Spontaneous and stimulated emission from free electrons

L. A. Vaĭnshteĭn

S.I. Vavilov Institute of Physics Problems, Academy of Sciences of the USSR, Moscow
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Relationships between spontaneous and stimulated emission from bound electrons (characterized by a discrete energy spectrum) are well known in quantum electrodynamics. We shall derive similar relationships for free electrons with a continuous energy spectrum, which are assumed to be traveling across a high-Q resonator with a discrete energy spectrum. An analysis is made of these relationships written in the classical and quantum forms, and examples of their applications are given.

INTRODUCTION

Stimulated emission of radiation from electrons moving in vacuum under the influence of static fields is related in several recent papers to their spontaneous emission, i.e., to the emission of radiation in the absence of alternating fields,[1–4] but a sufficiently full and general analysis of this topic is still lacking. We shall try to fill this gap in the case of resonant spontaneous oscillators, i.e., cavity or open resonators, traversed by electron fluxes. We shall first consider rectilinear electron fluxes and use them as a simple example to demonstrate the relationship between stimulated and spontaneous emission, on the one hand, and the relationship between the classical and quantum theories of emission and absorption, on the other. More general results (for curvilinear electron fluxes) will be obtained using quantum theory.

Application of quantum theory to the essentially classical problems of vacuum electronics is justified because all three processes of spontaneous emission, stimulated emission, and absorption are related firmly but simply in quantum electrodynamics, which is not true of the classical theory; moreover, quantum theory allows ready averaging, which in the classical approach requires cumbersome procedures. Since only the classical limit is important in the resultant quantum formulas, there is no need to apply the Dirac equation to relativistic electrons, but it is sufficient to employ the relativistic Schrödinger equation for zero-spin particles (or, which is equivalent, the Klein–Gordon or Klein–Fock equation), which simplifies the procedure very significantly. Further simplification is achieved by ignoring the interaction between electrons (in particular, by ignoring the space-charge field). However, the bunching of electrons in the resonator field is allowed for although only one electron is considered. In fact, an electron is ascribed a random initial phase and averaging is carried out over this phase. This makes it possible to calculate the coefficient representing the damping or growth of oscillations in a resonator with a continuous electron beam (Sec. 2).

1. CLASSICAL CALCULATION IN THE CASE OF RECTILINEAR ELECTRON TRAJECTORIES

As is known, an electron moving uniformly and rectilinearly in free space (vacuum) does not emit radiation. However, when an electron crosses a resonator, it excites wave fields and leaves behind some of its energy in the resonator. Both transition radiation and Vavilov–Cherenkov radiation (in the case of resonators filled with an insulator) or the similar Smith–Purcell radiation (in the case of resonators with a periodic structure) is then generated. In all cases the radiation can be calculated using a theory of excitation of resonators.

We shall assume that an electron moving uniformly along the z axis excites a cavity resonator in a section $0<z<L$, i.e., it generally excites all the natural oscillations of the resonator. We shall consider a specific oscillation with a complex eigenfrequency $\omega_s = \omega_s' - i\omega_s''$, where the attenuation coefficient ω_s'' is governed by the ohmic losses in the casing and by the transfer of energy to the load: when the Q factor $Q_s = \omega_s'/2\omega_s''$ is sufficiently high, the influence of these circumstances on the field distributions can be ignored and the complex vector amplitudes $\mathbf{E}_s(\mathbf{r})$ and $\mathbf{H}_s(\mathbf{r})$ are the same as if the sth natural oscillation had been undamped.

An electron moving uniformly along the z axis creates a current of density which has a single component

$$j_z = ev\delta(x)\delta(y)\delta[z-v(t-t_0)] = \frac{e}{\pi}\delta(x)\delta(y)\,\mathrm{Re}\int_0^\infty \exp\left[-i\omega\left(t-t_0-\frac{z}{v}\right)\right]d\omega, \qquad (1)$$

where e is the electron charge; v is the velocity of the electron; t_0 is the moment when the electron is at the point $z=0$, i.e., it enters the resonator (Fig. 1). Using the orthogonality of the natural oscillations

$$\int \mathbf{E}_s\mathbf{E}_{s'}^*\,dV = \int \mathbf{H}_s\mathbf{H}_{s'}^*\,dV = 4\pi N_s\delta_{ss'}, \qquad (2)$$

where

$$N_s = \frac{1}{4\pi}\int |\mathbf{E}_s|^2\,dV \qquad (3)$$

is the norm of the sth oscillation, we can discuss such oscillations in the form of traveling waves, as is usual in quantum electrodynamics. The electric field of the sth oscillation deduced from the theory of excitation of resonators (see, for example, Ref. 6) is given by the expression

$$\mathbf{E} = \frac{-1}{2\pi N_s}\mathrm{Re}\left\{ i\mathbf{E}_s\int_0^\infty \frac{e^{-i\omega t}\,d\omega}{\omega-\omega_s}\iint \mathbf{j}\mathbf{E}_s^* e^{i\omega t}\,dV\,dt\right\}. \qquad (4)$$

