Effect of focusing on photon correlation in parametric light scattering

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The correlation of photons in parametric and hyperparametric light scattering has been considered when a convergent lens is placed in front of one or both detectors. The space-time structure of the light intensity correlation function $G(x_1, x_2)$, which determines the probability to detect two photons at the points x_1 and x_2, has been calculated utilizing the Green's function (where $x = \{\mathbf{r}, t\}$). It is shown that the dependence of $G(x_1, x_2)$ on x_2 is determined by the effective field $\mathscr{E}(x_2)$ emitted from x_1 backwards in time and space. Consequently $G(x_1, x_2)$ has a sharp maximum where the points $\mathbf{r}_1$ and $\mathbf{r}_2$ are imaged onto each other by the lens, the size of the focal region being determined by the diffraction at the lens aperture or at the specimen.

Spontaneous parametric scattering (PS), i.e., the formation of correlated pairs of photons in piezocrystals with $\mathbf{k}_0 \to \mathbf{k} + \mathbf{k}'$, has found an ever expanding field of applications: in crystal spectroscopy,[1] for the absolute measurements of light intensity[2–4] and detector efficiency,[2,4–7,23] to disprove the Bell inequalities,[7] to generate light beams with a known number of photons,[2,4,8,9] with anti-bunching and the sub-Poisson photon statistics,[9] for message coding,[10] and to demonstrate the quantum interference of intensities.[11] This effect is also called spontaneous parametric frequency transformation and parametric fluorescence; however, the term "scattering," first used by Tang,[12] more suitably reflects the essence of the phenomenon (if only for the reason, that it goes over continuously into scattering on polaritons).[13,14]

All these applications, with the exception of Ref. 1, are based on the two-photon and coherent nature of PS, which is described by the intensity correlation function $G(x_1, x_2) = G_{12}$. In this paper the effect of a convergent lens on G_{12} is calculated. Even though quite a few papers devoted to the calculations of G_{12} have been produced,[2,6,14–19] the effect of focusing on G_{12} appears almost not to have been studied. Reference 6 presents an exception; it gives a brief description of the special case of two coincident points $\mathbf{r}_1 = \mathbf{r}_2$, on the lens axis; we also note Ref. 20, where the effect of a lens on the two-photon atomic emission field has been evaluated, and Ref. 21, where the focusing of G_{12} has been considered qualitatively in connection with the Einstein-Podolsky-Rosen experiment.

To clarify the nature of the problem we shall first consider an experiment, in which the correlation of a photon pair in hyperparametric scattering (HPS) is being observed in the case of a pumped standing wave (Fig. 1). The HPS-effect, or scattering of light on light (see, for example, Ref. 14), can be described by the elementary process $\mathbf{k}_0 + \mathbf{k}_0' \to \mathbf{k} + \mathbf{k}'$ which yields $\mathbf{k} = -\mathbf{k}'$ when $\mathbf{k}_0 = -\mathbf{k}_0$, so that the specimen emits photon pairs in every direction with opposite momenta and approximately the same energy ($\omega \approx \omega_0$).

As is shown in the calculations carried out in Sec. 2, the dependence of G_{12} on $\mathbf{r}_2$ at fixed $\mathbf{r}_1$ can be described by an effective field $\mathscr{E}(\mathbf{r}_2)$, emitted from $\mathbf{r}_1$ and passing backward through the lens and the medium into the area of displacement of the detector 2 (Fig. 1). Thus, if a photon was registered in $\mathbf{r}_1$, then the second photon of this pair will almost invariably pass through an area of a size which has an order of several Airy radii (the latter is determined by the diffraction on the lens aperture or the specimen). The presence of diffraction shows that the effective field is generated by coherent emission from the whole specimen, and consequently, it can not be assumed that photon-particle pairs diverge in opposite directions by turns from different points of the specimen.

A similar phenomenon should also be observed in PS (see Fig. 2). Here the momentum conservation law leads to the emission of photons (when $\omega \sim \omega' \sim \omega_0/2$) at roughly equal angles to the crystal surface, so that the crystal can be considered a mirror as far as the effective field is concerned. The specific connection between points in space, which is optimal in the sense of coincidence probability, is determined by the position and focal length of the lens. This connection can not be explained within the framework of "local realism" (See Ref. 21 for more details); it can be termed mutual focusing of photons.

