Fluctuation kinetics in superconductors at frequencies low compared with the energy gap

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A kinetic theory is developed for fluctuations in superconductors (including dirty ones) for frequencies that are low compared with the energy gap and for sufficiently smooth changes of the averaged and fluctuating quantities in space. The correlation functions are obtained for the extraneous Langevin fluxes in the Boltzmann equation for the fluctuations of the quasiparticle distribution function that depends on the energy variable ε and on the direction $\mathbf{p}/p$ of the quasimomentum. The spectral density of the voltage fluctuations and the frequency-dependent impedance of an S-c-S Josephson junction are calculated for currents lower than critical. The spectral density of the voltage fluctuations of this junction is obtained in the Josephson-generation regime at low voltages $U \ll \Delta/e$ corresponding to the plateau on the current-voltage characteristic.

1. INTRODUCTION

Spectral densities of fluctuations (SDF) of physical parameters (e.g., current or voltage) of systems in thermal equilibrium are expressed with the aid of the fluctuation-dissipation relations in terms of the response of the system to external perturbations, and the calculation of the SDF reduces to a calculation of the response. In nonequilibrium physical systems, on the other hand, calculation and measurement of the SDF is an isolated problem. Nonequilibrium states are realized in superconductors under various experimental conditions, viz., irradiation by a microwave field, quasiparticle injection, Josephson generation in S-c-S junctions (*c*—constriction), vortex motion, and others (see the survey anthology, Ref. 1). It is important that disequilibrium leads in superconductors to new effects (different from those in normal metals). It is therefore of interest to develop for fluctuations in superconductors a kinetic theory applicable to both equilibrium and nonequilibrium states.

An important application of such a theory can be the calculation of the SDF of the voltage at the end points of a narrow bridge between two superconductors, since these fluctuations determine the Josephson-generation linewidth.[2] When generation occurs in such a junctions, the particles acquire a nonequilibrium distribution,[3] and therefore calculation of the noise calls for knowledge of the correlation functions of the fluctuation sources in the nonequilibrium superconductors.

In pure superconductors (electron-collision frequency $\tau_p \ll \Delta/\hbar$) at low frequencies $\omega \ll \Delta/\hbar$ and in the case of continuous spatial inhomogeneities (inverse scale $q \ll \Delta/\hbar v$, where v is the Fermi velocity), one can describe the superconductor kinetics with the aid of the Boltzmann equation for the quasiparticle distribution function $f(\mathbf{p},\mathbf{r},t)$ ($\mathbf{p}$ is the quasimomentum).[4,5] An equilibrium-fluctuation theory for this case was developed in Ref. 6. In many cases of practical importance, however, for example in a Josephson junction, it is impossible to meet all three indicated conditions, and the kinetics cannot be described on the basis of the Boltzmann equation for $f(\mathbf{p},\mathbf{r},t)$. In a more general case, when the only restriction is that the inhomogeneity scale be large compared with the Fermi wavelength, the kinetics of the superconductor is described by a system of equations for the electron Green's functions (retarded, advanced, Keldysh's) integrated over $p^2/2m - \varepsilon_F$ (ε_F is the Fermi energy).[7,8] These equations can be used also for dirty superconductors, and also for any ratio $\hbar\omega/\Delta$. In those cases when $\hbar\omega \ll \Delta$, one of the Green's functions (Keldysh's) can be expressed in terms of the quasiparticle distribution function $f(\varepsilon,\mathbf{n},\mathbf{r},t)$, where ε is the energy variable and $\mathbf{n} = \mathbf{p}/p$ is a unit vector in the quasimomentum direction. This allows us to reduce the equations for the three Green's functions to a simpler form consisting of a kinetic equation for $f(\varepsilon,\mathbf{n},\mathbf{r},t)$ and the equations for the retarded $R(\varepsilon,\mathbf{n},\mathbf{r},t)$ and advanced $A(\varepsilon,\mathbf{n},\mathbf{r},t)$ Green's functions[8,9] (see also the review in Ref. 10). Even this system, however, is still difficult to solve in the case of strong spatial inhomogeneity Δ and of a phase order parameter χ, when R and A vary rapidly in space. This system is therefore usually solved for conditions that permit the use of a local approximation for R and A. The condition for its applicability in the pure limit is $L \gg \hbar v/\Delta$ (L is the characteristic scale of the inhomogeneity), while in the dirty limit, when the quasiparticle motion is diffuse, the limit should be $L \gg \eta = (\hbar D/\Delta)^{1/2}$, where D is the quasiparticle diffusion coefficient. It is important in the latter case that at temperatures T close to T_c the length is $\eta < \xi(T)$.[3,8]

The purpose of the present paper is the development of a kinetic theory for low-frequency and smooth fluctuations in a superconductor with not too strong a dependence of Δ and χ on the coordinates, using as the basis the kinetic equations for the quasiparticle-distribution fluctuations $\delta f(\varepsilon,\mathbf{n},\mathbf{r},t)$. The derivation of the expression for the correlation function of Langevine extraneous flows in the equation for $\delta f(\varepsilon,\mathbf{n},\mathbf{r},t)$ is based on the fact that separate transitions between states with different ε, $\mathbf{n}$, and $\mathbf{r}$ in scattering by impurities and phonons are not correlated. This property of the quasiparticle scattering processes is manifested in the fact that the collision integral in the kinetic equation consists of fluxes of quasiparticles between different states. The absence of correlation between the individual quasiparticle scattering processes was used earlier to develop a kinetic theory of nonequilibrium fluctuations in normal conductors.[11,12]

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In Sec. 2 we formulate a system of equations for the fluctuations $\delta f(\varepsilon\mathbf{n}\mathbf{r}t)$ of the distribution function, $\delta R(\varepsilon\mathbf{n}\mathbf{r}t)$ and $\delta A(\varepsilon\mathbf{n}\mathbf{r}t)$ of the retarded and advanced Green's functions, $\delta\Delta(\mathbf{r}t)$ of the energy gap, $\delta\mathbf{j}(\mathbf{r}t)$ of the current density, $\delta\mathbf{p}_s(\mathbf{r}t)$ of the gauge-invariant momentum, and $\delta\Phi(\mathbf{r}t)$ of the potential. We obtain also expressions for the correlation functions of the extraneous sources in the kinetic equation for δf. In Sec. 3 we obtain an expression for the impedance of an equilibrium long Josephson junction at frequencies $\omega \ll \Delta/\hbar$ when a superconducting current less than critical flows through the junction. We then obtain with the aid of the equations of Sec. 2 an expression for the SDF of the voltage under the same conditions and show that this SDF agrees with the Nyquist formula. In Sec. 4 is obtained the high-frequency SDF of the voltage in the same junction for currents higher than critical, i.e., under conditions of Josephson generation.

2. EQUATIONS FOR THE FLUCTUATIONS

The equation for the quasiparticle distribution function is derived from the equations for the Green's functions in Ref. 8 (see also Ref. 10). A somewhat different manner of expressing the Keldysh function in terms of the distribution function and of introducing $f(\varepsilon\mathbf{n}\mathbf{r}t)$ was proposed in Ref. 9. This method is more convenient for our purposes and we shall follow it (as well as the notation of Ref. 9). Let R_i and A_i be the components of the expansions of the matrix functions R and A in Pauli matrices $R = R_i\tau_i = \mathbf{R}\boldsymbol{\tau}$), $a_i = (R_i - A_i)/2$, $b_i = (R_i + A_i)/2$, $a^2 = a_i a_i = \mathbf{a}\mathbf{a}$, $p_s = \hbar/\nabla\chi/2 - e\mathbf{A}/c$ and $\Phi = 1/2\hbar\partial\chi/\partial t + e\varphi$ are the gauge-invariant momentum and potential, $\mathbf{A}$ and φ are vector and scalar potentials. We write the kinetic equation for the distribution function $f(\varepsilon\mathbf{n}\mathbf{r}t)$ in the form

$$a_z\frac{\partial f}{\partial t}+a^2 v\mathbf{n}\nabla f+(a_z v\mathbf{n}\dot{\mathbf{p}}_s+ia_y\dot{\Delta}+a^2\dot{\Phi})\frac{\partial f}{\partial \varepsilon}+I(\varepsilon\mathbf{n}\mathbf{r}t)=0. \tag{1}$$

