Spontaneous singularities in three-dimensional turbulence and the emission of sound during strong dynamical interaction between point vortex dipoles

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An "explosive" growth of the power of acoustic emission occurs after a finite time when two non-coaxial point vortex dipoles (infinitesimally small vortex rings) approach one another.

Onsager[1] was apparently the first to mention the fundamental problem of spontaneous singularities in three-dimensional turbulence; this problem has been studied intensively from various angles in present-day hydrodynamics.[2–6] In particular, the author[6] has obtained an exact solution of the dynamics of point vortex dipoles (infinitesimally small vortex rings) corresponding to an unbounded explosive growth of the localized vorticity in a finite time upon collapse (convergence into a single point) of two non-coaxial vortex dipoles. A particularly stimulating role is played here by experiments by the Stanford group of Klein[7] and by others,[8] (see also Ref. 3) who observed "bursts" of localized vorticity in turbulent boundary layers. The recorded finite (albeit relatively large) amplitude of the vorticity in Refs. 7, 8 during the time of the explosions is, apparently, caused by some dissipative mechanisms. For instance, the emission of acoustic waves by the turbulence[9,10] may be such a factor limiting the explosive growth of the local vortex field.

In the present paper we consider the possibility of an anomalously strong sound generation in a weakly compressible medium during the collapse of a pair of non-coaxial point vortex dipoles. In principle we define more precisely the existing ideas (see Refs. 9, 10) about the weak efficiency of turbulence as a sound emitter in the limit of small Mach numbers.

1. To sove the problem of the generation of vortex sound we use the method of the joining of asymptotic expansions[11,12] in which the Mach number $\mathbf{Ma} = v/c \ll 1$ is the small parameter, where $v(t)$ is the velocity of approach (along a logarithmic spiral trajectory[6]) non-coaxial vortex dipoles, and c the sound velocity in the weakly compressible medium.

Let the two non-coaxial vortex dipoles have Lamb momenta which are equal in absolute magnitude, but which have opposite directions, $\rho_0\gamma_1(t) = -\rho_0\gamma_2(t) \equiv \rho_0\gamma(t)$, and let they be at time t at a distance $l \equiv |\mathbf{l}| = |\mathbf{x}_1 - \mathbf{x}_2|$ from one another, where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the Cartesian coordinates of the first and the second vortex dipole, satisfying [like $\gamma(t)$] the dynamic set of equations given in Ref. 6. If initially at $t=0$ the vectors $\mathbf{l}$ and γ lie in the same (x, y) plane, it follows from the angular momentum conservation law $\mathbf{M} = \rho_0[\gamma \mathbf{l}] = \text{const}$ (ρ_0 is the unperturbed density of the medium) that they remain in the same plane also for any other $t > 0$. We shall start from this assumption about the initial conditions and characterize the direction of the vectors γ and $\mathbf{l}$ in the (x, y) plane by the polar angles $\varphi_1(t)$ and $\varphi_2(t)$, respectively.

The motion of the fluid outside the vortex dipoles is potential and is described by the velocity potential

$$\Phi = -\frac{\gamma(\mathbf{x}-\mathbf{x}_1)}{4\pi|\mathbf{x}-\mathbf{x}_1|^3} + \frac{\gamma(\mathbf{x}-\mathbf{x}_2)}{4\pi|\mathbf{x}-\mathbf{x}_2|^3}.$$

Choosing the origin of the spherical coordinate system (r, θ, φ) at the point

$$\mathbf{B} = [\mathbf{x}_1(t) + \mathbf{x}_2(t)]/2 = \text{const}$$

(into which the vortex dipoles collapse[6]) we get $\mathbf{x}_1 = \mathbf{l}(t)/2$, $\mathbf{x}_2 = -\mathbf{l}(t)/2$, and for the potential Φ we have in the limit $r \gg l$ the expression