In Secs. 1 and 2 below the corresponding correlation functions for PS and HPS have been computed, and in the Appendix we determine the commutator of the field operators with the lens effect taken into account, which is necessary for the calculations.

1. CORRELATION FUNCTIONS IN PS WITH FOCUSING

We shall proceed from the general formulas for correlation functions that provide the probabilities for single counts in the detectors and also the coincidence probability; they were derived in Refs. 6, 21:

$$G_1 = -i\hbar \hat{\chi}_x^{(-)} \hat{\chi}_{x'}^{(+)} D^*(x_1, x) D_0(x, x') D(x_1, x'), \quad (1.1)$$

$$G_{12} = |i\hbar \hat{\chi}_x^{(+)} D(x_1, x) D(x_2, x)|^2 \equiv |F_{12}|^2. \quad (1.2)$$

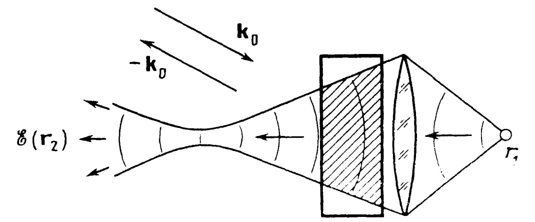

FIG. 1. The effect of a lens on photon correlation in degenerate hyperparametric scattering: the dependence of coincidence probability on the coordinate of one of the detectors $\mathbf{r}_2$ is described by the effective field $\mathscr{E}(\mathbf{r}_2)$ emitted from the point $\mathbf{r}_1$ where the second detector is located.

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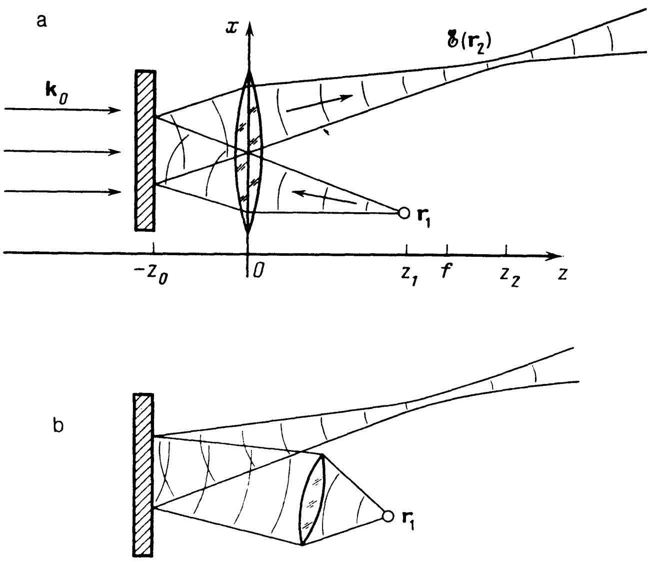

FIG. 2. In the case of parametric scattering the crystal serves as a mirror when determining the structure of the effective field; a) symmetrical lens configuration, b) asymmetrical lens configuration.

Here

$$\hat{\chi}_x^{(\pm)} f(x) \equiv \int \chi^{(2)} \mathscr{E}_0^{(\pm)}(x) f(x)\, d^4x, \tag{1.3}$$

$$D(x_1, x) \equiv \frac{i}{\hbar}\,[E_{x_1}^{(+)}, E_x^{(-)}], \tag{1.4}$$

$$D_0(x, x') \equiv \frac{i}{4\pi^2} \sum_k \omega_k \exp(ik(x-x')), \tag{1.5}$$

where $\chi^{(2)}$ is the squared susceptibility and $\mathscr{E}_0$ is the classical pumping field. Using Eq. (A11), we find the relationship

$$G_1 = \frac{c}{2\pi\hbar\bar{\omega}} \int G_{12} d^2\rho_2\, dt_2, \tag{1.6}$$

where $\rho = \{x, y\}$ and $\bar{\omega}$ is some average frequency (the photon flux density is assumed to be equal to $cG_1/2\pi\hbar\bar{\omega}$).

