c_Rc_t^4\alpha^2\gamma^2}\exp\left[-\frac{1}{2}\left(\frac{a\omega}{c_R}\right)^2\right]\times\left\{I_0(z)V_1+\frac{I_1(z)}{z}(zV_2-V_3)+3\frac{I_2(z)}{z^2}-\frac{1}{4}\frac{c_R^2}{c_t^2}I_3(z)\right\}, \tag{30}$$

where

$$z=\begin{cases}(1/2)(a\omega/c_R)^2(c_R/c_l)\sin\theta_s & \text{for } 1/l^{(l)},\\ (1/2)(a\omega/c_R)^2(c_R/c_t)\sin\theta_s & \text{for } 1/l^{(t)},\\ (1/2)(a\omega/c_R)^2 & \text{for } 1/l^{(R)},\end{cases}$$

and $I_n(z)$ are modified Bessel functions of order n; the explicit form of the expressions for $\Phi^{(l)}$, $\Phi^{(t)}$, and V_i ($i=1,2,3$) can be found in Ref. 11.

5. NUMERICAL CALCULATIONS. DISCUSSION OF RESULTS

We shall continue the analysis of Eqs. (28)–(30) by introducing new notation:

$$\frac{1}{l^{(l,t)}}=\frac{\delta^2a^2\omega^5}{\pi c_R^5}f_{B(l,t)}\left(\frac{a\omega}{c_R}\right),\quad \frac{1}{l^{(R)}}=\frac{\delta^2a^2\omega^5}{\pi c_R^5}f_R\left(\frac{a\omega}{c_R}\right), \tag{31}$$

where the function f_{Bl}, f_{Bt}, and f_R are dimensionless, whereas the factor $\delta^2a^2\omega^5/\pi c_R^5$ has the dimensions of the reciprocal length. We shall consider the limiting cases of long and short Rayleigh waves incident on a rough part of the surface.

In the case of long wavelengths $a\omega/c_R \equiv a/\lambda\!\!\!^- \ll 1$, we can use expansions of the Bessel functions as a Taylor series for small values of the argument and we then find from Eqs. (28)–(30) that the functions f_{Bl}, f_{Bt}, and f_R are independent of $a/\lambda\!\!\!^-$ and are expressed entirely in terms of the Poisson ratio σ of the medium. In this limiting case we obtain from Eq. (31) the reciprocals of the attenuation lengths $\sim\delta^2a^2\omega^5/c_R^5$, in agreement with the results of Refs. 1, 6, and 7.

In the limit of short wavelengths $a\omega/c_R \gg 1$, the Laplace method[13] yields the following asymptotic expressions:

$$\frac{1}{l^{(l)}}\approx\frac{\delta^2}{a^2}\omega\frac{c_R^4}{c_l^3c_t^2}\frac{G_l}{4P}\exp\left\{-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\left(1-\frac{c_R}{c_l}\right)^2\right\}, \tag{32}$$

$$\frac{1}{l^{(t)}}\approx\frac{\delta^2}{a^2}\omega\frac{c_R^4}{c_t^5}\frac{G_t}{4P}\exp\left\{-\frac{1}{4}\left(\frac{a\omega}{c_R}\right)^2\left(1-\frac{c_R}{c_t}\right)^2\right\}, \tag{33}$$

$$\frac{1}{l^{(R)}}\approx\frac{\pi^{1/2}}{R^2}\left(\frac{c_R}{c_t}\right)^4\frac{1}{\alpha^2\gamma^2}\left\{\frac{\delta^2a\omega^4}{c_R^4}G_{R1}+\frac{\delta^2\omega^2}{ac_R^2}G_{R2}+\frac{\delta^2}{a^3}G_{R3}-\frac{\delta^2c_R^2}{a^5\omega^2}G_{R4}+\dots\right\}, \tag{34}$$

where

$$G_{R1}=G_{R2}=0,\quad G_{R3}=\frac{3}{16}\left[4-\frac{c_R^2}{c_t^2}\right]^2,$$

$$G_{R4}=\frac{15}{8}\left[6-\frac{1}{8}\frac{c_R^4}{c_t^4}-\frac{c_R^2}{c_t^2}\right],$$

and G_l and G_t are constants dependent on σ (see Ref. 11).

It should be noted that the coefficients in front of ω^4 and ω^2 in Eq. (34) vanish, $G_{R1}=G_{R2}=0$, so that the final expression for $1/l^{(R)}$ is

$$\frac{1}{l^{(R)}}\approx\frac{\pi^{1/2}}{R^2}\left(\frac{c_R}{c_t}\right)^4\frac{G_{R3}}{\alpha^2\gamma^2}\frac{\delta^2}{a^3}\left\{1-\frac{(G_{R4}/G_{R3})}{(a\omega/c_R)^2}+\dots\right\}, \tag{35}$$

from which it follows that an increase in the frequency causes the reciprocal of the attenuation length of a Rayleigh wave to approach a constant value $C\delta^2/a^3$, where the constant C depends only on the Poisson ratio σ (for a medium with $\sigma = 0.25$, we find that $G_{R4}/G_{R3} \approx 5.089$).

