Electron cyclotron K modes in a plasma

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The spectrum of electron cyclotron waves of a homogeneous magnetized plasma with a Maxwellian electron velocity distribution is studied analytically and numerically in the long-wavelength region, i.e., at values of the transverse (with respect to the magnetic field) wavelength considerably greater than the electron Larmor radius. In addition to the solutions near the fundamental electron cyclotron frequency and its harmonics which correspond to ordinary, extraordinary, and plasma waves, there are two families of strongly damped waves on both sides of these other solutions. One of these families is similar to electrostatic waves. It has been found in previous numerical calculations for certain particular cases. The waves of the other family are definitely nonelectrostatic and are similar to magnetostatic waves. Analytic expressions are derived for the frequencies and damping rates of these two wave families.

1. INTRODUCTION

Numerical solutions[1–5] of the dispersion relation for electromagnetic waves in plasmas have revealed, in addition to known solutions,[6] several solutions near the fundamental electron cyclotron frequency and its harmonics, in the cases of both electrostatic waves[1,2,5] and nonelectrostatic waves.[3,4] In Ref. 7 both numerical calculations and analytic calculations in the long-wave limit showed that there are also numerous wave branches near the ion cyclotron frequency and its harmonics. The appearance of these additional wave branches, like Landau damping, has been linked with singularities (poles) of the perturbed distribution function of particles which are moving along the magnetic field at a velocity $v_\parallel = (\omega - n\omega_c)/k_\parallel$ ($n = 0; \pm 1; \pm 2,...; \omega_c$ is the electron cyclotron frequency) or, in the absence of a magnetic field, at a velocity $v = \omega/k$ along the wave propagation direction. These waves are thus naturally called "kinetic" (K) modes. The existence of K modes in the short-wave region, with $kr_D \gg 1$ (r_D is the electron Debye length), was pointed out in the classic study by Landau.[8] A numerical solution of the dispersion relation for these modes was given in Ref. 9 (see also Ref. 10).

A complete picture of the electron cyclotron K modes could not be drawn in Refs. 1–5 because of difficulties encountered in the numerical solution of the dispersion relation. In the present study we use the analytic and numerical methods of Ref. 7 to analyze the dispersion relation for electromagnetic waves in a plasma in a magnetic field in the long-wave limit, $k_\perp\rho_L \ll 1$, where ρ_L is the Larmor radius of electrons having the thermal velocity. In addition to the usual[6] wave branches near the electron cyclotron frequency and its harmonics, there are two families of K modes on the two sides of the cyclotron frequencies. One family is similar to electrostatic waves for $kc \gg \omega$ (the numerical solutions found in Refs. 1–5 for certain particular cases are part of this family). The other family is definitely nonelectrostatic, and for $kc \ll \omega$ it is similar to magnetostatic waves. We derive analytic expressions for the frequencies and damping rates of both of these families of electromagnetic K modes. For the first K modes, the frequency and damping rate are

$$\omega = n\omega_c + \eta_1 k_\parallel v_T, \quad \gamma = \eta_2 k_\parallel v_T, \tag{1}$$

where $\eta_{1,2} \sim 1$ or $\eta_{1,2} \sim [\ln(1/k_\perp\rho_L)]^{1/2}$. With increasing mode index, the coefficients $\eta_{1,2}$ increase. At $k_\perp\rho_L \sim 1$ these waves are strongly damped: $\gamma \sim \omega \sim \omega_c$. For a plasma with a Maxwellian velocity distribution, the number of solutions in the limit $k_\parallel \to 0$ is infinite. The point $k_\parallel = 0$ is an essential singularity (because the point $v_\parallel = \infty$ is an essentially singular point for a Maxwellian distribution in the longitudinal velocity). For a nonequilibrium plasma with a power-law particle velocity distribution $f_0(v_\parallel) \propto (v_\parallel^2 + v_0^2)^{-(l+i)}$ ($l \geqslant 1$), the components of the tensor ε_{ij} have simple poles at the points $(\omega - n\omega_c)/k_\parallel = -iv_0$. In this case the dispersion relation has a limited number of additional solutions with $\omega \to n\omega_c$ as $k_\parallel \to 0$. With $l = 0$, in a plasma without a magnetic field, $\varepsilon(\omega, k)$ has a second-order pole at $\omega/k = -iv_0$, and there are no additional solutions corresponding to K modes.[11] Similar K modes would evidently also exist in semiconductor and metal plasmas.