We shall define the photon effective field at the point x_2 as follows:

$$\mathscr{E}_2^{(+)} \equiv C_1 F_{12}, \qquad C_1 \equiv G_1^{-1/2}. \tag{1.7}$$

It follows from Eq. (1.6) that the total energy of the effective field is equal to $\hbar\bar{\omega}$. According to Eq. (1.2), the effective field can be determined as a result of interaction between the advanced wave emitted at x_1 and the pumping field (the details are discussed in Ref. 21).

The probability of the independent detection of a photon by a wide band detector is proportional to the function G_1 and is weakly dependent on its position $\mathbf{r}_1$. At the same time, the function $|F_{12}|$ which determines the form of the effective field and the coincidence probability, changes drastically in the focal region. We find this dependence in the case of the geometry depicted in Fig. 2a, by means of the Green's function for an ideal lens with Gaussian apodization (see Appendix).

Assume that the radius of a pumping beam and the transverse size of a non-linear crystal greatly exceed the lens radius R; then from Eqs. (1.2) and (A9), by substituting

$$\sum_{\mathbf{k}} = \int d^3k \approx \int d^2q\, d\omega/c$$

we will obtain (neglecting factors of order unity):

$$F_{12} = \frac{i\hbar\chi^{(2)}\mathscr{E}_{00}}{2\pi c^2} \int d^2q \int_0^{\omega_0} d\omega\,\omega\tilde{\omega}\, D_{\mathbf{r}_1\mathbf{k}} D_{\mathbf{r}_2\tilde{\mathbf{k}}} \times \int_{-z_0-l}^{-z_0} dz \exp i\,[\omega(t_2-t_1) - \omega_0 t_2 - \Delta_z z], \tag{1.8}$$

where

$$\tilde{\omega} = \omega_0 - \omega, \quad \tilde{\mathbf{k}} = \{-\mathbf{q},\ (k^2(\tilde{\omega}) - q^2)^{1/2}\},$$

$$\mathbf{q} = \{k_x, k_y\}, \quad \Delta_z = k_z + \tilde{k}_z - k_0,$$

$z_0 > 0$ is the distance from the lens to the crystal, l is the length of the crystal, $D_{\mathbf{rk}}$ is the lens Green's function, and it is assumed that

$$\mathscr{E}_{0x}^{(+)} = 2\,\mathrm{Re}\,\mathscr{E}_{00} \exp(ik_0 x).$$

To simplify the formulas, we assume that one detector registers only the "signal" photons ($\omega > \omega_0/2$), while the other registers only "blanks" ($\omega < \omega_0/2$). Assume that for the frequencies ω_1, $\omega_2 = \tilde{\omega}_1$ not too distant from $\omega_0/2$ there is a collinear synchronism $k_1 + k_2 = k_0$, so that in quasioptical and quasimonochromatic approximations we have

$$\begin{aligned} k_z(\omega, \mathbf{q}) &\approx k(\omega) - q^2/2k(\omega) \approx k_1 + (\omega - \omega_1)/u_1 - q^2/2k_1, \\ \tilde{k}_z(\omega, \mathbf{q}) &\approx k(\tilde{\omega}) - q^2/2k(\tilde{\omega}) \approx k_2 - (\omega - \omega_1)/u_2 - q^2/2k_2, \\ \Delta_z(\omega, \mathbf{q}) &\approx \alpha(\omega - \omega_1) - \beta q^2, \end{aligned} \tag{1.9}$$

where

$$\alpha \equiv u_1^{-1} - u_2^{-1}, \quad \beta \equiv (k_1^{-1} + k_2^{-1})/2, \quad u \equiv d\omega/dk.$$

Thus, it is assumed that the radiation spectrum consists of two regions around ω_1 and ω_2, symmetrical with respect to $\omega_0/2$. This is precisely how the PS spectrum looks at a definite orientation of the crystal.[5] The width of these bands $\Delta\omega$ is much larger than $\Delta\omega_l = 2\pi/|\alpha| l$, the width of the spectrum observed in a narrow interval of directions in the far zone. The average relative photon retardation $\Delta\tau = |t_1 - t_2|$ in one experiment, which is of the order of the reciprocal of the spectrum width, varies accordingly: $1/\Delta\omega_l \sim 1$ ps in the far zone and $1/\Delta\omega \ll 1$ ps in the near zone.[6]