It is interesting to compare the frequency dependence given by Eq. (35) with results of other investigations. For example, the average energy flux in a wave is also calculated in Ref. 2, but in contrast to our results an allowance is made for multiple scattering. It is shown there that in the range of high correlation radii a the situation is quite complex and there are several limiting cases. In particular, if $a < \tau c_R$ (τ is the attenuation or decay time of a Rayleigh wave), the following estimate applies:

$$\frac{1}{l^{(R)}} \sim \frac{\delta^2}{a} \frac{\omega^2}{c_R^2}. \tag{36}$$

The attenuation was investigated in Refs. 1, 6, and 7 using the dispersion law of Rayleigh surface acoustic waves on a free weakly rough boundary

$$\omega(k) \approx \omega_0(k) + \Delta\omega(k), \qquad \Delta\omega(k) = \mathrm{Re}\,\Delta\omega - i|\mathrm{Im}\,\Delta\omega|, \tag{37}$$

where $\omega_0 = kc_R$ is the dispersion law of a Rayleigh wave propagating along an ideally smooth surface. The imaginary part $\Delta\omega(k)$ is known to govern the attenuation of a surface wave. Since the energy flux is a quadratic function of the displacement, the attenuation coefficient is defined as follows:

$$\frac{1}{l} = 2\frac{|\mathrm{Im}\,\Delta\omega(k)|}{c_R}. \tag{38}$$

Then, in the limit of short Rayleigh wavelengths $a/\lambda\!\!\!^{-} \gg 1$, the following estimate is obtained in Ref. 1:

$$\frac{1}{l^{(R)}} \sim \frac{\delta^2 a\omega^4}{c_R^4}. \tag{39}$$

In Ref. 6 the asymptotic form of $1/l^{(R)}$ was not investigated, but we carried out such calculations independently [using Eqs. (4.24)–(4.29) of Ref. 6] and we obtained an expression completely identical with Eq. (35). Finally, the same limiting case is considered in Ref. 7 and it is found that

$$\frac{1}{l^{(R)}} \approx \frac{\pi^{1/2}}{Q^2}\left(1-\frac{c_R^2}{c_t^2}\right)\left(1-\frac{c_R^2}{c_l^2}\right) \times G_{R3}\frac{\delta^2}{a^3}\left\{1-\frac{(G_{R4}/G_{R3})}{(a\omega/c_R)^2}+\dots\right\}, \tag{40}$$

where Q is a constant (see Ref. 7), whereas G_{R3} and G_{R4} are given above [see Eq. (34)]. In the limit of high frequencies both Eqs. (40) and (35) tend to a constant value $C'\delta^2/a^3$, but the constant C' differs from the constant C used in the present study (for a medium with $\sigma = 0.25$ we have $C'/C \approx 6.5$).

Going back to an analysis of the asymptotic expressions (32)–(35), we note that whereas $1/l^{(R)}$ tends to a constant on increase in the frequency, both $1/l^{(l)}$ and $1/l^{(t)}$ decrease exponentially. Therefore, at high frequencies $a\omega/c_R \gg 1$ the attenuation due to the scattering into secondary Rayleigh waves becomes the main process in respect of the parameter

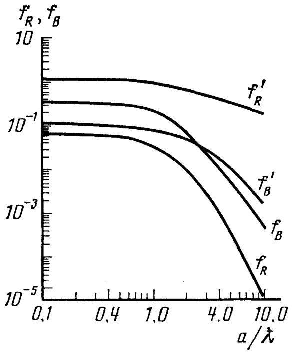

FIG. 1. Results of a numerical calculation of the functions f_B and f_R for an elastic medium characterized by $c_t/c = 1/\sqrt{3}$ (i.e., by $\sigma = 0.25$). The dependences of f'_B and f'_R are taken from Ref. 3.

$a/\lambda\!\!\!^{-}$, in agreement with the results given in Refs. 1 and 7.