The energy transferred by the current of Eq. (1) to the field of the sth oscillation during the whole transit time $T=L/v$ is

$$\Delta_0 W_s = -\int dV\int_{t_0}^{t_0+T} j_z E_z\,dt. \qquad (5)$$

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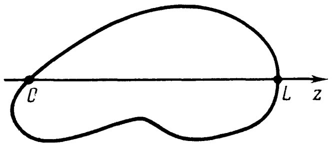

FIG. 1. Electron in a cavity resonator (part of its casing may be in the form of a periodic structure).

Elementary transformations make it possible to write down $\Delta_0 W_s$ in the form

$$\Delta_0 W_s = \int_0^\infty R_s(\omega) F_s(\omega)\, d\omega, \quad (6)$$

where

$$R_s(\omega) = \frac{1}{\pi\omega_s''(\xi_s^2+1)}, \quad \xi_s = \frac{\omega-\omega_s'}{\omega_s''} \quad (7)$$

is the resonator function which in the limit $\omega_s'' \to 0$ becomes $\delta(\omega - \omega_s')$, whereas $F_s(\omega)$ is the electron function which has the following form in the case of uniform motion along the z axis:

$$F_s(\omega) = \frac{e^2 |I_s(\omega)|^2}{2N_s}, \quad I_s(\omega) = \int_0^L E_{s,z} \exp\left(-i\frac{\omega}{v}z\right) dz, \quad (8)$$

where $E_{s,z}$ is taken along the z axis ($x = y = 0$).

Since

$$\int_0^\infty R_s(\omega)\, d\omega = 1 \text{ for } Q_s \gg 1, \quad (9)$$

application of the theorem on averages to the integral of Eq. (6) gives a simple expression

$$\Delta_0 W_s = F_s(\omega), \quad (10)$$

where $\omega \to \omega_s'$ in the limit when $\omega_s'' T \to 0$; if $\omega_s'' T \ll 1$, the function $R_s(\omega)$ varies much more slowly than the function $R_s(\omega)$ [see Fig. 2]. The condition $\omega_s'' T \ll 1$ means that up to the moment when the electron leaves the resonator only a slight fraction of the energy supplied to a given oscillation is lost. Therefore, such an oscillation can be regarded as a loss-free harmonic oscillator.

The quantity $\Delta_0 W_s$ describes spontaneous radiation emitted by an electron in the absence of a field inside the resonator. Applying the same reasoning to arbitrary motion characterized by $\mathbf{r} = \mathbf{r}(t)$ in a resonator (in the time interval $0 < t < T$), we obtain the same Eqs. (6), (8), and (10), except now that the integral $I_s(\omega)$ becomes

$$I_s(\omega) = \int_0^T \mathbf{E}_s(\mathbf{r}(t))\, \dot{\mathbf{r}}(t)\, e^{-i\omega t}\, dt. \quad (11)$$

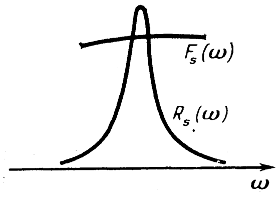

FIG. 2. Functions R_s and F_s in the limit $\omega_s'' T \ll 1$.

Therefore, if we know the electron motion due to static fields, we find that the calculation in the case of spontaneous emission reduces to quadratures provided we ignore the reaction of a spontaneously emitted field on the motion of an electron. Estimates indicate that $\Delta_0 W_s$ represents a negligible fraction of 1 eV so that the reaction of the radiation can be ignored.