Compared with Ref. 9, we have omitted here the quantities $[\mathbf{a}\times\mathbf{b}]_{y,z}$ in the coefficients of the time derivatives, since they are small when the local approximation is applicable. Under the same conditions, the expression obtained in Ref. 9 for the collision integral $I(\varepsilon\mathbf{n}\mathbf{r}t)$ can be written in the form

$$I(\varepsilon\mathbf{n}\mathbf{r}t)=\int d\varepsilon'\,d\mathbf{n}'\{J_1(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')-J_1(\varepsilon'\mathbf{n}';\varepsilon\mathbf{n})+J_2'(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')-J_2''(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')\}. \tag{2}$$

Here $\int dn'... = \int d\Omega'/4\pi$ denotes integration over the angles that determine the direction of $\mathbf{n}'$; J_1, J_2', and J_2'' are quasiparticle fluxes equal to

$$J_1(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')=K_1(\varepsilon\mathbf{n},\varepsilon'\mathbf{n}')f(\varepsilon\mathbf{n})[1-f(\varepsilon'\mathbf{n}')]\times\left\{\frac{1}{\tau}\delta(\varepsilon-\varepsilon')+\delta(\mathbf{n}-\mathbf{n}')\frac{\pi\lambda\hbar^2}{2(sp_F)^2}\times\int_0^\infty d\omega\,\omega^2[(N_\omega+1)\delta(\varepsilon-\varepsilon'-\hbar\omega)+N_\omega\delta(\varepsilon-\varepsilon'+\hbar\omega)]\right\}, \tag{3a}$$

$$J_2'(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')=K_2(\varepsilon\mathbf{n},\varepsilon'\mathbf{n}')f(\varepsilon\mathbf{n})f(\varepsilon'\mathbf{n}')\times\left\{\frac{1}{\tau}\delta(\varepsilon+\varepsilon')+\delta(\mathbf{n}+\mathbf{n}')\frac{\pi\lambda\hbar^2}{2(sp_F)^2}\times\int_0^\infty d\omega\,\omega^2[(N_\omega+1)\delta(\varepsilon+\varepsilon'-\hbar\omega)+N_\omega\delta(\varepsilon+\varepsilon'+\hbar\omega)]\right\}, \tag{3b}$$

$$J_2''(\varepsilon\mathbf{n};\varepsilon'\mathbf{n}')=K_2(\varepsilon\mathbf{n},\varepsilon'\mathbf{n}')[1-f(\varepsilon\mathbf{n})][1-f(\varepsilon'\mathbf{n}')]\times\left\{\frac{1}{\tau}\delta(\varepsilon+\varepsilon')+\delta(\mathbf{n}+\mathbf{n}')\frac{\pi\lambda\hbar^2}{2(sp_F)^2}\int_0^\infty d\omega\,\omega^2[(N_\omega+1)\delta(\varepsilon+\varepsilon'+\hbar\omega)+N_\omega\delta(\varepsilon+\varepsilon'-\hbar\omega)]\right\}. \tag{3c}$$

Here λ is the electron-phonon interaction constant, s is the speed of sound, τ is the time of momentum relaxation due to scattering by impurities and is assumed isotropic, N_ω is the number of phonons in one mode with energy $\hbar\omega$, and K_1 and K_2 are coherence factors equal to

$$K_{1,2}(\varepsilon\mathbf{n},\varepsilon'\mathbf{n}')=\{\mathbf{a}(\varepsilon\mathbf{n})\mathbf{a}(\pm\varepsilon',\pm\mathbf{n}')\pm a^2(\varepsilon\mathbf{n})a^2(\varepsilon'\mathbf{n}')\}/2. \tag{4}$$

In (4), just as in (1), the terms with vector products are omitted.

When interpreting the quasiparticle fluxes in the collision integral, it must be borne in mind that at $\varepsilon > 0$ the function $f(\varepsilon\mathbf{n})$ is a quasielectron distribution function, and at $\varepsilon < 0$ the function $1 - f(\varepsilon\mathbf{n})$ is the hole distribution function. At equilibrium f is the usual Fermi function $f_0 = (e^{\varepsilon/T} + 1)^{-1}$.

In the Langevin theory of fluctuations of quasiparticle occupation states (see Ref. 11) the left-hand side of the equation for the fluctuations δf (Boltzmann-Langevin equations) is obtained by linearizing the kinetic equation (1), while the source of the fluctuations (of the right-hand side of the equation) of the extraneous random flux $\delta J(\varepsilon\mathbf{n}\mathbf{r}t)$ due to the random character of the quasiparticle scattering:

$$\delta\left[a_z\frac{\partial f}{\partial t}+a^2 v\,\mathbf{n}\nabla f+(a_z v\mathbf{n}\dot{\mathbf{p}}_s+ia_y\dot{\Delta}+a^2\dot{\Phi})\frac{\partial f}{\partial\varepsilon}+I(\varepsilon\mathbf{n})\right]=\delta J(\varepsilon\mathbf{n}\mathbf{r}t). \tag{5}$$

Here $\delta[\ldots]$ denotes linearization of the expression in the brackets.

The form of the correlation function of the extraneous fluxes is determined by the fact that each elementary quasiparticle scattering process is correlated only with itself.[11] In addition, accurate to the collision time, the correlation function of the extraneous fluxes, taken at the instants t_1 and t_2, is proportional to $\delta(t_1 - t_2)$ and, accurate to distances shorter than the characteristic inhomogeneity scales (see above), it is proportional to $\delta(\mathbf{r}_1-\mathbf{r}_2)$. Let us find this correlation function.

By virtue of the Poisson character of the quasiparticle scattering processes, the contribution of each type of scattering to the correlation function $\langle \delta J(\varepsilon_1\mathbf{n}_1\mathbf{r}_1t_1)\delta J(\varepsilon_2\mathbf{n}_2\mathbf{r}_2t_2)\rangle$ of the extraneous scattering is proportional to the corresponding average flux, just as in shot noise. For $\varepsilon_1=\varepsilon_2$ and $\mathbf{n}_1=\mathbf{n}_2$ the fluctuations $\delta J(\varepsilon_1\mathbf{n}_1)$ and $\delta J(\varepsilon_2\mathbf{n}_2)$ are obviously of the same sign, so that the corresponding contribution to the correlation function is positive and is proportional to the sum of all the average fluxes due to the appearance of a quasiparticle in the state $\varepsilon_1\mathbf{n}_1$ or to its departure from this state.

A contribution to the flux $J_1(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)$ with $\varepsilon_1\mathbf{n}_1\neq\varepsilon_2\mathbf{n}_2$ is made by quasiparticle transitions within one mode (ε_1 and ε_2 of the same sign), annihilation of quasielectron-quasihole states ($\varepsilon_1>0,\ \varepsilon_2<0$), production of quasielectron-quasihole states ($\varepsilon_1<0,\ \varepsilon_2>0$). All have a common property: the corresponding fluxes into the states $\varepsilon_1\mathbf{n}_1$ and $\varepsilon_2\mathbf{n}_2$ are of opposite sign, so that these processes make a negative contribution to $\langle \delta J(\varepsilon_1\mathbf{n}_1)\delta J(\varepsilon_2\mathbf{n}_2)\rangle$, proportional to $J_1(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)$. Contributions to $J_2'(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)$ are made by transitions between branches (ε_1 and ε_2 of opposite sign), annihilation of quasielectron pairs ($\varepsilon_1>0,\ \varepsilon_2>0$) and creation of quasihole pairs ($\varepsilon_1<0,\ \varepsilon_2<0$). In each such process the changes of the occupation of the states $\varepsilon_1\mathbf{n}_1$ and $\varepsilon_2\mathbf{n}_2$ are of the same sign, therefore the contributions of these processes to the correlation function of the extraneous fluxes are positive and proportional to the average flux $J_2'(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)$. The same holds for processes corresponding to $J_2''(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)$. Thus, the correlation function of the extraneous quasiparticle fluxes in a superconductor is equal to

$$\begin{aligned}\langle \delta J(\varepsilon_1\mathbf{n}_1\mathbf{r}_1t_1)\delta J(\varepsilon_2\mathbf{n}_2\mathbf{r}_2t_2)\rangle &=\frac{1}{N_F}\delta(t_1-t_2)\delta(\mathbf{r}_1-\mathbf{r}_2)\\ &\times\Big\{\delta(\mathbf{n}_1-\mathbf{n}_2)\delta(\varepsilon_1-\varepsilon_2)\\ &\times\int d\varepsilon'\,d\mathbf{n}'[J_1(\varepsilon_1\mathbf{n}_1;\ \varepsilon'\mathbf{n}')+J_1(\varepsilon'\mathbf{n}';\ \varepsilon_1\mathbf{n}_1)\\ &+J_2'(\varepsilon_1\mathbf{n}_1;\ \varepsilon'\mathbf{n}')+J_2''(\varepsilon_1\mathbf{n}_1;\ \varepsilon'\mathbf{n}')]\\ &-J_1(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)-J_1(\varepsilon_2\mathbf{n}_2;\ \varepsilon_1\mathbf{n}_1)\\ &+J_2'(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)+J_2''(\varepsilon_1\mathbf{n}_1;\ \varepsilon_2\mathbf{n}_2)\Big\}.\end{aligned} \quad (6)$$

Here N_F is the density of states on the Fermi surface. Note that when the signs of both their arguments are reversed, the quantities $a_i(\varepsilon\mathbf{n})$ either do not change at all (a_z), or change only in sign (a_y), therefore the factor K_2 given by (4), and the right-hand side of (6) as a whole, are symmetric with respect to the permutation $\varepsilon_1\mathbf{n}_1\leftrightarrow\varepsilon_2\mathbf{n}_2$.