The relative contribution of the attenuation of surface acoustic waves due to the scattering into volume and secondary Rayleigh waves throughout the frequency range was found by numerical calculation of the functions $f_B = f_{Bl} + f_{Bt}$ and f_R depending on $a\omega/c_R$. In these calculations we used Eqs. (28)–(35) for an elastic medium characterized by $\sigma = 0.25$ ($c_t/c_l = 1/\sqrt{3}$). The results of these calculations are presented in Figs. 1 and 2. It is clear from these numerical calculations that in the range of long wavelengths of Rayleigh surface acoustic waves $a/\lambda\!\!\!^{-} \ll 1$ the attenuation due to the scattering into volume waves is approximately 6 times stronger than the attenuation due to the scattering into secondary Rayleigh waves, i.e., the attenuation coefficients are quantities of the same order of magnitude. However, at short wavelengths $a/\lambda\!\!\!^{-} \gg 1$ the surface attenuation is the principal process in respect of the parameter $a/\lambda\!\!\!^{-}$. These results are also in good agreement with the estimates given in Refs. 1 and 7. In comparing our results with those of Ref. 6, we must bear in mind the following point. In the range of wavelengths $0.1 \leqslant a/\lambda\!\!\!^{-} \leqslant 10.0$ (or $\lambda > 2\pi a/10 \approx 0.6a$), which was the only range considered in Ref. 6, the results of the present work and of Ref. 6 were compared numerically and found to be in agreement. However, the range of wavelengths $a/\lambda\!\!\!^{-} > 10$ was not considered in Ref. 6. It is in this range already when $a/\lambda\!\!\!^{-} \approx 70$ ($\lambda \approx 0.09a$) that the scattering into secondary Rayleigh waves is an order of magnitude stronger than the scattering into volume acoustic waves (Fig. 2). It should be noted that the values of f_R for $a/\lambda\!\!\!^{-} > 20$ (Fig. 2) were calculated

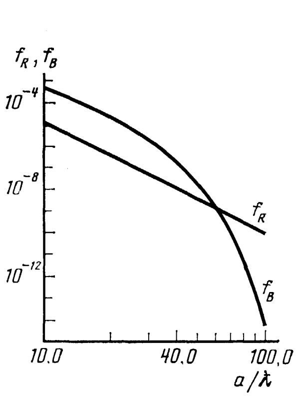

FIG. 2. Same as in Fig. 1, but for higher values of $a\omega/c_R \equiv a/\lambda\!\!\!^{-}$.

TABLE I. Dependences of functions f_{Bl}, f_{Bt}, and f_R on the Poisson ratio σ of a medium in the case of long ($a/\lambda \ll 1$) Rayleigh waves.

σ	f_{Bt}	f_{Bl}	f_B	f_R	f_B/f_R
0.05	0.388	0.113	0.501	0.202	2.48
0.17	0.342	0.074	0.416	0.104	4.00
0.25	0.312	0.046	0.358	0.063	5.68
0.34	0.277	0.021	0.298	0.034	8.76
0.45	0.230	0.003	0.233	0.014	16.6
0.49	0.212	0.0003	0.212	0.010	21.2

using an asymptotic expression (35), which—as pointed out above—is deduced from analytic expressions in Ref. 6 in the limit $a/\bar{\lambda} \gg 1$. Therefore, in the range $a/\bar{\lambda} > 10$ the results agree. Hence, we may conclude that the results of the present work and of Ref. 6 obtained by different methods agree for any wavelength of Rayleigh surface acoustic waves. The agreement between these results demonstrates that the conclusion reached by Eguiluz and Maradudin[6] on predominant scattering of Rayleigh acoustic waves of any frequency into volume waves is not quite correct.

Finally, a comparison with Ref. 3 demonstrates that the results obtained in the present study differ fundamentally from those given in Ref. 3 (compare dependences of f_R' and f_R' in Fig. 1) and the latter are in error. The error of Ref. 3 was admitted by the authors in Ref. 6 and they tried to account for the errors made in the Green function method.[3] They compared the expression obtained for the displacement field of scattered waves with an analogous expression in Ref. 3 (Appendix B in Ref. 6) and concluded that there is an "extra" term in Ref. 3. The reason for the appearance of this term is not established in Ref. 6. We demonstrated above that if the conditions (9)–(11) are satisfied, this extra term is balanced out and the displacement fields become identical. This identity of the displacement fields of scattered waves obtained in the present study and in Ref. 6, and the agreement between the final results shows that the corrections introduced by us into the Green function method developed in Ref. 3 are appropriate.

We can therefore conclude that the Green function method gives the correct results when dealing with the scattering of elastic waves on a weakly rough boundary if in going over from a semiinfinite homogeneous medium to a semiinfinite inhomogeneous space we assume that the density of the medium is, like the elastic moduli, a function of the spatial coordinates.

We also investigated the dependences of the attenuation coefficients of Rayleigh surface acoustic waves on the nature of the material. It follows from our numerical calculations that the attenuation coefficients depend strongly on the Poisson ratio σ. At long wavelengths $\bar{\lambda} \gg a$ in the case of media with $0 < \sigma \lesssim 0.25\sigma$ the attenuation due to the scattering into volume acoustic waves and secondary Rayleigh surface acoustic waves is of the same order of magnitude (in agreement with Ref. 1). However, in the case of media characterized by $0.25 < \sigma < 0.5$ the attenuation due to the scattering into volume waves is considerably (by an order of magnitude) stronger than the attenuation due to the scattering into secondary Rayleigh surface acoustic waves (Table I). In the case of short wavelengths $\bar{\lambda} \ll a$, the scattering into secondary Rayleigh surface acoustic waves is the dominant mechanism in any elastic medium.

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