If there is always a weak field inside the resonator

$$\mathbf{E} = \mathrm{Re}\,\{C_s \mathbf{E}_s e^{-i\omega t}\}, \quad (12)$$

it may change the motion of an electron and give rise to an additional exchange of energy between the electron and the field. We shall assume that in alternating fields the electron moves along the z axis (which is the usual assumption for type O devices and is justified by the presence of a strong magnetostatic field or periodic focusing). Then, assuming that $t = t_0 + z/v + \delta t$, we find that the function $\delta t(z)$ is described by the following equation[6]:

$$\frac{d^2\delta t}{dz^2} = -\frac{e}{mv^3} \mathrm{Re}\left\{C_s E_{s,z} \exp\left[-i\omega\left(t_0 + \frac{z}{v}\right)\right]\right\},$$

where the right-hand side is calculated for unperturbed uniform motion along the Z axis because of the smallness of the amplitude C_s. Subject to the initial conditions $\delta t = 0$, $d\delta t/dz = 0$ at $z = 0$, we obtain the following expression:

$$\delta t(z) = -\frac{e}{mv^3} \mathrm{Re}\left\{C_s \exp(-i\omega t_0) \int_0^z (z-z') E_{s,z'} \times \exp\left(-i\frac{\omega}{v}z'\right) dz'\right\}. \quad (13)$$

The refined value of E_z at the point z is given by Eq. (12) in which we have to allow for δt. Then, in the second approximation with respect to the small amplitude C_s, we find the average field

$$\overline{\delta E_z} = \frac{1}{2\pi} \int_0^{2\pi} \delta E_z\, d(\omega t_0),$$

$$\delta E_z = \mathrm{Re}\left\{-i\omega C_s E_{s,z} \exp\left[-i\omega\left(t_0 + \frac{z}{v}\right)\right]\right\} \delta t(z)$$

in the form

$$\overline{\delta E_z} = -\frac{e\omega}{2mv^2} |C_s|^2 \frac{\partial}{\partial\omega} \times \mathrm{Re}\left\{E_{s,z} \int_0^z E^{*}_{s,z'} \exp\left[-i\frac{\omega}{v}(z-z')\right] dz'\right\},$$

which is obtained because of phase bunching in accordance with Eq. (13). The corresponding energy increment W_s for the field of Eq. (12) can be calculated from

$$\Delta_1 W_s = -e \int_0^L \overline{\delta E_z}\, dz; \quad (14)$$

it is assumed to be equal to the work done (relative to an initial phase ωt_0) by the field on one electron in the segment $0 < z < L$, but the sign of the work has to be reversed. We thus obtain

$$\Delta_1 W_s = \frac{W_s}{mv^2}\,\omega\,\frac{dF_s}{d\omega}(\omega), \tag{15}$$

where the function $F_s(\omega)$ is given by Eq. (8). The quantity $\Delta_1 W_s$, proportional to the energy of an oscillation $W_s = \frac{1}{2}|C_s|^2 N_s$, gives the induced radiation emitted by an electron (averaged over its initial random phase ωt_0) in the first nonvanishing approximation of perturbation theory. The total increment in the energy found in this approximation is equal to the sum

$$\Delta W_s = \Delta_0 W_s + \Delta_1 W_s.$$

2. MAIN RELATIONSHIP IN CLASSICAL AND QUANTUM FORMS

The relationship (15) between stimulated and spontaneous processes of emission of radiation can be simplified bearing in mind that the function F_s depends not only on the frequency, but also on the energy of an electron $\mathscr{E} = \frac{1}{2}mv^2$. Writing down F_s in the form $F_s(\omega, \mathscr{E})$ and using Eq. (8), we can readily transform Eq. (15) to

$$\Delta_1 W_s = -W_s \frac{\partial F_s}{\partial \mathscr{E}}(\omega, \mathscr{E}). \tag{16}$$

In this form it applies also to relativistic electrons because in Eq. (15) we need to replace m with $m\gamma^3$ (longitudinal mass) and in Eq. (16) we have to assume that $\mathscr{E} = mc^2\gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$.

We shall show later that the relationship (16) applies also to electrons which cross a resonator along an arbitrary trajectory. The quantum form of Eq. (16) is

$$\Delta W_s = (n_s+1) F_s(\omega, \mathscr{E}) - n_s F_s(\omega, \mathscr{E}+\hbar\omega), \tag{17}$$

where n_s is the number of photons in the sth oscillation and $\mathscr{E}$ is the energy of an electron in static fields shaping its trajectory; $\mathscr{E} = \text{const}$ applies in the absence of alternating fields. In the limit $\hbar\omega \to 0$ the quantum relationship (17) reduces to the classical relationships (10) and (16) if we bear in mind that $W_s = n_s \hbar\omega$. Using Eq. (16) and the general expression (11), we can find the stimulated emission (or absorption, if $\partial F_s/\partial \mathscr{E} > 0$) without going into details of the interaction of an electron with the field.