2. WAVES NEAR THE FUNDAMENTAL ELECTRON CYCLOTRON FREQUENCY

Let us consider electromagnetic waves near the electron cyclotron resonance in a plasma with a Maxwellian electron velocity distribution. We assume that the ions are at rest. It is then easy to derive expressions for the components of the dielectric tensor, working from Ref. 6 (for example):

$$\begin{gathered}
\varepsilon_{11} = 1 - \frac{q}{2}\frac{x}{x+1} + q\frac{e^{-\mu}I_1}{\mu} i\pi^{1/2} z_0 w(z), \\
\varepsilon_{22} = 1 - \frac{q}{2}\frac{x}{x+1} + qe^{-\mu}\left[\frac{I_1}{\mu} + 2\mu(I_1 - I_1')\right] i\pi^{1/2} z_0 w(z), \\
\varepsilon_{12} = -i\frac{q}{2}\frac{x}{x+1} + iqe^{-\mu}(I_1 - I_1') i\pi^{1/2} z_0 w(z), \\
\varepsilon_{13} = qx\frac{k_\perp}{k_\parallel}\frac{e^{-\mu}I_1}{\mu}[1 + i\pi^{1/2} z w(z)], \\
\varepsilon_{23} = -iqx\frac{k_\perp}{k_\parallel} e^{-\mu}(I_1 - I_1')[1 + i\pi^{1/2} z w(z)], \\
\varepsilon_{33} = 1 - q + q\frac{k_\perp^2}{k_\parallel^2}\frac{e^{-\mu}I_1}{\mu}(x-1)x[1 + i\pi^{1/2} z w(z)], \\
x = \frac{\omega}{\omega_c}, \quad q = \frac{\omega_p^2}{\omega^2}, \quad z_0 = \frac{\omega}{2^{1/2}k_\parallel v_T}, \\
z = \frac{\omega - \omega_c}{2^{1/2}k_\parallel v_T}, \quad \mu = \frac{k_\perp^2 v_T^2}{\omega_c^2},
\end{gathered} \tag{2}$$

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$$v_T=\left(\frac{T_e}{m_e}\right)^{1/2},\quad \omega_p{}^2=\frac{4\pi e^2n_0}{m_e},$$
$$\omega_c=\frac{eB_0}{m_ec}>0,\quad \operatorname{tg}\vartheta=\frac{k_\perp}{k_\parallel},$$

where ϑ is the angle between the wave vector of $\mathbf{k}$ and the magnetic field $\mathbf{B}_0$. In deriving (2) we assumed $\mu\ll 1$ and $|\omega-n\omega_c|\gg 2^{1/2}k_\parallel v_T (n=-1;\pm 2)$; however, the Bessel function I_1 and its derivative I_1' have not been expanded in μ because of possible cancellations in the dispersion relation,

$$w(z)=e^{-z^2}\left(1+\frac{2i}{\pi^{1/2}}\int_0^z e^{t^2}dt\right).$$

For definiteness we assume $k_\parallel>0$. Substituting expressions (2) into the dispersion relation for electromagnetic waves, we find

$$\frac{q}{2}i\pi^{1/2}z_0w(z)\Lambda_0+\Lambda_1+i\pi^{1/2}zw(z)\Lambda_2$$
$$+q^2[i\pi^{1/2}z_0w(z)]^2(\mu^2/8)(1-q-\sin^2\vartheta N^2)=0. \quad (3)$$