Let us substitute the Green's function (A6) into Eq. (1.8). The integrals over z and ω are:

$$I = \omega_1\omega_2 \int_0^l dz \int_0^{\omega_0/2} d\omega \exp i[(\omega - \omega_1)\alpha z - \omega\Delta t - \beta q^2 (z + z_0)], \tag{1.10}$$

where $\Delta t \equiv t_1 - t_2 - r_1/c + r_2/c$, and we have taken $\alpha = 0$ into account outside the crystal. If the limits of integration with respect to ω are taken to be infinite, then the result of integration is $2\pi\delta(\alpha z - \Delta t)$. This means[17] that, when the delay Δt is fixed, only the radiation emitted by an infinitely thin layer at depth $\Delta t/\alpha \geqslant 0$[1)] is detected. For the δ-approximation to be applicable, the actual thickness of that active layer $1/\Delta\omega|\alpha|$ has to be much less than $k/q_m^2 \sim \lambda/\vartheta_m^2$, which is the typical scale for the functions of q ($\vartheta_m = q_m/k$ is the maximal significant scattering angle). Therefore we assume that $\vartheta_m^2 \ll |\alpha| c(\Delta\omega/\omega) \sim 0.1(\Delta\omega/\omega)$. Then

$$I=2\pi\omega_1\omega_2|\alpha|^{-1}\exp(-i\omega_1\Delta t-i\beta z_0^* q^2), \quad (1.11)$$

where $z_0^* \equiv z_0 + \Delta t/\alpha$ is the distance from the lens to the active layer and the condition $0 < \Delta t/\alpha < l$ is satisfied.

It remains only to integrate functions of the type $\exp(\mathbf{a}\cdot\mathbf{q} + bq^2)$. In order to account for possible additional limiting of the angular spectrum, we will introduce the factor $\exp(-q^2/q_m^2)$. As a result

$$F_{12}=-\frac{\pi\hbar k_1 k_2\chi^{(2)}\mathscr{E}_{00}\tilde{z}_1\tilde{z}_2}{|\alpha|z_1 z_2 A_{12}}e^{i\varphi_{12}}, \quad (1.12)$$

where

$$\varphi_{12}\equiv k_0 r_2-\omega_0 t_2-\omega_1\Delta t-\frac{1}{4A_{12}}\left(\frac{\tilde{z}_1}{z_1}\boldsymbol{\rho}_1-\frac{\tilde{z}_2}{z_2}\boldsymbol{\rho}_2\right)^2+\frac{k_1\tilde{z}_1}{2z_1^2}\rho_1^2+\frac{k_2\tilde{z}_2}{2z_2^2}\rho_2^2,$$

$$A_{12}\equiv\frac{\tilde{z}_1-z_0^*}{2k_1}+\frac{\tilde{z}_2-z_0^*}{2k_2}+\frac{i}{q_m^2}, \quad (1.13)$$

$$\tilde{z}_n\equiv\left(\frac{1}{f}-\frac{1}{z_n}-\frac{i}{k_n R^2}\right)^{-1}\equiv\left(\frac{1}{z_n'}-\frac{i}{k_n R^2}\right)^{-1}.$$

The parameters q_m and R determine the diffraction dimensions of the focal region, i.e., the longitudinal and transverse width of the maximum of $|F_{12}|$ with respect to z_2 and $\boldsymbol{\rho}_2$ when z_1, $\boldsymbol{\rho}_1$ are fixed (and vice versa). We will assume in the future that R plays the main role (i.e., $q_m = \infty$).