In electronics Eq. (16) can be used to calculate the threshold current. If we multiply both sides of Eq. (16) by the number J_e of electrons reaching a resonator per unit time and characterized by random phases ($J_e = J/e$, where J is the current from a continuously operating electron gun), we find the power transferred by an electron beam to a given oscillation. Since the power dissipated in the casing and in the load is $2\omega_s'' W_s$, the energy attenuation coefficient of the sth oscillation in the presence of an electron beam is

$$\varkappa_s = J_e \partial F_s / \partial \mathscr{E} + 2\omega_s''.$$

Lasing is possible if $\varkappa_s < 0$ because then the oscillation grows. The condition $\varkappa_s = 0$ determines the threshold current.

A theory of excitation of oscillations in open resonators is formally the same (see Ref. 7) as the above theory for closed (cavity) resonators. Therefore, all the results can be applied to electrons in open resonators. We must simply bear in mind that the damping is governed also by the emission of radiation in the lateral directions and that the norm given by Eq. (3) is due to the field between the mirrors. In particular, the above relationships can be applied to an orotron.[8]

The classical theory predicts spontaneous emission in the constant-current approximation [Eq. (1)] and stimulated emission in the constant-field approximation [Eq. (11)]. The quantum theory, i.e., Eq. (17) describes the combined spontaneous and stimulated radiation.

3. QUANTUM THEORY FOR A ONE-DIMENSIONAL MODEL

If the trajectory is fixed, then in classical mechanics the motion of an electron along this trajectory (apart from its direction) is governed by its energy $\mathscr{E}$, as if it had one degree of freedom. In quantum mechanics the corresponding electron state is denoted by $|\mathscr{E}\rangle$. A multidimensional model in which one can allow not only for the energy but also for other parameters is discussed in Sec. 5.

We shall consider a resonator casing impermeable to an alternating current, but completely transparent to electrons. We shall discuss an interaction between an electron and an sth oscillation with n_s phonons at a moment $t = 0$. Then, if $t > 0$, the state of the electron-field system is given by the following expression (if we limit our analysis to the first approximation of perturbation theory)

$$\Psi = C|n_s\rangle|\mathscr{E}\rangle + C_+|n_s+1\rangle|\mathscr{E}-\hbar\omega\rangle + C_-|n_s-1\rangle|\mathscr{E}+\hbar\omega\rangle, \tag{18}$$

where $C = 1$ and $C_+ = C_- = 0$ at $t = 0$, and the coefficients C_+ and C_- are obtained from the equation

$$i\hbar\dot{C}_+ = (n_s+1)^{1/2}\left\langle \mathscr{E}-\hbar\omega \left| -\frac{e}{mc}\mathbf{p}_e\mathbf{A}_s^* \right| \mathscr{E} \right\rangle,$$

$$i\hbar\dot{C}_- = n_s^{1/2}\left\langle \mathscr{E}+\hbar\omega \left| -\frac{e}{mc}\mathbf{p}_e\mathbf{A}_s \right| \mathscr{E} \right\rangle \tag{19}$$

$$= n_s^{1/2}\left\langle \mathscr{E} \left| -\frac{e}{mc}\mathbf{p}_e\mathbf{A}_s^* \right| \mathscr{E}+\hbar\omega \right\rangle .$$

The matrix elements on the right-hand side of Eq. (19) are independent of time since the states on the right-hand side of Eq. (18) have the same energy. The Hamiltonian of the interaction of an electron with the wave field can be written in the form $-(e/mc)\mathbf{p}_e\mathbf{A}$, where $\mathbf{A} = a_s\mathbf{A}_s + a_s^+\mathbf{A}_s^*$ is the vector potential operator of this field,

$$\mathbf{p}_e = \frac{mc^2}{\mathscr{E}}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right), \quad \mathbf{A}_s = \frac{c}{i\omega}\left(\frac{\hbar\omega}{2N_s}\right)^{1/2}\mathbf{E}_s,$$

$\mathscr{E}$ is the energy of an electron in static (classical) fields with a vector potential $\mathbf{A}_0(\mathbf{r})$ and a scalar potential $\Phi(\mathbf{r})$, where $\mathbf{p} = -i\hbar\,\text{grad}$ is the momentum operator; in the case of nonrelativistic electrons the expression for $\mathbf{p}_e$ should include $\mathscr{E} = mc^2$.

The problem of emission of radiation by a free electron differs from the corresponding problem for a bound electron because the field now has a discrete energy spectrum and an electron has a continuous spectrum; moreover, the interaction occurs during a finite time T taken by an electron to cross the resonator.