If the superconductor is in a state of thermodynamic equilibrium, the correlation function of the extraneous fluxes can be represented in the form

$$\begin{aligned}\langle \delta J(\varepsilon_1\mathbf{n}_1\mathbf{r}_1t_1)\delta J(\varepsilon_2\mathbf{n}_2\mathbf{r}_2t_2)\rangle &=\frac{1}{N_F}\delta(t_1-t_2)\delta(\mathbf{r}_1-\mathbf{r}_2)\int d\varepsilon'\,d\mathbf{n}'\{\tilde{I}(\varepsilon_1\mathbf{n}_1;\ \varepsilon'\mathbf{n}')\\ &\times\delta(\varepsilon'-\varepsilon_2)\delta(\mathbf{n}'-\mathbf{n}_2)+\tilde{I}(\varepsilon_2\mathbf{n}_2;\ \varepsilon'\mathbf{n}')\delta(\varepsilon'-\varepsilon_1)\delta(\mathbf{n}'-\mathbf{n}_1)\}\\ &\times f_0(\varepsilon')[1-f_0(\varepsilon')].\end{aligned} \quad (7)$$

Here $\tilde{I}(\varepsilon\mathbf{n};\ \varepsilon'\mathbf{n}')$ is the operator of the linearized collision integral (2). Relations of the same form as (7) hold also in the case of normal conductors[12] and pure superconductors.[6]

The equations for the fluctuations δR and δA are obtained by linearizing the corresponding equations for R and A.[8,9] In the low-frequency limit the equation for δR takes the form (the equation for δA is similar)

$$\begin{aligned}&iv\mathbf{n}\nabla\delta R+\frac{1}{\hbar}[(\varepsilon\tau_z-v\mathbf{n}\mathbf{p}_s\tau_z-i\tau_y\Delta-\Phi),\ \delta R]-\frac{1}{2i\tau}[\overline{R},\delta R]\\ &+\frac{1}{2i\tau}[R,\overline{\delta R}]+i[\hat{\gamma}^R,\delta R]=\frac{1}{\hbar}[v\mathbf{n}\delta\mathbf{p}_s\tau_z+i\tau_y\delta\Delta+\delta\Phi,R].\end{aligned} \quad (8)$$

The square brackets denote here as usual a commutator, a superior bar denotes averaging over the angles $\mathbf{n}$, and the matrix $\hat{\gamma}^R$ describes the damping due to energy scattering by phonons.

It is necessary to add to (5) and (8) the neutrality condition

$$\delta\Phi(\mathbf{r}t)=\int_{-\infty}^{+\infty}d\varepsilon\int\frac{d\Omega}{4\pi}\left\{a^2\delta f+\left(f-\frac{1}{2}\right)\delta a^2\right\}, \quad (9)$$

the linearized self-consistency condition for the energy gap

$$\delta\Delta(\mathbf{r}t)=\lambda\int_{-\infty}^{+\infty}d\varepsilon\int\frac{d\Omega}{4\pi i}(a_y\delta f+f\delta a_y), \quad (10)$$

and an expression for the current-density fluctuation

$$\delta\mathbf{j}(\mathbf{r}t)=eN_Fv\int_{-\infty}^{+\infty}d\varepsilon\int\frac{d\Omega}{4\pi}\mathbf{n}(a_z\delta f+f\delta a_z). \quad (11)$$

Equations (5) and (8–11), together with the correlation functions (6), make up a complete system for the determination of all the needed correlation functions in the limit of low frequencies and in the local approximation.

In the case of dirty superconductors, all the quantities that depend on $\mathbf{n}$ can be represented as a sum of an isotropic part independent of $\mathbf{n}$ and a part proportional to $\mathbf{n}$.[8] In particular, the fluctuation of the distribution function and the fluctuational extraneous flux are equal to

$$\delta f(\varepsilon\mathbf{n})=\delta f(\varepsilon)+\mathbf{n}\delta\mathbf{f}(\varepsilon),\quad \delta J(\varepsilon\mathbf{n})=\delta J(\varepsilon)+\mathbf{n}\delta\mathbf{J}(\varepsilon). \quad (12)$$

It follows from (2) that the fluctuational integral for collisions with impurities is equal to $\delta I_{\rm imp}=a^2\mathbf{n}\delta\mathbf{f}/\tau$. The expression for the fluctuational integral for collisions with phonons becomes simpler in the energy region $\varepsilon\sim\Delta$, of greatest interest, in the case $\Delta\ll T$. We then have $\delta I_{\rm ph}=\gamma a_z(\varepsilon)\delta f(\varepsilon)$, where $\gamma=\tau_\varepsilon^{-1}$ is the reciprocal time of energy scattering by the electrons.

We use the subscript s to designate those parts of the distribution function and of the extraneous fluxes that are even in ε, and the subscript a for the odd ones. We confine ourselves to processes in which there is no unbalance between the quasiparticle-spectrum branches and f_s is equal to its equilibrium value 1/2. It is convenient to replace the odd $f_a(\varepsilon)$ by $F(\varepsilon) = -2f_a(\varepsilon)$. At equilibrium we have $F(\varepsilon) = \tanh(\varepsilon/2T)$.

From (5) we get equations for δf_s and δF (path length $l = v\tau$)

$$v\delta\left[\dot{\mathbf{p}}_s a_z \frac{\partial F}{\partial \varepsilon}\right] - 2a^2 \frac{\delta \mathbf{f}_s}{\tau} = -2\delta \mathbf{J}_s(\varepsilon),$$

$$a_z\left(\frac{\partial}{\partial t} + \gamma\right)\delta F - D\nabla(a^2 \nabla \delta F) + i\frac{D}{\hbar}\delta\left[(R_y{}^2 - A_y{}^2)\mathbf{p}_s \dot{\mathbf{p}}_s \frac{\partial F}{\partial \varepsilon}\right]$$

$$+ \dot{F}\delta a_z + \delta\left[i\dot{\Delta} a_y \frac{\partial F}{\partial \varepsilon}\right] = -2\delta J_a(\varepsilon) + \frac{2l}{3}\operatorname{div}\delta \mathbf{J}_a. \tag{13}$$

As follows from (6) and (12) (α and β number the δJ components),

$$\langle(\delta J_\alpha(\varepsilon_1 \mathbf{r}_1 t_1))_{s,a} (\delta J_\beta(\varepsilon_2 \mathbf{r}_2 t_2))_{s,a}\rangle = \frac{3}{4\tau N_F}\delta_{\alpha\beta}\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2)$$

$$\times[\delta(\varepsilon_1 - \varepsilon_2) \pm \delta(\varepsilon_1 + \varepsilon_2)]a^2(\varepsilon_1)[1 - F^2(\varepsilon_1)], \tag{14}$$

$$\langle \delta J_a(\varepsilon_1 \mathbf{r}_1 t_1)\delta J_a(\varepsilon_2 \mathbf{r}_2 t_2)\rangle = \frac{\gamma}{2N_F} a_z(\varepsilon_1)\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\times[\delta(\varepsilon_1 - \varepsilon_2) - \delta(\varepsilon_1 + \varepsilon_2)], \quad \varepsilon_1, \varepsilon_2 \ll T. \tag{15}$$

We consider the case of temperatures close to T_c, when $\Delta \ll T$. If we calculate $a_y(\varepsilon \mathbf{n} \mathbf{r} t)$ with the aid of the equations for R and A, accurate to the second derivatives with respect to the coordinates and with respect to terms proportional to p_s^2, and substitute in the self-consistency equation for $\Delta(\mathbf{r}t)$, we obtain the Ginzburg-Landau equation generalized to the case of a nonequilibrium quasiparticle distribution function $F(\varepsilon)$ (see Ref. 3):

$$\frac{\pi\hbar D}{8T}s(\Delta)\left(\nabla^2 - \frac{4p_s{}^2}{\hbar^2}\right)\Delta + |\tau|\Delta\left(1 - \frac{\Delta^2}{\Delta_0{}^2(T)}\right)$$

$$+\Delta\int_\Delta^\infty d\varepsilon\left[F(\varepsilon) - \operatorname{th}\frac{\varepsilon}{2T}\right](\varepsilon^2 - \Delta^2)^{-1/2} = 0. \tag{16}$$

We have introduced here the notation

$$s(\Delta) = \frac{T}{\Delta}\left[\frac{d}{d\varepsilon}(\varepsilon F(\varepsilon))\right]_{\varepsilon=\Delta},$$

$|\tau| = (T_c - T)/T_c$, and $\Delta_0(T)$ is the equilibrium value of Δ. Linearization of (16) leads to the fluctuational Ginzburg-Landau equation, which is the expanded expression (10) in the dirty limit.