Here

$$\Lambda_0=\sin^2\vartheta N^4-N^2\left[(1-q)(1+\cos^2\vartheta)+2\left(1-q\frac{x}{x+1}\right)\sin^2\vartheta\right]$$
$$+2(1-q)\left(1-q\frac{x}{x+1}\right),$$
$$\Lambda_1=\left\{1-q\cos^2\vartheta+\frac{q}{2}\sin^2\vartheta\left[x(x+1)-\frac{x}{x+1}\right]\right\}N^4$$
$$+N^2\left[-\left(1-\frac{q}{2}\frac{x}{x+1}\right)(1-q)(1+\cos^2\vartheta)\right.$$
$$+\frac{q^2}{4}x^2\operatorname{tg}^2\vartheta(1+\cos^2\vartheta)$$
$$\left.-\left(1-q\frac{x}{x+1}\right)\sin^2\vartheta-\sin^2\vartheta qx\left(1-q\frac{x}{x+1}\right)\right]$$
$$+(1-q)\left(1-q\frac{x}{x+1}\right)$$
$$-\left(1-q\frac{x}{x+1}\right)\frac{q^2}{2}x^2\operatorname{tg}^2\vartheta,$$
$$\Lambda_2=q\sin^2\vartheta\frac{x(x+1)}{2}N^4$$
$$+N^2\left[\frac{q^2}{4}x^2\operatorname{tg}^2\vartheta(1+\cos^2\vartheta)-qx\sin^2\vartheta\left(1-q\frac{x}{x+1}\right)\right]$$
$$-\left(1-q\frac{x}{x+1}\right)\frac{q^2}{2}x^2\operatorname{tg}^2\vartheta,$$
$$N=kc/\omega.$$

a) Longitudinal propagation ($k_\perp=0$). In this case Eq. (3) splits into two familiar equations:

$$N^2=1-qx/(x+1),\quad N^2=1+i\pi^{1/2}qz_0w(z). \quad (4)$$

The first of them, which determines the refractive index for the ordinary wave, has no solutions corresponding to K modes. Let us examine solutions of Eq. (4), which determines the refractive index of the slow extraordinary wave, in the case of a dense plasma in the region of the anomalous skin effect, with $q\gg v_T/c$. In this case we can write

$$N^2=i\pi^{1/2}qz_0w(z). \quad (5)$$

In the region $|z|\ll 1$ this equation has the solution 6

$$N=\frac{3^{1/2}+i}{2}\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{c}{v_T}\frac{\omega_p{}^2}{\omega_c{}^2}\right]^{1/3}, \quad (6)$$

which corresponds to a large value of the modulus of the refractive index ($|N|\gg 1$), strong damping ($|\mathrm{Re}N|\sim|\mathrm{Im}N|$), and a penetration depth $\delta_a\sim 1/\mathrm{Im}k\sim(c^2v_r/\omega_p^2\omega_c)^{1/2}$. It is not difficult to see that under the condition $|z|\gtrsim 1$ Eq. (5) has an infinite set of other solutions with $|\mathrm{Re}N|\sim|\mathrm{Im}N|\sim 1$, corresponding to deeper penetration of the field into the plasma. Assuming $|N|\sim 1$, we find from (5), in the zeroth approximation,

$$w(z)=0. \quad (7)$$

We will first give an analytic solution of Eq. (7) for high-index modes ($m\gg 1$). For this purpose we make use of the asymptotic form of the probability integral of complex argument:

$$w(z)=2e^{-z^2}+(i/\pi^{1/2}z)(1+\ldots). \quad (8)$$

This expression is valid for $|z|\gg 1$ and $|\mathrm{Im}z|\gg 1$. For the solutions of Eq. (7), $z=z_m=x_m-iy_m$ ($m=1,2,3\ldots$), we then find

$$x_m=\pm(y_m+\Delta_m), \quad (9)$$
$$y_m=\left\{\frac{1}{2}\ln\frac{\exp(2m\pi-\pi/4)}{2[\pi\ln(2^{-1/2}\pi^{-1/2}\exp(2m\pi-\pi/4))]^{1/2}}\right\}^{1/2}, \quad (10)$$
$$\Delta_m=(2m\pi-2y_m{}^2-\pi/4)(2y_m)^{-1}. \quad (11)$$