The probability that as a result of the interaction an oscillation acquires (loses) one photon will be described by P_+ (P_-). It follows from the system (19) that

$$P_+=(n_s+1)\left(\frac{eT}{mc\hbar}\right)^2 |\langle\mathscr{E}-\hbar\omega|\mathbf{p}_e\mathbf{A}_s^*|\mathscr{E}\rangle|^2,$$

$$P_-=n_s\left(\frac{eT}{mc\hbar}\right)^2 |\langle\mathscr{E}|\mathbf{p}_e\mathbf{A}_s^*|\mathscr{E}+\hbar\omega\rangle|^2, \tag{20}$$

and, if we assume that

$$P_+=(n_s+1)F_s(\omega,\mathscr{E})/\hbar\omega,\quad P_-=n_sF_s(\omega,\mathscr{E}+\hbar\omega)/\hbar\omega, \tag{21}$$

we obtain the main relationship of Eq. (17) because the average increment is

$$\Delta W_s=(P_+-P_-)\hbar\omega. \tag{22}$$

We shall apply Eqs. (20) and (21) to a cylindrical resonator with cross-section area S and length L along the z axis, and we shall assume that

$$|\mathscr{E}\rangle=(LS)^{-1/2}\exp[ik(\mathscr{E})z],\quad k(\mathscr{E})=(2m\mathscr{E})^{1/2}/\hbar. \tag{23}$$

Then, at each moment there is inside the resonator a charge e, and during an interval $T=L/v$ (v is the electron velocity $1/v=\hbar dk/d\mathscr{E}$) the same charge e crosses each $z=\text{const}$ cross section. The function F_s is described by the first expression in Eq. (8) where

$$I_s(\omega)=\frac{k(\mathscr{E})+k(\mathscr{E}-\hbar\omega)}{2k(\mathscr{E})S}$$

$$\times\int E_{s,z}\exp\{-i[k(\mathscr{E})-k(\mathscr{E}-\hbar\omega)]z\}dV$$

$$\to\frac{1}{S}\int E_{s,z}\exp\left(-i\frac{\omega}{v}z\right)dV\quad\text{for}\quad\hbar\omega\to0. \tag{24}$$

We obtain the classical expression (8) by assuming $|\mathscr{E}\rangle$ to be not the plane wave of Eq. (23), but a thin wave beam localized near the z axis within the resonator. This can be done because of the smallness of the de Broglie wavelength $2\pi/k(\mathscr{E})$. In the case of relativistic electrons we can use Eq. (24) if we assume that $k(\mathscr{E})=(1/\hbar)(\mathscr{E}^2/c^2-m^2c^2)^{1/2}$.

In addition to the relationship between the stimulated and spontaneous radiations, Madey[2] deduced the relationship (see also Ref. 5)

$$\overline{\mathscr{E}}_T-\mathscr{E}=\frac{1}{2}\frac{\partial}{\partial\mathscr{E}}\overline{(\mathscr{E}_T-\mathscr{E})^2}, \tag{25}$$

between the change in the average energy of an electron and the variance of its energy. Here, $\mathscr{E}_T$ is the final value of the electron energy and the superior bar represents averaging over ωt_0 (see Sec. 1).

The relationship (25) is readily deduced from the expressions in Eq. (21). We shall use $P=1-P_+-P_-$ for the probability that the number of photons does not change. Then,

$$\overline{\mathscr{E}}_T=\mathscr{E}P+(\mathscr{E}-\hbar\omega)P_++(\mathscr{E}+\hbar\omega)P_-$$

$$=\mathscr{E}-(P_+-P_-)\hbar\omega=\mathscr{E}-\Delta W_s,$$

which gives the energy conservation law

$$\mathscr{E}-\overline{\mathscr{E}}_T=\Delta W_s. \tag{26}$$

Since

$$\overline{\mathscr{E}_T^2}=\mathscr{E}^2P+(\mathscr{E}-\hbar\omega)^2P_++(\mathscr{E}+\hbar\omega)^2P_-=\mathscr{E}^2+2\mathscr{E}(\overline{\mathscr{E}}_T-\mathscr{E})$$
$$+[(n_s+1)F_s(\omega,\mathscr{E})+n_sF_s(\omega,\mathscr{E}+\hbar\omega)]\hbar\omega,$$

it follows that

$$\overline{(\mathscr{E}_T-\mathscr{E})^2}=[(n_s+1)F_s(\omega,\mathscr{E})+n_sF_s(\omega,\mathscr{E}+\hbar\omega)]\hbar\omega \tag{27}$$

or, in the classical limit

$$\overline{(\mathscr{E}_T-\mathscr{E})^2}=2W_sF_s(\omega,\mathscr{E}). \tag{28}$$

Differentiating the above expression with respect to $\mathscr{E}$ and using Eqs. (16) and (26), we obtain Eq. (25).

All these relationships are naturally valid in the first approximation of perturbation theory when the probabilities P_+ and P_- are small and we have $P\approx1$.