At low frequencies $\omega \ll \Delta/\hbar$ expression (11) for the current-density fluctuation takes in the dirty limit the form

$$\delta \mathbf{j}(\mathbf{r}t) = \frac{eN_F v}{3}\int_0^\infty d\varepsilon\,(2a_z \delta \mathbf{f}_s + 2\mathbf{f}_s \delta a_z - F\delta \mathbf{a}_z - \mathbf{a}_z \delta F). \tag{17}$$

Calculation with the aid of (1), (13), and the equations for R and A yields

$$\delta \mathbf{j} = \delta \mathbf{j}_s + \delta \mathbf{j}_n + \delta \mathbf{j}^{ext}. \tag{18}$$

The fluctuational superconducting current, the normal current of the quasiparticles, and the extraneous random current, which make up the total fluctuational current, are respectively equal in the homogeneous case to (σ_n is the conductivity of the normal metal)

$$\delta \mathbf{j}_s(\mathbf{r}t) = \frac{eN_F D}{\hbar}\int_0^\infty d\varepsilon\,\delta[-i(R_y{}^2 - A_y{}^2)F\mathbf{p}_s], \tag{19a}$$

$$\delta \mathbf{j}_n(\mathbf{r}t) = \sigma_n \int_0^\infty d\varepsilon\,\delta\left[\frac{a_z{}^2}{a^2}\frac{\partial F}{\partial \varepsilon}\frac{\dot{\mathbf{p}}_s}{e}\right], \tag{19b}$$

$$\delta \mathbf{j}^{ext}(\mathbf{r}t) = 2eN_F v\tau \int_0^\infty d\varepsilon \frac{a_z}{a^2}\delta \mathbf{J}_s(\varepsilon \mathbf{r}t). \tag{19c}$$

It follows from (14) and (19c) that the extraneous-currents correlation function is

$$\langle \delta j_\alpha^{ext}(\mathbf{r}_1 t_1)\delta j_\beta^{ext}(\mathbf{r}_2 t_2)\rangle$$

$$= \delta_{\alpha\beta}\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2)\sigma_n \int_0^\infty d\varepsilon \frac{a_z{}^2(\varepsilon)}{a^2(\varepsilon)}[1 - F^2(\varepsilon)]. \tag{20}$$

Comparing the expressions for the normal current [see (19b) and (20)] we verify that in the thermodynamic-equilibrium state, when $F(\varepsilon) = \tanh(\varepsilon/2T)$, the Nyquist theorem is valid:

$$\langle \delta j_\alpha^{ext}(\mathbf{r}_1 t_1)\delta j_\beta^{ext}(\mathbf{r}_2 t_2)\rangle$$

$$= 2\delta_{\alpha\beta}\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2)T\sigma_n \int_0^\infty d\varepsilon \frac{a_z{}^2}{a^2}\frac{\partial F}{\partial \varepsilon}. \tag{21}$$

3. JOSEPHSON-JUNCTION VOLTAGE FLUCTUATION AT CURRENTS BELOW CRITICAL

Using the method described in Sec. 2, we obtain the spectral density S_U of the voltage fluctuations of a Josephson junction with direct conductivity of the S-c-S type, i.e., of the type of a bridge between bulky superconductors (shores). We consider the so-called long junction, whose length L satisfies the inequalities $\eta \ll L \ll \xi$ (ξ is the coherence length and $\eta = (\hbar D/\Delta_0(T))^{1/2}$). The theory of the Josephson effect in such a junction was developed by Aslamazov and Larkin.[3,13] They have found that in the resistive state, i.e., under Josephson generation conditions, the quasiparticle distribution function $F(\varepsilon)$ deviates greatly from equilibrium, and this stimulates the superconductivity. The reason for the disequilibrium is that the order parameter and the energy gap $\Delta(\mathbf{r})$ in the junction are smaller than the gap $\Delta_0(T)$ in the shores. Quasiparticles with energy $\varepsilon < \Delta_0(T)$ turn out to be trapped in a well and cannot diffuse directly into the shores. For these quasiparticles, the time to establish equilibrium coincides with the energy relaxation time τ_ε, and is long at low temperatures. As a result, $F(\varepsilon)$ deviates noticeably from $\tanh(\varepsilon/2T)$ already in a weak electric field.

We consider first the case when the superconducting current I_s is weaker than the critical I_c. If $I_s \neq 0$, then $\Delta < \Delta_0(T)$ in the junction. As already indicated, the kinetics of quasiparticles with $\varepsilon < \Delta_0(T)$ differs from that of particles with $\varepsilon > \Delta_0(T)$, namely, the relaxation time of the trapped particles is τ_ε and that of the untrapped ones is equal to the diffusion time $\tau_D = L^2/D \ll \tau_\varepsilon$. Therefore, although the time-averaged distribution function $F(\varepsilon)$ does not differ from its equilibrium value, the fluctuations of the number of quasiparticles with $\varepsilon < \Delta_0(T)$ are large at low frequencies, owing to the large value of τ_ε. By virtue of the Ginzburg-Landau equation (16), the fluctuations $\delta F(\varepsilon)$ lead to fluctuations of the order parameter in the junction, and in the presence of an average super-conducting current and at $\mathbf{p}_s \neq 0$ this fluctuation leads to fluctuations δI_s of the superconducting current. If the total current is fixed, the fluctuations δI_s cause a fluctuation δI_n of the normal current, and by the same token a fluctuation δU of the junction voltage. Since τ_ε is long, this mechanism can make an appreciable contribution to $S_U(f)$ at frequencies $f < \gamma = \tau_\varepsilon^{-1}$ (compared with the spectral density obtained if account is taken only of the extraneous fluctuation currents $\delta \mathbf{j}^{\text{ext}}(\mathbf{r}t)$ that result from the random character of the quasiparticle elastic scattering).

We assume for simplicity that the transverse dimensions of the constriction are small compared with its length L and that all the quantities in the bridge vary only in one direction (X). According to (19a),

$$\frac{\delta j_s}{j_s} = 2\frac{\delta\Delta}{\Delta} + \frac{\delta p_s}{p_s} + \frac{2T}{\Delta}\delta F_\Delta. \tag{22}$$

We have introduced here the symbol

$$\delta F_\Delta = \frac{1}{i\pi\Delta}\int_0^\infty d\varepsilon\,(R_y^2 - A_y^2)\,\delta F(\varepsilon). \tag{23}$$

Since $\tau_D \ll \tau_\varepsilon$, the fluctuation $\delta F(\varepsilon)$ can be regarded as constant (independent of the coordinates) in the entire interval in which quasiparticles with a given energy ε can move, i.e., within the interval in which $\Delta(x) < \varepsilon$. Moreover, owing to the frequent Andreev reflections from the well walls, the spectrum branches become strongly intermixed, so that we can neglect the fluctuations of the function f_s which is even in ε and put $f_s = 1/2$. In the derivation of the kinetic equation for δF from (13) it must also be kept in mind that $\dot{p}_s = 0$, $\dot{F} = 0$, $\dot{\Delta} = 0$. Let us integrate the kinetic equation term by term within the limits of the junction. This integration is carried out in fact for each ε over a segment on which $\Delta(x) < \varepsilon$. By virtue of the conditions imposed on the ends of this segment, the diffusion term vanishes. We express now δp_s with the aid of (22) and (23). Finally, we neglect the term $(2l/3)\operatorname{div}\delta\mathbf{J}_a$ in the right-hand side of (13), inasmuch as allowance for it, in the dirty limit and under equilibrium conditions, results in a small correction. For the Fourier transform of the fluctuation $\delta F(\varepsilon\omega)$ we obtain the equation

$$(-i\omega\langle a_z\rangle^+ + \gamma\langle a_z\rangle)\,\delta F(\varepsilon\omega) = -2\langle \delta J_a(\varepsilon\omega)\rangle - \frac{i\omega}{2T}\left\langle \frac{Dp_s^2}{\hbar j_s}\frac{1}{i}(R_y^2 - A_y^2)\,\delta j_s\right\rangle + \frac{i\omega}{2T}\langle i a_y^+ \delta\Delta\rangle. \tag{24}$$

The angle brackets denote here integration over the region of the junction, and we have introduced the notation

$$\langle a_z\rangle^+ = \langle a_z\rangle + \frac{4\hbar T^2 j_s^2}{\pi e^2 N_F^2 D\varepsilon^4 |\partial\Delta/\partial x|_\varepsilon},$$
$$i a_y^+ = i a_y + \frac{2Dp_s^2}{i\hbar\Delta}(R_y^2 - A_y^2), \tag{25}$$

the subscript ε of $|\partial\Delta/\partial x|_\varepsilon$ means that the derivative is taken at the point where $\Delta(x) = \varepsilon$. We took it into account in the derivation that $R_y^2 - A_y^2 \approx i\pi\Delta\delta(\varepsilon - \Delta)$, and used the connection between j_s and p_s.