Let us first consider the special case of $z_1 = z_2 = f$; then $\tilde{z}_n = ik_n R^2$ and $A = iR^2$ so that the envelope of the correlation function is

$$|F_{12}|=F\exp[-(k_1\boldsymbol{\rho}_1+k_2\boldsymbol{\rho}_2)^2R^2/4f^2], \quad (1.14)$$

where

$$F\equiv\hbar|\alpha|^{-1}k_1^2k_2^2\chi^{(2)}|\mathscr{E}_{00}|\Delta\Omega, \quad \Delta\Omega\equiv\pi R^2/f^2.$$

Thus, the coincidences in the focal plane are mostly observed in the vicinity of points which satisfy the condition

$$k_1\boldsymbol{\rho}_1+k_2\boldsymbol{\rho}_2=0 \quad (1.15)$$

and are defined with the diffraction precision $\Delta\rho\sim\lambda/\vartheta_m$, where $\vartheta_m = R/f$. This relation, which can be readily obtained from the condition $q_1 + q_2 = 0$ and the approximate identity $\rho_n/f = q/k_n$, has been confirmed experimentally.[5]

Next we shall exclude the area $z_n \sim f$ and disregard the difference between k_1 and k_2; then it follows from Eq. (1.12) when $kR^2 \gg z_n'$, that

$$|F_{12}|=\frac{2f^2F}{kRa_2(z_1^2+z_2^2/\mu^2)^{1/2}}\exp\left[-\frac{(\boldsymbol{\rho}_2-\mu\boldsymbol{\rho}_1)^2}{2a_2^2}\right], \quad (1.16)$$

where

$$a_2\equiv a_2^{(0)}(1+\delta^2)^{1/2}, \quad a_2^{(0)}\equiv(\mu^2z_1^2+z_2^2)^{1/2}/kR,$$
$$\delta\equiv kR^2(z_1'+z_2'-2z_0^*)/(z_1'^2+z_2'^2),$$
$$z_n'\equiv(f^{-1}-z_n^{-1})^{-1}, \quad \mu\equiv(z_2-f)/(z_1-f).$$

Thus the condition for the coincidences in the transverse plane to have a maximum takes the form $\boldsymbol{\rho}_2 = \mu\boldsymbol{\rho}_1$; the width of the maximum is determined by the function $a_2(z_1, z_2)$, which has the meaning of beam radius for the effective field $\mathscr{E}_2$. The maximum as a function of the longitudinal coordinates of the detectors occurs at $\delta = 0$. It can be readily verified that these conditions describe the effect of mutual focusing of photons: they connect the points $\mathbf{r}_1$ and $\mathbf{r}_2$ according to the rules of geometric optics, with the light passing twice through an ideal lens located in the plane $z = 0$ and reflected from a flat mirror in the plane $-z_0^*$. It is assumed here that the conditions $0 < \Delta t/\alpha < l$ are satisfied by choice of an appropriate difference between detection times $t_1 - t_2$; in practice, this is automatically assured owing to the large value of the time constant for the detectors, as a result of which the whole volume of the crystal contributes, and the "depth of field" will be determined by the crystal length l.

2. THE EFFECT OF FOCUSING ON DEGENERATE HPS

The decay of two pumped photons into two photons can be described using the above formulas when $\chi^{(3)}\mathscr{E}_0^{(+)2}$ is replaced by $\chi^{(2)}\mathscr{E}_0^{(+)2}$, where $\chi^{(3)}$ is the cubic susceptibility.[2)] In the case of a pumped standing wave, $\mathscr{E}_0^{(+)2}$ contains the uniform term $2\mathscr{E}_{00}^2\exp(-i2\omega_0 t)$, which is responsible for the wave front conjugation. The latter is accompanied by spontaneous Rayleigh scattering and also by HPS, which is almost isotropic and monochromatic (for more details concerning parametric effects in elastic scattering see Ref. 22).

Let us consider the experiment depicted in Fig. 1. The following substitutions are now in order in Eq. (1.8):

$$\chi^{(2)}\mathscr{E}_{00}\to 2\chi^{(3)}\mathscr{E}_{00}^2, \quad \omega_0\to 2\omega_0, \quad D_{\mathbf{r}_2\tilde{\mathbf{k}}}\to\exp(i\tilde{\mathbf{k}}\mathbf{r}_2),$$
$$\Delta_z\to k_z+\tilde{k}_z\approx 2(\omega-\omega_0)/c;$$

here $k_z > 0$, $z_1 > 0$, $\tilde{k}_z < 0$, $z_2 < -z_0 - l$. Instead of (1.12), we now obtain