4. UNDULATOR RADIATION IN AN OPEN RESONATOR AND OTHER TOPICS

We shall consider an open resonator formed by two mirrors separated by a distance $\tilde{L}$ from one another. The field of a natural oscillation near the z axis will be taken in the form of a circularly polarized plane wave:

$$E_{s,x}=E_0\exp\left(i\frac{\omega}{c}z\right)+\dots,$$

$$E_{s,y}=iE_0\exp\left(i\frac{\omega}{c}z\right)+\dots\quad(E_0>0), \tag{29}$$

where the three dots denote the opposite wave which hardly interacts with electrons but does influence the norm $N_s=\pi^{-1}E_0^2\tilde{L}S$, where S is the effective transverse cross section of a Gaussian beam wave in the interaction space $0<z<L$ (Fig. 3); in this segment the beam is assumed to be homogeneous along the z axis.

An electron moves in an undulator where the electric field near the z axis is

$$H_x=H_0\cos h_0z,\quad H_y=H_0\sin h_0z\quad(h_0=2\pi/l) \tag{30}$$

along a helix

$$x(t)=\frac{v_\perp}{vh_0}\sin(h_0vt),$$

$$y(t)=-\frac{v_\perp}{vh_0}\cos(h_0vt),\quad z(t)=vt$$

(see, for example, Ref. 9), where

$$v_\perp=c\frac{K}{\gamma},\quad v=c\left(1-\frac{1+K^2}{\gamma^2}\right)^{1/2},\quad K=\frac{eH_0}{h_0mc^2},$$

and $mc^2\gamma$ is the energy of an electron. Apart from an important phase factor, we find from Eq. (11) that

$$I_s=\frac{v_\perp}{v}E_0L\frac{\sin\Phi/2}{\Phi/2},$$

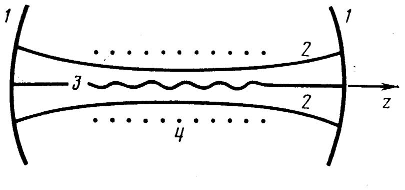

FIG. 3. Electron in an open resonator: 1) mirrors; 2) wave-beam caustic; 3) electron trajectory; 4) undulator.

$$\Phi=\left(h_0+\frac{\omega}{c}-\frac{\omega}{v}\right)L,$$
$$F_s(\omega,\mathscr{E})= \qquad (31)$$
$$=\pi\mathscr{N}\frac{L}{\tilde{L}}\frac{K^2}{1+K^2}\frac{e^2\lambda}{S}\left(\frac{\sin\Phi/2}{\Phi/2}\right)^2,$$

where $\lambda = 2\pi c/\omega$ is the wavelength corresponding to the frequency ω; $\mathscr{N} = L/l$ is the number of periods of the magnetic field (30); it is assumed that $v = c$ everywhere where this is possible (it is assumed that $\gamma \gg 1$) and a term in the expression for I_s characterized by a large phase $\Phi' = (\omega/c + \omega/v - h_0)L$ is dropped. The derivative

$$\frac{\partial F_s}{\partial \mathscr{E}}=4\pi\mathscr{N}^2\frac{L}{\tilde{L}}\frac{K^2}{1+K^2}\frac{e^2\lambda}{S\mathscr{E}}\frac{d}{d\Phi}\left(\frac{\sin\Phi/2}{\Phi/2}\right)^2 \quad (\mathscr{E}=mc^2\gamma), \qquad (32)$$

where the relationship $\partial\Phi/\partial\mathscr{E} = 4\pi\mathscr{N}/\mathscr{E}$ is included, determines the relative reduction in the energy of an oscillation because of its interaction with one electron. If $\tilde{L} = L$, Eq. (32) agrees with Eq. (36) derived in Ref. 9; this last expression is obtained directly, i.e., like the expression for $\Delta_1 W_s$ in Sec. 1.

It should be stressed that in these and similar problems it is not the emission of radiation in a free state which is important, but the radiation attuned to the natural oscillations of the resonator. In the cited investigations[1–4] the presence of a resonator is practically ignored, so that in Ref. 1 an indirect expression is obtained for the gain (which is related simply to the threshold current, see Sec. 2).