The Ginzburg-Landau fluctuation equation is obtained by linearizing (16). We eliminate δp_s and δF_s with the aid of (22) and (24). We put $\tilde{\Delta} = \Delta/\Delta_0$ and $\tilde{j}_s = j_s/j_c$, where $j_c = \pi e N_F D\Delta_0^2/4LT$. Assume the operator

$$\hat{H} = -L^2\nabla^2 - 3\tilde{j}_s^2/\tilde{\Delta}^4. \tag{26}$$

The Ginzburg-Landau equation for $\delta\tilde{\Delta}$ can then be represented in the form (the terms $\sim L^2/\xi^2 \ll 1$ were left out of Eq. (36) for H)

$$\hat{H}\delta\tilde{\Delta} + \frac{4\omega}{\pi}\frac{L^2}{\eta^2}\int_0^{\Delta_0}\frac{d\varepsilon}{\Delta_0}\frac{i a_y^+\langle i a_y^+ \delta\tilde{\Delta}\rangle}{i\langle a_z\rangle\gamma + \omega\langle a_z\rangle^+} = -\frac{1}{2}\frac{\tilde{j}_s}{\tilde{\Delta}^3}\delta\tilde{j}_s + \frac{\omega}{\pi}\int_0^{\Delta_0}\frac{d\varepsilon}{\Delta_0} i a_y^+\left\langle\frac{\tilde{j}_s}{\tilde{\Delta}^4}\frac{R_y^2 - A_y^2}{i}\delta\tilde{j}_s\right\rangle\frac{1}{i\gamma\langle a_z\rangle + \omega\langle a_z\rangle^+} - \frac{16T}{\pi\Delta_0}\frac{L^2}{\eta^2}\int_0^{\Delta_0}\frac{d\varepsilon}{\Delta_0}\frac{i a_y^+\langle\delta J_a\rangle}{\gamma\langle a_z\rangle - i\omega\langle a_z\rangle^+}. \tag{27}$$

The boundary condition for (27) is $\delta\Delta = 0$ in the shores.

The solution of Eq. (27) must be substituted in Eq. (22) for δj_s, in which δF is taken from (14). This makes it possible to express δj_s in terms of δp_s and of the extraneous current $\delta J_a(\varepsilon)$. Expression (18) for the fluctuation of the total current $\delta I = S\delta j$ (S the cross section area of the junction), and the connection between $\delta\dot{p}_s$ and the fluctuation of the junction voltage,

$$\delta U(t) = \int_{-L/2}^{L/2} dx\,\delta\dot{p}_s/e, \tag{28}$$

make it possible to relate the fluctuations δI and δU with the extraneous sources $\delta I^{\text{ext}} = S\delta j^{\text{ext}}$ and δJ_a. It is possible next to obtain both the junction impedance $Z(\omega)$ and $S_U(f)$ for a given current ($\delta I = 0$). Independently of the possibility of solving Eq. (27), it is possible to verify that at currents lower than critical the Nyquist relation holds, i.e., $S_U(f) = 4T\,\mathrm{Re}\,Z(\omega)$.

An approximate solution of (27) can be obtained for currents j_s close to the critical current j_c, and for frequencies $\omega \ll \gamma$. For small j_s all the eigenvalues of the operator $\hat{H}$ in (26) are positive. As j_s approaches j_c, the minimal eigenvalue tends to zero and vanishes at $j_s = j_c$, when it is impossible to solve (27) for $\omega = 0$ and $\delta J_a = 0$ but $\delta j_s \neq 0$. For j_s close to j_c, the eigenfunction ψ_0 corresponding to the minimum eigenvalue is proportional to $\partial\Delta/\partial j_s$ and can be regarded as known. The solution of the inhomogeneous equation (27) is determined at small $j_c - j_s$ by the lowest eigenvalue and by the corresponding eigenfunction ψ_0. Calculation yields in this case

$$Z(\omega)=R_n\frac{\omega}{i\omega_c}\frac{1}{1-i\omega\tau_J}\frac{1}{(1-I^2/I_c^2)^{1/2}}. \tag{29}$$

Here $\omega_c = \pi\Delta_0^2/2\hbar T$ and

$$\tau_J=\frac{9(2\ln 2-1)}{2^{13/2}}\frac{L^2}{\eta^2}\left(1-\frac{I^2}{I_c^2}\right)^{-1/2}\tau_\varepsilon. \tag{30}$$

It can be seen that the variance of $Z(\omega)$ sets in already at frequencies $\omega \ll \gamma$. The characteristic time τ_J is longer than τ_ε by a factor $L^2/\eta^2 \gg 1$, and increases furthermore as the current approaches the critical value I_c.

4. FLUCTUATIONS IN JOSEPHSON JUNCTIONS ON SATURATION OF THE STIMULATION OF THE SUPERCONDUCTING CURRENT

One interesting application of the theory of fluctuations in superconductors is to a Josephson junction with direct conduction in the generation regime. The spectral density $S_U(f)$ of the voltage fluctuations in such junctions was calculated in Ref. 14, using a resistive model in which it was assumed that the fluctuation source was only the Nyquist fluctuation currents corresponding to the normal resistance R_n of the junction. It is of interest to determine how $S_U(f)$ changes when account is taken of other fluctuation mechanisms. We shall use below the method described in Sec. 2 to calculate $S_U(f)$ of a junction of length L satisfying the condition $\eta \ll L \ll \xi$ at temperatures T close to T_c and at voltages corresponding to saturation of the current on the current-voltage characteristic (IVC), and compare the results with the resistive model and with the experimental data.

As shown by Aslamazov and Larkin[3,13] (see also the review in Ref. 15). The distribution function $F(\varepsilon)$ in the considered junctions loses equilibrium in the region of the well ($\varepsilon < \Delta_0$) under Josephson generation conditions ($I > I_c$). Owing to the collisions of the quasiparticles present in the well, the jittering well walls cause diffusion of the quasiparticles towards higher energies, so that $F(\varepsilon)$ increases compared with the equilibrium value $\tanh(\varepsilon/2T)$ and reaches in the limit a value corresponding to equilibrium at $\varepsilon = \Delta_0$, i.e., $\tanh(\Delta_0/2T)$. This situation corresponds to fewer quasiparticles than their equilibrium number, and this leads to stimulation of the superconducting current. It is manifested by an appreciable increase of I_s over I_c, by a factor $\sim L/\eta \gg 1$. A plateau appears on the IVC of the transition (the current I hardly increases with the voltage U), starting with $\overline{U}^* \ll \Delta/e$.

Calculation of the junction kinetics in the superconducting-current regime is substantially simplified because not only F is independent of the coordinates inside the junction (see Sec. 4), but also the order parameter Δ is independent of the coordinates in almost the entire junction (except in narrow regions of width $\sim\eta \ll L$ near the edges).[13] Therefore, in almost the entire junction, the phase of the order parameter increases linearly with the coordinate x, and $p_s = \hbar\chi/2L$, where χ is the phase difference of the order parameter over the entire junction.