$$F_{12}=F\frac{f\tilde{z}_1}{2k_0 R A_{12}}e^{i\varphi_{12}}, \quad (2.1)$$

where

$$F\equiv 2\pi\hbar c k_0^3\chi^{(3)}\mathscr{E}_{00}^2 R/fz_1, \quad A_{12}\equiv(\tilde{z}_1+z_2)/2k_0+i/q_m^2,$$
$$\varphi_{12}\equiv k_0(z_1-z_2)-\omega_0(t_1+t_2)+k_0\rho_1^2/2(z_1-\tilde{f})-(\boldsymbol{\rho}_2+\tilde{z}_1\boldsymbol{\rho}_1/z_1)^2/4A_{12},$$
$$\tilde{f}^{-1}\equiv f^{-1}-i/kR^2.$$

The photon registration points should satisfy the obvious inequalities:

$$-2l<ct_2-ct_1+z_1+z_2+2z_0<0. \quad (2.2)$$

If the detector 1 is located in the lens focus ($\boldsymbol{\rho}_1 = 0$, $z_1 = f$), then

$$|F_{12}|=Ffa_2^{-1}\exp(-\rho_2^2/2a_2^2), \quad (2.3)$$

where $a_2(f, z_2) = R[1 + (z_2/k_0R^2)^2]^{1/2}$. Thus the effective field $\mathscr{E}_2$ to the left of the layer represents a Gaussian beam with initial radius equal to the lens radius.

Now suppose $\mathbf{r}_1$ is located outside the focal region, i.e., $|z_1'| \ll k_0R^2$, then we will obtain from (2.1)

$$|F_{12}|=Ffa_2^{-1}\exp[-(\boldsymbol{\rho}_2+\boldsymbol{\rho}_1 z_1'/z_1)^2/2a_2^2], \quad (2.4)$$

where the radius of the beam to the left equals:

$$a_2(z_1, z_2)=|z_1'|(1+\delta^2)^{1/2}/k_0R, \quad \delta=k_0R^2(z_1'+z_2)/z_1'^2 \quad (2.5)$$

(the radius of a_1 of the beam of the effective field $\mathscr{E}_1$ equals $a_2 z_1/|z_1'|$). Equation (2.4) describes the effect of mutual focusing of two photons in HPS: the probability of coincidence is a maximum when $\delta = 0$ and $\boldsymbol{\rho}_2/z_2 = \boldsymbol{\rho}_1/z_1$, i.e., when $-z_2 = z_1' = (f^{-1} - z_1^{-1})^{-1}$.

Next with the help fo Eq. (1.6) we will estimate the total intensity of the degenerate HPS. Integrating $|F_{12}|$ with respect to t_1 (or t_2) according to Eq. (2.2), yields the factor $2l/c$, while integrating with respect to $\boldsymbol{\rho}_1$ (or $\boldsymbol{\rho}_2$) gives the factor πa_1^2 (or πa_2^2). The result is

$$I_{1,2} = 8\pi^2\hbar k_0^5 \chi^{(3)2} I_0^2 l \Delta\Omega_{1,2}, \tag{2.6}$$

where $I_0 = c\mathscr{E}_{00}^2/2\pi$, $\Delta\Omega_1 = \pi R^2/z_1^2$, $\Delta\Omega_2 = \pi R^2/z_1'^2$.

From Eqs. (2.6) and (1.7) we find the effective field to the left of the layer

$$|\mathscr{E}_2^{(\pm)}|^2 = \frac{\hbar\omega_0}{la_2^2(z_1, z_2)} \exp\left[-\frac{(\boldsymbol{\rho}_2 + \boldsymbol{\rho}_1 z_1'/z_1)^2}{a_2^2(z_1, z_2)}\right] \tag{2.7}$$