In many cases the field of a given oscillation differs radically from the plane-wave field. For example, in the case of an orotron a plane-parallel electron beam passes above a comb located on a plane mirror of an open resonator.[8] The electric field of a natural oscillation has a component

$$E_{s,z}=E_0\exp(-h_0y+ih_0z)+\ldots, \qquad h_0=2\pi/l,$$

where $y = 0$ is the plane of the comb and l is the period of the comb ($l \ll \lambda$); the three dots denote non-phase-matched fields which hardly interact with electrons; the norm for this oscillation will be written in the form $N_s = (1/4\pi)E_0^2 V_0$, where V_0 is a volume which is of the same order as the volume occupied by the field. If we assume that electrons occupy a volume $0 < x < x_0$, $0 < y < y_0$, and $0 < z < L$ in the interaction space, and if we use $f(y)$ for the distribution of a transient (alternating) current along the y axis, then instead of Eq. (24) we obtain an expression

$$I_s=\chi\frac{\sin\Phi/2}{\Phi/2}e^{-i\Phi/2}, \qquad \chi=\int_0^{y_0} f(y)e^{-h_0y}\,dy,$$

$$\Phi=\left(h_0-\frac{\omega}{v}\right)L,$$

and hence

$$F_s(\omega,\mathscr{E})=2\pi\psi\mathscr{N}\frac{e^2l}{S_0}\left(\frac{\sin\Phi/2}{\Phi/2}\right)^2, \qquad \psi=\frac{LS_0}{V_0}|\chi|^2 \qquad (31')$$

and

$$\frac{\partial F_s}{\partial\mathscr{E}}=\pi^2\psi\mathscr{N}^2\frac{e^2l}{S_0\mathscr{E}}\frac{d}{d\Phi}\left(\frac{\sin\Phi/2}{\Phi/2}\right)^2 \quad \left(\mathscr{E}=\frac{1}{2}mv^2\right), \qquad (32')$$

where $S_0 = x_0 y_0$ is the transverse cross-sectional area of the electron beam and $\mathscr{N} = L/l$ is the number of periods; in the case of an orotron, we have $\partial\Phi/\partial\mathscr{E} = \pi\mathscr{N}/\mathscr{E}$.

The threshold current calculated using Eq. (32') agrees with the result of Rusin[8] obtained ignoring the space charge. However, it is clear that there is a close analogy between undulator and orotron radiations, i.e., between Eqs. (31) and (31'), and (32) and (32'); the analogy is close (see also Ref. 9).

We shall give also without derivation the solutions of two problems obtained by a classical method. If an electron moves along a circle in a homogeneous magnetostatic field H_0 at an angular frequency $\Omega = eH_0/mc\gamma$ and if the alternating field in the resonator is homogeneous within this circle, the stimulated radiation of an electron which leaves the resonator after a time is related to a function $F_s(\omega,\mathscr{E})$ governing, in accordance with Eq. (10), the emission of spontaneous radiation

$$\Delta_1 W_s=-\frac{\Omega}{\omega}W_s\frac{\partial F_s}{\partial\mathscr{E}}. \qquad (33)$$

If a nonrelativistic electron oscillates in a potential well at a fundamental frequency ω_0 and it interacts with a homogeneous alternating field inside the resonator, then

$$\Delta_1 W_s=-W_s\omega_0\frac{\partial}{\partial\mathscr{E}}\left(\frac{F_s}{\omega_0}\right). \qquad (34)$$

Here, the frequency ω_0 depends generally on the energy $\mathscr{E}$; ω_0 is independent of $\mathscr{E}$ only for a parabolic well when all the higher harmonics $2\omega_0$, $3\omega_0$, ... are absent from the motion.

The two relationships (33) and (34) differ from the main relationship of Eq. (16). From the practical point of view, the difference is slight: the relationship (33) is usually applied at $\Omega \approx \omega$ and the right-hand side of Eq. (33) has an additional term proportional to $d\omega_0/d\mathscr{E}$, but small compared with the main term. However, this difference is of fundamental importance: the motion is finite so that quantization leads to a discrete energy spectrum in which an electron cannot be regarded as free in accordance with the above definition. For nonrelativistic motion in a magnetic field and for harmonic oscillations in a parabolic well the energy levels are described by the same expressions

$$\mathscr{E}_n=(n+{}^1/_2)\hbar\Omega \text{ and } \mathscr{E}_n=(n+{}^1/_2)\hbar\omega, \qquad n=0,\ 1,\ 2,\ \ldots. \qquad (35)$$

In more general cases the spectrum is nonequidistant but still discrete.

Therefore, the classical theory "senses" that the energy spectrum is discrete, which leads to the modified relationships (33) and (34). Gaponov[10] demonstrated that the classical theory is also unsuitable in the case of equidistant spectra of Eq. (35) used in generation of stimulated radiation. In fact, if Ω and ω_0 are independent of $\mathscr{E}$, then Eqs. (33) and (34) always yield $\Delta_1 W_s < 0$, because $\partial F_s/\partial\mathscr{E} > 0$.