In the considered regime of superconductivity-stimulation saturation, only the term proportional to $4p_s^2 = \hbar^2\chi^2/L^2$, and the nonequilibrium term are significant in the Ginzburg-Landau equation.[16] This equation can be written in the form

$$\frac{\eta^2}{L^2}s(\Delta)\chi^2=\Psi(\Delta;F), \tag{31}$$

$$\Psi(\Delta;F)=\frac{8T}{\pi\Delta_0}\int_\Delta^{\Delta_0}\frac{d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\left[F(\varepsilon)-\mathrm{th}\frac{\varepsilon}{2T}\right]. \tag{32}$$

In accordance with (19a) and with the fact that $-i(R_y^2 - A_y^2) \approx \pi\Delta\delta(\varepsilon-\Delta)$, the superconducting current in the junction is equal to

$$I_s=I_c\frac{\Delta}{\Delta_0}\frac{2TF(\Delta)}{\Delta_0}\chi,\quad I_c=\frac{\pi\Delta_0^2}{4eR_nT}. \tag{33}$$

It must be borne in mind, in the derivation of the relation between the fluctuation δI_s of the superconducting current and the fluctuation δF of the distribution function, that the total fluctuation of each of the quantities Ψ and s consists of a part directly due to the fluctuation $\delta\Delta$, and a part connected with δF:

$$\delta\Psi=\left(\frac{\partial\Psi}{\partial\Delta}\right)_F\delta\Delta+\delta\Psi',\quad \delta\Psi'=\frac{8T}{\pi\Delta_0}\int_\Delta^{\Delta_0}\frac{d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\delta F(\varepsilon),$$

$$\delta s=\left(\frac{\partial s}{\partial\Delta}\right)_F\delta\Delta+\delta s',\quad \delta s'=\frac{T}{\Delta}\frac{d}{d\varepsilon}[\varepsilon\delta F(\varepsilon)]_{\varepsilon=\Delta}. \tag{34}$$

Since $F \approx \Delta_0/2T$ in the stimulation saturation regime,

$$\Psi=\frac{4}{\pi}\left\{\ln\left[\frac{1+(1-\Delta^2/\Delta_0^2)^{1/2}}{\Delta/\Delta_0}\right]-(1-\Delta^2/\Delta_0^2)^{1/2}\right\},$$
$$s(\Delta)=\Delta_0/2\Delta. \tag{35}$$

From (31)–(34) we get for the fluctuations the equations:

$$2\frac{\delta\chi}{\chi}+\frac{\partial\ln(s/\Psi)}{\partial\Delta}\delta\Delta=\frac{\delta\Psi'}{\Psi}-\frac{\delta s'}{s},$$
$$\delta I_s=I_s\left(1+\frac{\partial\ln F(\Delta)}{\partial\ln\Delta}\right)\delta\Delta+I_s\frac{\delta\chi}{\chi}+I_s\frac{\delta F'(\Delta)}{F(\Delta)}. \tag{36}$$

After eliminating $\delta\Delta$ we get

$$\delta I_s=\frac{dI_s}{d\chi}\delta\chi+\delta I_s^{ext}. \tag{37}$$

The extraneous fluctuating superconducting current is due entirely to the fluctuations of the quasiparticle distribution function:

$$\delta I_s^{ext}=I_s\left\{\frac{\delta F'(\Delta)}{F(\Delta)}+\left[\frac{\partial\ln(s/\Psi)}{\partial\ln\Delta}\right]^{-1}\left(\frac{\delta\Psi'}{\Psi}-\frac{\delta s'}{s}\right)\right\}. \tag{38}$$

To express now the fluctuation of the distribution function in terms of the extraneous fluctuating currents, whose correlation functions are known (see Sec. 2), we must use the kinetic equation. Since the noise spectrum density is measured at frequencies $f \ll \omega_J/2\pi$ (ω_J is the Josephson frequency), it is determined by fluctuations that are smoothedout over a time interval $t_0 \gg \omega_J^{-1}$. In addition, the fluctuating quantities measured in an external circuit depend only on fluctuations averaged over the volume of the crystal. It is necessary to derive from the general equation (13) a kinetic equations for the fluctuation $\langle \overline{\delta F} \rangle$ averaged over the time (superior bar) and over the volume of the junction (angle brackets). This problem is solved in the Appendix, and we present here only the result:

$$\langle \bar{a}_z \rangle \frac{\partial}{\partial t} \langle \overline{\delta F} \rangle + \langle \overline{\delta I_{ph}} \rangle - \frac{\partial}{\partial \varepsilon}\left[D_\varepsilon \frac{\partial}{\partial \varepsilon} \langle \overline{\delta F} \rangle \right]$$
$$- \frac{\partial}{\partial \varepsilon}\left[\delta D_\varepsilon \frac{\partial}{\partial \varepsilon} \langle \bar{F} \rangle \right] + \frac{\partial \langle \bar{F} \rangle}{\partial \varepsilon} \frac{\partial}{\partial t} \langle \overline{\delta \xi} \rangle = \langle \overline{\delta Q} \rangle - \frac{\partial}{\partial \varepsilon} \langle \overline{\delta j_\varepsilon} \rangle. \tag{39}$$

Here $\xi = (\varepsilon^2 - \Delta^2)^{1/2}$, D_ε is the energy-diffusion coefficient[3,13,15]:

$$D_\varepsilon = -\left\langle \overline{\frac{\partial \xi_{11}}{\partial t} (D\nabla^2)^{-1} \frac{\partial \xi_{11}}{\partial t}} \right\rangle, \tag{40}$$

δD_ε is the fluctuation of this coefficient and an expression for it is obtained by linearizing (40). The subscript 11 means taking only that part of the quantity which is averaged over space and time and whose averaging (the part) over either time or space yields zero. In particular,

$$\xi_{11} = \xi - \bar{\xi} - \langle \xi \rangle + \langle \bar{\xi} \rangle. \tag{41}$$

The extraneous flux is $\delta Q = -2\delta J_a + (2l/3)\mathrm{div}\delta \mathbf{J}_a$ [see (13)], $\langle \overline{\delta j_\varepsilon} \rangle$ is the extraneous energy-diffusion fluctuation flux and is equal to

$$\langle \overline{\delta j_\varepsilon} \rangle = \left\langle \overline{\frac{\partial \xi_{11}}{\partial t} (D\nabla^2)^{-1} \delta Q_{11}} \right\rangle. \tag{42}$$

The action of the operator $(D\nabla^2)^{-1}$ is defined in Refs. 3, 13, and 15.

Let us compare (39) with the kinetic equation derived in Refs. 3 and 13 for the average (nonfluctuating) distribution function $\langle \bar{F}(\varepsilon) \rangle$. It can be seen that the left-hand side of (39) is obtained by linearizing the equation for $\langle \bar{F} \rangle$, while two different extraneous sources appear in the right-hand side. The flux $\langle \overline{\delta Q} \rangle$ is due entirely, as we shall show, to scattering by phonons; $\langle \overline{\delta j_\varepsilon} \rangle$ is the extraneous part of the fluctuating energy-diffusion flux due in final analysis to the jitter of the well in which are trapped the quasiparticles with $\varepsilon < \Delta_0$ and with random character of the scattering by impurities.

In the calculation of the correlation functions $\langle \overline{\delta Q} \rangle$ and $\langle \overline{\delta j_\varepsilon} \rangle$ with the aid of Eqs. (14), (15), and (42) it must be borne in mind that, averaged over the junction volume, we have $\langle \mathrm{div}\, \delta \mathbf{J}_a \rangle = 0$, since scattering by impurities does not change the number of the quasiparticles. This is why $\langle \overline{\mathrm{div}\, \delta \mathbf{J}_a} \rangle = 0$ and $\langle \delta Q \rangle$ contains only the extraneous flux $-2\langle \delta J_c \rangle$, which is due to scattering by phonons. The Fourier transform of the corresponding correlation function is

$$\langle \langle \overline{\delta Q(\varepsilon_1 t_1)} \rangle \langle \overline{\delta Q(\varepsilon_2 t_2)} \rangle \rangle_\omega$$
$$= \frac{\Upsilon}{V(\varepsilon_1) N_F} \langle \bar{a}_z(\varepsilon_1) \rangle [\delta(\varepsilon_1 - \varepsilon_2) - \delta(\varepsilon_1 + \varepsilon_2)]. \tag{43}$$

Here $V(\varepsilon)$ is the junction volume accessible to quasiparticles having a given ε.