[the effective field to the right of the lens differs from the above by the factor $(z_1'/z_1)^2$]. This quasi-monochromatic field, not vanishing in a time interval of order $2l/c$ [see (2.2)], determines the conditional probability to detect a photon at the point x_2 when a photon is detected at x_1, according to the general rules. Eq. (2.7) describes the spatial structure of the plane-wave "packet," which corresponds to the emitted photon in the intuitive interpretation. According to (2.7), the structure of the photon (for $z_2 < -z_0 - l$) corresponds to the structure of the field which has been emitted from the point $\mathbf{r}_1$ and then has propagated back through the lens and the specimen. For $z_1 > f$ the photon is focused by the lens (through which it does not pass!) into the point $\{\boldsymbol{\rho}_2, z_2\} = \{-\boldsymbol{\rho}_1 z_1'/z_1, -z_1'\}$. Due to the condition $|\mathbf{k} + \mathbf{k}'| < 2\pi/l$ (which also leads to wavefront conjugation effect), the spatial spectrum of the advanced spherical δ-pulse remains practically unchanged under parametric transformation, while a band $\omega_0 \pm \pi c/l$ is filtered from the frequency spectrum when the pulse length is appropriately increased. According to Eq. (2.7), the photon field is localized in a region of length of $2l$ and cross-section a_2, which depends on z_1 and z_2.

Let us now compare the intensities of degenerate HPS and Rayleigh scattering. In terms of $\chi^{(3)}$, the latter is characterized by the following dispersion coefficient (specific pumping losses per unit distance and unit solid angle):

$$R_{\mathrm{Ral}} = \frac{I}{I_0 l \Delta\Omega} = k_0^4 \varkappa T \chi^{(3)} \sim 10^{-4}\ \mathrm{cm}^{-1}, \tag{2.8}$$

where $\varkappa$ is the Boltzmann constant and it is assumed that $\lambda = 0.5\ \mu\mathrm{m}$; $T = 300\ ^\circ\mathrm{K}$; $\chi^{(3)} = 10^{-11}\ \mathrm{cm}^3/\mathrm{erg}$ (for carbon disulfide when the field amplitudes are determined without the factor of 1/2). According to (2.6), on the other hand, when $I_0 = 1\ \mathrm{W/cm}^2$,

$$R_{\mathrm{HPS}} = 8\pi^2\hbar k_0^5 \chi^{(3)2} I_0 \sim 10^{-15}\ \mathrm{cm}^{-1}. \tag{2.9}$$

Note here that even though HPS is much weaker than PS, the photon focusing effect manifests itself in a purer form—without the "reflection" which diffuses the focal region due to chromatic aberration and the finite length l of the specimen.

APPENDIX

Green's function for the focused field

Let the amplitude of the monochromatic field to the left of the lens, which is located on the plane $z = 0$, be equal $E_\omega(\boldsymbol{\rho}, -0)$, where $\boldsymbol{\rho} = \{x, y\}$. An ideal lens introduces a phase shift $-k\rho^2/2f$, where f is the focal length and $k = \omega/c$; therefore, the amplitude immediately behind the lens is

$$E_\omega(\boldsymbol{\rho}, +0) = E_\omega(\boldsymbol{\rho}, -0)\exp(-ik\rho^2/2f). \tag{A1}$$

To account for the finite size of the lens aperture a factor $\exp(-\rho^2/2R^2)$ is introduced (R is the effective radius), equivalent to replacing f *by* $\tilde{f} \equiv (1/f - i/kR^2)^{-1}$. Such a smooth apodization significantly simplifies the structure of the focal region (when compared to the case with sharp boundaries) but it violates the unitarity of transformation of the field by the lens. This last argument is not essential to the problem under consideration, though.

In quasi-optical approximation the further field propagation can be described by the Fresnel transformation ($z_1 > 0$).

$$E_\omega(\mathbf{r}_1) = \frac{k}{2\pi iz} e^{ikz_1} \int d^2\rho\, E_\omega(\boldsymbol{\rho}, +0)\exp\left[i\frac{k(\boldsymbol{\rho}_1 - \boldsymbol{\rho})^2}{2z_1}\right]. \tag{A2}$$

Let us expand the incident field in plane waves (assume that $\omega > 0$):

$$E_\omega(\mathbf{r}) = \sum_{\mathbf{k}} E_{\mathbf{k}}^{(+)} e^{i\mathbf{k}\mathbf{r}} \delta(\omega - \omega_{\mathbf{k}}), \tag{A3}$$

where $z \leqslant 0$, $k_z \geqslant 0$. Substituting (A1) and (A3) when $z = 0$ into (A2), we will express the field which has passed through the lens in terms of the incident field in the $\mathbf{k}$-representation.