5. QUANTUM THEORY FOR A MULTIDIMENSIONAL MODEL

We shall use $|\mathscr{E}, a\rangle$ to denote the state of an electron in a multidimensional model; here, a is a set of parameters which can vary under the influence of wave fields; $\int\ldots da$ is understood to be integration (or summation) over all possible values of these parameters.

The examples of continuous parameters of a are as follows: in the absence of static fields these parameters are the distance of a rectilinear trajectory from the z axis or the angle it forms with this axis. In type O devices a longitudinal magnetic field or some other focusing system induces translational motion and finite transverse motion, the energy levels of which are labeled by the parameters a (see end of Sec. 4); in the absence of a resonance, the transverse motion is passive. In type M devices (with crossed fields) we understand a to be the quantum energy of orbital motion superimposed on the drift and not attaining a resonance with an alternating field.

In the multidimensional model Eq. (18) should be replaced with

$$\Psi = C|n_s\rangle|\mathscr{E}, a\rangle + |n_s+1\rangle \int C_+(b,a)\,|\mathscr{E}-\hbar\omega, b\rangle db + |n_s-1\rangle \int C_-(b,a)\,|\mathscr{E}+\hbar\omega, b\rangle db, \quad (36)$$

where $C = 1$, $C_+ = 0$ at $t = 0$, whereas for $t = T$, we have

$$C_+(b,a) = (n_s+1)^{1/2}\frac{ieT}{mc\hbar}\langle\mathscr{E}-\hbar\omega, b|\mathbf{p}_e\mathbf{A}_s^*|\mathscr{E}, a\rangle,$$

$$C_-(b,a) = n_s^{1/2}\frac{ieT}{mc\hbar}\langle\mathscr{E}+\hbar\omega, b|\mathbf{p}_e\mathbf{A}_s|\mathscr{E}, a\rangle.$$

Hence the probabilities (probability densities)

$$P_+(|\mathscr{E}, a\rangle \to |\mathscr{E}-\hbar\omega, b\rangle) = (n_s+1)\left(\frac{eT}{mc\hbar}\right)^2 |\langle\mathscr{E}-\hbar\omega, b|\mathbf{p}_e\mathbf{A}_s^*|\mathscr{E}, a\rangle|^2,$$

$$P_-(|\mathscr{E}, a\rangle \to |\mathscr{E}+\hbar\omega, b\rangle) = n_s\left(\frac{eT}{mc\hbar}\right)^2 |\langle\mathscr{E}, a|\mathbf{p}_e\mathbf{A}_s^*|\mathscr{E}+\hbar\omega, b\rangle|^2, \quad (37)$$

describe transitions accompanied by the addition (+) or removal (−) of a photon and satisfying the relationship

$$\frac{P_-(|\mathscr{E}, a\rangle \to |\mathscr{E}+\hbar\omega, b\rangle)}{n_s} = \frac{P_+(|\mathscr{E}+\hbar\omega, b\rangle \to |\mathscr{E}, a\rangle)}{n_s+1}. \quad (38)$$

Summation (integration) of all the final parameters b and averaging over all the possible initial parameters a shows that such double averaging gives the probabilities P_+ and P_- dependent only on ω and $\mathscr{E}$ and satisfying Eq. (21). In this way we obtain the quantum expression (17) and the classical expression (16) in which the quantities ΔW_s and $\Delta_1 W_s$ are subjected to additional averaging. If initially an oscillation in a resonator is not in a state $|n_s\rangle$ with a specific number of photons, but in some other state, then after a further averaging the value of n_s changes to $\bar{n}_s$, which is the average number of photons.

Similar averaging is used in the theory of radiation emitted by bound electrons, which have a discrete energy spectrum (see, for example, Ref. 11), and the relationship (38) is similar to that between the coefficients A and B obtained by Einstein[12] in 1916 and justified by Dirac,[13] which represents the foundation of quantum electrodynamics.

CONCLUSIONS

In spite of the simplicity of our results, they are clearly novel. The difference between this investigation and the earlier ones is that we are not dealing with radiation emitted in free space but with radiation in a resonator (closed or open) with any structure of the field. This difference has important consequences, particularly negation of all the laws of conservation in the electron + field system, apart from the energy conservation law. Our quantum treatment of the oscillations in a high-Q resonator has made it possible to treat from the same point of view the coupling between spontaneous and stimulated radiations in various systems, going over to the classical limit ($\hbar\omega \to 0$). There is no classical way of deriving the corresponding general expression, as demonstrated by the last two examples in Sec. 4.

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