$$\Phi \approx \frac{\gamma(t)l(t)}{4\pi r^3}\left(\frac{4\pi}{5}\right)^{1/2}\Big[-Y_{2,0}(\theta)\cos(\varphi_1(t)-\varphi_2(t)) + \frac{(4!)^{1/2}}{4}\left(Y_{2,2}(\theta,\varphi)e^{-i(\varphi_1+\varphi_2)} + Y_{2,-2}(\theta,\varphi)e^{i(\varphi_1+\varphi_2)}\right)\Big] \times\left[1 + O\left(\frac{l^2}{r^2}\right)\right], \tag{1}$$

where $Y_{n,m}(\theta, \varphi)$ are spherical functions, $\gamma(t) \equiv |\gamma|$, $r^2 = x^2 + y^2 + z^2$.

Under the influence of the non-stationary pressure field corresponding to (1) the point vortex dipoles can generate acoustic oscillations Ψ, the propagation of which in the wave zone $r \gg \lambda$ (λ is the wavelength of Ψ) is described by the equation $c^{-2}\partial^2\Psi/\partial t^2 - \Delta\Psi = 0$, where Ψ is the sound potential and Δ the three-dimensional Laplace operator. We shall apply a standard technique,[11,12] which uses an expansion of Ψ in a series in the spherical functions $Y_{n,m}$ and the radial Hankel functions $H^{(1)}_{n+1/2}(r/\lambda)$, to look for a solution Ψ of this equation which satisfies the emission conditions as $r \to \infty$ and which is the same as the potential (1) in the vortex zone $\lambda \gg r \gg l$ (as $\lambda \approx O(l/\mathbf{Ma})$ when $\mathbf{Ma} \ll 1$). We then get from the equation $p = -\rho_0\partial\Psi/\partial t$ for the oscillations of the pressure in the acoustic wave which is emitted by the pair of vortex dipoles in the wave zone $r \gg \lambda$

$$p\left(t + \frac{r}{c}, r\right) \approx -\frac{5\rho_0 \sin^2\theta \tilde{M}}{4\pi^3 r c^2 l^{10}(t)}[A_1(t)\cos 2(\varphi - \varphi_2(t)) + A_2(t)\sin 2(\varphi-\varphi_2(t))], \tag{2}$$

where

$$\tilde{M} \equiv \frac{M}{\rho_0}, \quad A_1 = -4H\tilde{M} + \frac{\tilde{M}[\tilde{M}^2 - 65(\gamma\mathbf{l})^2]}{10\pi l^5(t)},$$

$$A_2 = \gamma\mathbf{l}\left[-10H + \frac{7\tilde{M}^2 - 5(\gamma\mathbf{l})^2}{\pi l^5(t)}\right], \quad H \equiv \frac{T'}{\rho_0},$$

$$T' = \frac{\rho_0\gamma^2}{4\pi l^3}\left[1 - \frac{3(\gamma\mathbf{l})^2}{\gamma^2 l^2}\right]$$

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is the invariant interaction energy of the vortex dipoles. In agreement with Ref. 6

$$\boldsymbol{\gamma}\mathbf{l}=5Ht+\boldsymbol{\gamma}_0\mathbf{l}_0, \quad \gamma^2(t)=4\pi Hl^3+\frac{3}{l^2}(\boldsymbol{\gamma}\mathbf{l})^2,$$

$$l^5(t)=l_0^5-\frac{5}{\pi}\left(t(\boldsymbol{\gamma}_0\mathbf{l}_0)+\frac{5Ht^2}{2}\right), \quad \omega\equiv\frac{d\varphi_2}{dt}=-\frac{M}{2\pi l^5(t)},$$

$$\boldsymbol{\gamma}_0\equiv\boldsymbol{\gamma}(t=0).$$

There is therefore in this approximation with respect to the small parameter $\mathbf{Ma}\ll 1$ no emission in the direction $\theta=0$ [i.e., in the (x,y) plane] and the frequency of the p oscillations increases without bounds in the time of the collapse of the vortex dipoles, i.e., $\omega(t)\to\infty$ as $l(t)\to 0$.