We calculate first that part of the correlation function of the quantities $\langle \overline{\delta j_\varepsilon} \rangle$ which is due to impurity scattering. From the equality $\langle \mathrm{div}\delta \mathbf{J}_a \rangle = 0$ it follows that $(\mathrm{div}\delta \mathbf{J}_a)_{11} = \mathrm{div}\delta \mathbf{J}_a - \overline{\mathrm{div}\delta \mathbf{J}_a}$ [see (41)]. Expression (42) for $\langle \overline{\delta j_\varepsilon} \rangle$ contains an alternating quantity $\partial \xi_{11}/\partial t$ having the Josephson frequency and containing no time-independent part. This means that we need substitute in the expression for $\langle \delta j \rangle$ only $\mathrm{div}\delta \mathbf{J}_a$ and discard $\overline{\mathrm{div}\delta \mathbf{J}_a}$. With the aid of (14) we obtain the following expression for the Fourier transform of the impurity-scattering contribution to the correlation function $\langle \overline{\delta j_\varepsilon} \rangle$:

$$\langle \langle \overline{\delta j_\varepsilon(\varepsilon_1 t_1)} \rangle \langle \overline{\delta j_\varepsilon(\varepsilon_2 t_2)} \rangle \rangle_\omega$$
$$= \frac{D_\varepsilon}{V(\varepsilon_1) N_F} [\delta(\varepsilon_1 - \varepsilon_2) - \delta(\varepsilon_1 + \varepsilon_2)][1 - F^2(\varepsilon_1)]. \tag{44}$$

The contribution of the background scattering to this correlation function is relatively small.

Under conditions of stimulation saturation of the superconducting equation, two terms in the left-hand side of the kinetic equation vanish because $\partial \langle \bar{F} \rangle / \partial \varepsilon = 0$ at $\varepsilon < \Delta_0$. The first two terms in the left-hand side are small compared with the term due to energy diffusion. In the right-hand side, the extraneous source $\langle \overline{\delta Q} \rangle$ is small compared with the source that is connected with the same energy diffusion. The kinetic equation reduces in this regime to the statement that the total fluctuation energy-diffusion flux is independent of ε:

$$\frac{\partial}{\partial \varepsilon}\left[D_\varepsilon \frac{\partial}{\partial \varepsilon} \langle \overline{\delta F} \rangle + \langle \overline{\delta j_\varepsilon} \rangle \right] = 0. \tag{45}$$

Since the flux is zero at the bottom of the well and since $\langle \overline{\delta F} \rangle = 0$ at $\varepsilon = \Delta_0$, we get

$$\langle \overline{\delta F(\varepsilon)} \rangle = \int_\varepsilon^{\Delta_0} d\varepsilon' \frac{\langle \overline{\delta j_\varepsilon(\varepsilon')} \rangle}{D_\varepsilon(\varepsilon')}. \tag{46}$$

An expression for $D_\varepsilon(\varepsilon)$ was obtained in Ref. 13. Under conditions when the nonequilibrium term in the kinetic equation for $\langle \bar{F} \rangle$ is large, we have

$$D_\varepsilon(\varepsilon) = c \frac{\Delta_0}{\hbar} \overline{(\hbar \dot{\chi})^2} \left(\frac{\Delta}{\Delta_0} \right)^4 \frac{\Delta^2}{\xi^2}. \tag{47}$$

In this expression, c is a coefficient of order unity, $\overline{(\hbar \dot{\chi})^2} = 4e^2 \overline{U}^2$, and $U(t)$ is the instantaneous value of the junction voltage.

We proceed now to calculate the spectral density of the junction-voltage fluctuations. At nonzero average junction voltage the time dependence of the phase difference is determined by the equation for the total current

$$I = \frac{\hbar}{2eR_n} \dot{\chi} + I_s(\chi) + \delta I_s^{ext} + \delta I_n^{ext}. \tag{48}$$

Here δI_n^{ext} is the extraneous normal current whose correlation function is determined by the Nyquist formula, since the main contribution to the normal current is made by the quasiparticle energy region $\varepsilon \sim T$, and in this region the distribution function is close to equilibrium:

$$\langle \delta I_n^{ext}(t_1) \delta I_n^{ext}(t_2) \rangle = (2T/R_n) \delta(t_1 - t_2). \tag{49}$$

In the regime of strong superconductivity stimulation, the amplitude of the oscillations of the order parameter in the junction is small compared with the mean value of Δ. This allows us to express the fluctuation current δI_s^{ext} [see (34) and (38)] in terms of the smoothly varying part of the distribution-function fluctuation $\langle \overline{\delta F} \rangle$ given by Eq. (46). The correlation function δI_s^{ext} follows from (44). It must also be recognized that in the strong superconductivity-stimulation regime[15] the average current is $I_s = 0{,}4eN_F SD^{1/2}\Delta_0^{5/2}/\hbar^{1/2}T$, the energy diffusion coefficient D_ε is given by (47), and the volume $V(\varepsilon)$ practically coincides with the junction volume V at all energies $\varepsilon < \Delta_0$. The Fourier transform of the correlation function of the extraneous superconducting currents is equal (apart from a coefficient of order unity) to

$$\langle \delta I_s^{ext}(t_1)\delta I_s^{ext}(t_2)\rangle_\omega = \frac{2T}{R_n}\frac{\Delta_0}{T}\frac{\Delta_0^2}{4e^2\overline{U^2}}. \qquad (50)$$

The fluctuations δI_n^{ext} and δI_s^{ext} are produced by different mechanisms, so that there is no correlation between them, and the corresponding spectral voltage fluctuation densities $S_U^{(n)}(f)$ and $S_U^{(s)}(f)$ are additive. Calculation of each of these spectral densities calls for a different approach, since the times of the correlations δI_n^{ext} and δI_s^{ext} are entirely different. The δI_n^{ext} correlation time is $\tau_p \ll \omega_J^{-1}$, and the δI_s^{ext} correlation time is of the order of $\Delta_0^2/D_\varepsilon \gg \omega_J^{-1}$. The spectral density $S_U^{(n)}(0)$ was calculated in Ref. 14 in the framework of the resistive model. Using the method described in Ref. 16, we can represent this density, for arbitrary $I_s(\chi)$ dependence, in the form

$$S_U^{(n)}(0) = 4T\frac{R_d^2}{R_n}\left[1 - \frac{\overline{U}}{2R_d^2}\frac{dR_d}{dI}\right]. \qquad (51)$$

Since the fluctuations δI_s^{ext} are slow, the response of the voltage to δI_s^{ext} is simply $\delta U^{(s)} = -R_d \delta I_s^{\text{ext}}$, and therefore

$$S_U^{(s)}(0) = 2R_d^2\langle \delta I_s^{ext}(t_1)\delta I_s^{ext}(t_2)\rangle_{\omega=0}. \qquad (52)$$

The total spectral density of the voltage fluctuations at low frequencies is equal to the sum of (51) and (52):

$$S_U(0) = 4T\frac{R_d^2}{R}\left\{1 - \frac{U}{2R_d^2}\frac{dR_d}{dI} + \frac{\Delta_0}{T}\frac{\Delta_0^2}{4e^2}(\overline{U^2})^{-1}\right\}. \qquad (53)$$

In the region of the plateau on the IVC of the junction, the quantity $\Delta_0^2/4e^2\overline{U}^2$ increases with decrease of the voltage. In the region of the "shoulder" (the start of the plateau), i.e., at $D_\varepsilon/\Delta_0^2 \sim \gamma$, its order of magnitude is ω_c/γ, where $\omega_c = \pi\Delta_0^2/2\hbar T$, i.e., it can be much larger than unity. This means that the noise due to the energy diffusion can exceed by many times the usual resistive noise. Since $S_U(0)$ as a function of the voltage is the product of R_d^2 and a decreasing function $\overline{U}$, the maximum of $S_U(0)$ should be shifted away from the maximum of R_d towards lower $\overline{U}$, by amounts on the order of the width of the shoulder on the IVC. It can also be seen that the ratio $S_U^{(s)}(0)/R_d^2$ decreases with decrease of Δ_0 when the temperature T appraoches the critical T_c.

In a number of experiments (see, e.g., Refs. 17–20) it was noted that the spectral density $S_U(0)$ in S-c-S junctions is several (3–10) times larger at $\overline{I} > I_c$ than the value calculated on the basis of the resistive model[14] [see (51)]. On the basis of the foregoing it can be assumed that this difference between the experiment and the theory of Ref. 14 is due to the noise $S_U^{(s)}(0)$ produced by the fluctuations of the isotropic part of the quasiparticle distribution functions, both through scattering by phonons and through fluctuations due to the superconducting-current fluctuations. It is also pointed out in Ref. 17 that the maximum of $S_U(f)$ measured at low frequencies as a function of the time-averaged current $\overline{I}$ is shifted away from the maximum of $R_d(\overline{I})$ towards smaller $\overline{I}$. It was shown above that if the IVC $\overline{I}(\overline{U})$ and the noise $S_U(0)$ are determined by superconductivity stimulation by energy diffusion of the quasiparticles, such an effect does indeed take place. A decrease of the spectral density $S_U(0)$ of the voltage fluctuations at low frequencies (see Ref. 17) and a decrease of the Josephson generation linewidth connected with $S_U(0)$ (see Ref. 19) were also observed when the temperature approached T_c as the critical current I_c of the junction decreased. This also agrees with (53).