$$E_\omega(\mathbf{r}_1) = \sum_{\mathbf{k}} D_{\mathbf{r}_1\mathbf{k}} E_{\mathbf{k}}^{(+)} \delta(\omega - \omega_{\mathbf{k}}), \tag{A4}$$

$$E_{x_1}^{(+)} = \int_0^\infty d\omega\, E_\omega(\mathbf{r}_1) e^{-i\omega t_1} = \sum_{\mathbf{k}} D_{x_1\mathbf{k}} E_{\mathbf{k}}^{(+)}. \tag{A5}$$

The following notations are used here:

$$D_{\mathbf{r}\mathbf{k}} \equiv -\frac{\tilde{z}}{z}\exp\left[ikr + i\frac{k\tilde{z}}{2}\left(\frac{\mathbf{q}}{k} - \frac{\boldsymbol{\rho}}{z}\right)^2\right]$$
$$= \frac{1}{1 - z/\tilde{f}}\exp\left[ikz - \frac{i}{1 - z/\tilde{f}}\left(\frac{k\rho^2}{2\tilde{f}} - \mathbf{q}\boldsymbol{\rho} + \frac{q^2 z}{2k}\right)\right], \tag{A6}$$

$$D_{x\mathbf{k}} \equiv D_{\mathbf{r}\mathbf{k}}\exp(-i\omega_{\mathbf{k}} t), \tag{A7}$$

$$\frac{1}{\tilde{z}} \equiv \frac{1}{\tilde{f}} - \frac{1}{z}, \quad r \equiv z + \frac{\rho^2}{2z}, \quad \mathbf{q} \equiv \{k_x, k_y\}. \tag{A8}$$

According to (A4) and (A6), the plane wave is transformed by an ideal apodized lens into a Gaussian packet (TEM_{00} wave) with an axis parallel to $\mathbf{k}$, and its "waist" in the plane $z = f$.

This rule of field transformation by a lens also holds in quantum theory, if applied to the normal moments. In order to determine the Green's function $D(x_1, x)$, i.e., the field at the point x_1 created by a point-source at the point x in front of the lens, we shall express the incident field as $E_x^{(+)} = \Sigma E_{\mathbf{k}}^{(+)} e^{ikx}$ and form the commutator of the incident and the transmitted fields:

$$D(x_1,x)=\frac{i}{\hbar}[E_{x_1}^{(+)},E_x^{(-)}]$$

$$=\frac{i}{4\pi^2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}D_{x_1k}e^{-ikx},\quad z<0,\quad z_1>0,\quad k_z>0. \quad \text{(A9)}$$

Here we have used the Eqs. (A5) and $[E_{\mathbf{k}}^{(+)},E_{\mathbf{k}}^{(-)}]=\hbar\omega_k/4\pi^2$ (the normalization length is assumed to be 2π).

These functions $D_{\mathbf{rk}}$ are roughly orthogonal for different values of $\mathbf{q}$. In fact, it follows from (A6) that (for arbitrary values of z and $\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}$):

$$\int d^2\rho\, D_{\mathbf{rk}}{}^{*}D_{\mathbf{rk}'}=\pi R^2\exp[-(\mathbf{q}-\mathbf{q}')^2R^2/4]\approx4\pi^2\delta^{(2)}(\mathbf{q}-\mathbf{q}'). \quad \text{(A10)}$$

Thus, in the approximation $R=\infty$

$$c\int dt\, d^2\rho D_{x\mathbf{k}}{}^{*}D_{x\mathbf{k}'}=(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \quad \text{(A11)}$$

Using (A9) and (1.5) we obtain the resulting relationship:

$$D_0(x,x')=\frac{ic}{2\pi\overline{\omega}}\int dt_1\, d^2\rho_1 D^{*}(x_1,x)D(x_1,x'),$$

$$z,\ z'<0,\quad z_1>0, \quad \text{(A12)}$$

which leads to (1.6).

[1)]This layer serves as a mirror which reflects the advanced wave.
[2)]Here we neglect the dispersion $\chi^{(3)}$ in the region $\omega\sim\omega_0$.

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