2. In particular, for almost coaxial merging vortex dipoles the energy flux of the acoustic emission

$$I=r^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\frac{p^2}{\rho_0 c}$$

(see Ref. 10) through the surface of a sphere of radius $r\gg\lambda$ has, in accordance with (2), the form (in the limit as $t\to t_0$)

$$I\left(t+\frac{r}{c},r\right)\approx\frac{\varepsilon\,\mathrm{Ma}_0^5}{(1-t/t_0)^{12}}[1+O(\varphi_0^2)], \quad (3)$$

where $\varphi_0\ll 1$, but $\varphi_0\neq 0$ when

$$\cos\varphi_0\equiv\boldsymbol{\gamma}_0\mathbf{l}_0/\gamma_0 l_0>0, \quad t_0=2\pi l_0^4/5\gamma_0>0,$$

$\mathbf{Ma}_0=v_0/c\ll 1$, $v_0=\gamma_0/l_0^3$, $\varepsilon=\rho_0 v_0^3 l_0^2\varphi_0^8/60\pi^7$ is the magnitude of the vortex energy flux. In this limit $\mathbf{Ma}=|v(t)|/c\approx\mathbf{Ma}_0(1-t/t_0)^{-3/5}$ and the applicability of (3) is clearly justified under the condition

$$(1-t/t_0)^{9/5}\lesssim\mathrm{Ma}(t)\ll 1$$

(i.e., $\mathbf{Ma}_0^{5/3}\ll|1-t/t_0|\lesssim\mathbf{Ma}_0^{5/12}$), when the acoustic efficiency

$$K=\frac{I}{\varepsilon}\approx\frac{\mathrm{Ma}^5(t)}{(1-t/t_0)^9},$$

corresponding to (3) becomes already close to unity. The situation is not changed quantitatively for larger φ_0, since we have, for instance for $\varphi_0=\pi/2$,

$$I\approx O\left(\mathrm{Ma}_0^5\left[1-\left(\frac{t}{\sqrt{2}t_0}\right)^2\right]^{-6}\right).$$

At the same time we have for coaxial vortex dipoles ($\varphi_0=0$ or $\varphi_0=\pi$) already $I\approx O(\mathbf{Ma}^9)$. We note that the estimate $I\approx O(\mathbf{Ma}^5)$ in Ref. 11 (see also Ref. 9) for the sound emission intensity of two coaxial vortex rings of finite radius $R(t)$ is obtained in the limit when $l(t)\ll R(t)$—of small distances $l(t)$ between the centers of the rings—and is determined by the effect of the periodic time dependence of $R(t)$ in the "vortex leap-frogging" process. In the present paper, however, we consider essentially the opposite limit $l(t)\gg R(t)$, (e.g., when $R\sim l_0\mathbf{Ma}_0^{2/3}\equiv R_0$, since $|1-t/t_0|\gg\mathbf{Ma}_0^{5/3}$ and $l(t)\approx O((1-t/t_0)^{2/5})$) simulated by the dynamics of point vortex dipoles which do not change their structure even as $l(t)\to 0$.[6]

Equation (3) thus shows that the emission of vortex sound at times t close to t_0 can be very efficient notwithstanding that the magnitude of the acoustic efficiency $K\approx O(\mathbf{Ma}^5)$, as is usually the case for sound emission by turbulence in a weakly compressible medium.[9,10] The possibility of similar, although appreciably weaker, effects for the magnification of K was obtained for point vortices in two-dimensional hydrodynamics,[13] and also in Ref. 14 for vorton dynamics (vorton dynamics itself, however, does not satisfy all conservation laws of the three-dimensional equations of hydrodynamics, in contrast to the dynamics of point vortex dipoles[6]).

In connection with the results obtained above there is interest in developing experimental studies related to those described in Ref. 15: of acoustic radiation by small vortex rings (with $R\sim R_0$ and $\varphi_0\neq 0$) which collide at a nonzero angle, and also the realization of acoustic time measurements of the vorticity bursts observed in a turbulent boundary layer.[3,7,8]

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