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APPENDIX

When the distribution function and the order parameter change little over distances $\sim\eta$, the kinetic equation (13) for the fluctuation of the isotropic part of the distribution function δF can be expressed, if $\varepsilon > \Delta$, in the form (δ denotes linearization of the expressions in the square brackets)

$$\delta\left[-D\nabla^2 F + \frac{\partial\xi}{\partial\varepsilon}\frac{\partial F}{\partial t} - \frac{\partial F}{\partial\varepsilon}\frac{\partial\xi}{\partial t} + I_{ph}\right] = \delta Q, \qquad (A.1)$$

where

$$\xi = (\varepsilon^2 - \Delta^2)^{1/2}, \quad \delta Q = -2\delta J_a + (2l/3)\,\mathrm{div}\,\delta\mathbf{J}_a.$$

The boundary condition for δF at $\varepsilon = \Delta$ is the nonpenetrability condition (equation of the derivative in the direction of the well wall to zero).

We represent δF in the form (see the text for the notation)

$$\delta F = \delta F_{00} + \delta F_{01} + \delta F_{10} + \delta F_{11}, \qquad (A.2)$$

where

$$\delta F_{00} = \langle\overline{\delta F}\rangle, \quad \delta F_{01} = \overline{\delta F} - \langle\overline{\delta F}\rangle, \quad \delta F_{10} = \langle\delta F\rangle - \langle\overline{\delta F}\rangle,$$
$$\delta F_{11} = \delta F - \overline{\delta F} - \langle\delta F\rangle + \langle\overline{\delta F}\rangle.$$

We use a procedure similar to that used in Refs. 3 and 13 to obtain a closed energy-diffusion equation for the average (nonfluctuational) distribution function $\langle\overline{F}\rangle$ (see the review in Ref. 15). We average each term of (A.1) over the volume and smooth out the time dependence. We use the vanishing of the time averages of the time derivatives of rapidly oscillating quantities. In the upshot we get

$$\frac{\partial\xi_{00}}{\partial\varepsilon}\frac{\partial\delta F_{00}}{\partial t}+\left\langle\frac{\partial\delta F_{01}}{\partial t}\frac{\partial\xi_{01}}{\partial\varepsilon}\right\rangle+\frac{\partial}{\partial\varepsilon}\overline{\frac{\partial\delta F_{10}}{\partial t}\xi_{10}}$$

$$-\frac{\partial}{\partial\varepsilon}\left\langle\overline{\delta F_{11}\frac{\partial\xi_{11}}{\partial t}}\right\rangle+\langle\overline{\delta I_{ph}}\rangle=\delta Q_{00}+\frac{\partial\delta\xi_{00}}{\partial t}\frac{\partial F_{00}}{\partial\varepsilon}$$

$$+\left\langle\frac{\partial\delta\xi_{01}}{\partial t}\frac{\partial F_{01}}{\partial\varepsilon}\right\rangle-\frac{\partial}{\partial\varepsilon}\overline{\frac{\partial F_{10}}{\partial t}\delta\xi_{10}}+\frac{\partial}{\partial\varepsilon}\left\langle\overline{F_{11}\frac{\partial\delta\xi_{11}}{\partial t}}\right\rangle. \tag{A.3}$$

For the quantities δF_{01}, δF_{10} and δF_{11} we obtain, taking the inequality $D/L^2\gg\gamma$ into account, the expressions

$$\frac{\partial\delta F_{10}}{\partial t}=\left(\frac{\partial\xi_{00}}{\partial\varepsilon}\right)^{-1}\times\left[\delta Q_{10}-\frac{\partial F_{10}}{\partial t}\frac{\partial\delta\xi_{00}}{\partial\varepsilon}+\frac{\partial\delta F_{00}}{\partial\varepsilon}\frac{\partial\xi_{10}}{\partial t}+\frac{\partial F_{00}}{\partial\varepsilon}\frac{\partial\delta\xi_{10}}{\partial t}\right], \tag{A.4}$$

$$\delta F_{01}=(-D\nabla^2)^{-1}\delta Q_{01},$$

$$\delta F_{11}=(-D\nabla^2)^{-1}\left[\frac{\partial\delta F_{00}}{\partial\varepsilon}\frac{\partial\xi_{11}}{\partial t}+\frac{\partial F_{00}}{\partial\varepsilon}\frac{\partial\delta\xi_{11}}{\partial t}+\delta Q_{11}\right].$$

The following expressions for F_{10}, F_{01} and F_{11} were obtained in Ref. 15:

$$\frac{\partial F_{10}}{\partial t}=\left(\frac{\partial\xi_{00}}{\partial\varepsilon}\right)^{-1}\frac{\partial F_{00}}{\partial\varepsilon}\frac{\partial\xi_{10}}{\partial t},\qquad F_{01}=0,$$

$$F_{11}=(-D\nabla^2)^{-1}\left(\frac{\partial F_{00}}{\partial\varepsilon}\frac{\partial\xi_{11}}{\partial t}\right). \tag{A.5}$$

Substituting (A.4) and (A.5) in (A.3) and discarding small terms, we obtain the closed equation (39) for δF_{00} and expressions (40) and (42).

[1] *Non-equilibrium Superconductivity*, D. N. Langenburg and B. V. Larkin, eds., Elsevier, 1986.

[2] K. K. Likharev, Introduction to the Dynamics of Josephson Junctions [in Russian], Nauka, 1985.

[3] L. G. Aslamazov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **70**, 1340 (1976) [Sov. Phys. JETP **43**, 698 (1976)].

[4] A. G. Aronov and V. L. Gurevich, *ibid.* **65**, 1111 (1973) [**38**, 550 (1974)]. Fiz. Tverd. Tela (Leningrad) **16**, 2656 (1974) [Sov. Phys. Solid State **16**, 1722 (1975)].

[5] A. G. Aronov, Yu. M. Galperin, V. L. Gurevich, and V. I. Kozub, Adv. Phys. **30**, 539 (1981).

[6] A. G. Aronov and R. Katilyus, Zh. Eksp. Teor. Fiz. **68**, 2208 (1975) [Sov. Phys. JETP **41**, 1106 (1975)].

[7] A. I. Larkin and Yu. N. Ovchinnikov, *ibid.* **68**, 1915 (1975) [**41**, 960 (1975)].

[8] A. I. Larkin and Yu. N. Ovchinnikov, *ibid.* **73**, 299 (1977) [**46**, 155 (1977)].

[9] A. L. Shelankov, *ibid.* **78**, 2359 (1980) [**51**, 1186 (1980)].

[10] A. Schmidt, in: *Non-equilibrium Superconductivity, Phonons and Kapitza Boundaries*, NATO Adv. Institute Series B. V. 65, Ed. by K. E. Gray, Plenum, 1981, p. 423.

[11] Sh. M. Kogan and A. Ya. Shul'man, Zh. Eksp. Teor. Fiz. **56**, 862 (1969) [Sov. Phys. JETP **29**, 467 (1969)].

[12] Sh. M. Kogan and A. Ya. Shul'man, Fiz. Tverd. Tela (Leningrad) **12**, 1119 (1970) [Sov. Phys. Solid State **12**, 874 (1970)].

[13] L. G. Aslamazov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **74**, 2184 (1978) [Sov. Phys. JETP **47**, 1136 (1978)].

[14] K. K. Likharev and V. K. Semenov, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 625 (1972) [JETP Lett. **15**, 442 (1972)].

[15] L. G. Aslamazov and A. F. Volkov, in Ref. 1, p. 66.

[16] K. E. Nagaev, Fiz. Nizk. Temp. **12**, 1244 (1986) [Sov. J. Low-Temp. Phys. **12**, 700 (1986)].

[17] S. K. Decker and J. E. Mercereau, Appl. Phys. Lett. **27**, 466 (1975).

[18] J. Mygind, W. F. Pedersen, O. H. Sorensen, *et al.*, Int. Conf. SQUID-1980, Conference Digest, 1980, p. 139.

[19] D. B. Schwartz, P. M. Mankiewich, A. K. Jain, and J. E. Luckens, IEEE Trans. **MAG-17**, 92 (1981).

[20] D. B. Schwartz and J. E. Luckens, Proc. LT-16, Physica (Utrecht) **B+C**, **108**, 1297 (1981).

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