Table I compares the first six pairs of analytic and numerical solutions of Eq. (7). We see from this table that although Eqs. (9)–(11) are strictly applicable only at large values of m they can be used within an error $\leqslant 4\%$ even at $m=1$. The error decreases with increasing mode index. [The data in this table show that the roots $z_1^\pm$, $z_2^\pm$, and $z_6^\pm$, of Eq. (7) were found in Refs. 1 and 2, while the solutions given in Ref. 3 for $\vartheta=0$ correspond to z_2^-, z_4^-, and z_6^-. The parameter values which were were chosen in Ref. 3 were such that solutions of Eq. (4) can be found from (7) within 0.01.] Using the zeroth-approximation solutions (9)–(11), we find the following expressions for the refractive indices of the K modes in the case $q\gg v_T/c$:

$$N_m=N_m{}^{(0)}[1-(N_m{}^{(0)2}-1)/2qz_0z_m], \quad (12)$$

where

TABLE I.

Root	Eqs. (9)–(11)	Numerical calculation	Root	Eqs. (9)–(11)	Numerical calculation
$z_1^\pm$	$\pm 2.07-i1.33$	$\pm 1.99-i1.35$	$z_4^\pm$	$\pm 3.71-i3.28$	$\pm 3.69-i3.29$
$z_2^\pm$	$\pm 2.73-i2.16$	$\pm 2.69-i2.18$	$z_5^\pm$	$\pm 4.12-i3.72$	$\pm 4.11-i3.73$
$z_3^\pm$	$\pm 3.26-i2.77$	$\pm 3.24-i2.78$	$z_6^\pm$	$\pm 4.48-i4.12$	$\pm 4.48-i4.12$

$$N_m^{(0)}=(\omega-\omega_c)c/2^{1/2}\omega v_T z_m,\quad m=1, 2, \ldots$$

From (12) we find the following order-of-magnitude expression for the penetration depths:

$$\delta_m\sim\frac{c}{\omega\,\mathrm{Im}\,N}\sim\frac{v_T}{(\omega-\omega_c)\,\mathrm{Im}(1/z_m)}. \tag{13}$$

The condition $|N|\sim 1$ at $|z|\sim 1$ is reached under the condition $(\omega-\omega_c)/\omega\sim v_T/c$. We accordingly have the following expression for the ratio δ_m/δ_a:

$$\delta_m/\delta_a\sim(qc/v_T)^{1/3}/\mathrm{Im}\,(1/z_m). \tag{14}$$

In other words, the K modes penetrate a distance into the plasma considerably greater than the depth of the anomalous skin effect. The penetration depth of the K modes at a given frequency increases with increasing mode index. The solutions which have been found for K modes, (12), should accordingly be taken into account in the problem of the field distribution in a plasma under conditions of the anomalous skin effect.

We turn now to the case of a low-density plasma: $q\ll v_T/c$. If $q|z_0|\ll 1$, then it follows from (4) that in the zeroth approximation we have $N=1$, and in the next approximation, $N=1+\Delta N(\Delta N\ll 1)$, we find the known expression[6] for the complex increment in the refractive index for the extraordinary wave:

$$\Delta N=(i\pi^{1/2}/2)\,qz_0w(z). \tag{15}$$

If the wave frequency is given, expression (15) determines the damping rate $\varkappa=\mathrm{Im}\Delta N$ and the increment caused in the real part of N by the presence of the plasma. If, on the other hand, the wave number k is given, then we have $\omega(k)=kc$ in the zeroth approximation, and in the next approximation we have $\omega=kc+\Delta\omega$, $\Delta N=\Delta\omega/\omega$, and $z=(kc-\omega_c+\Delta\omega)/2^{1/2}kv_T$. Equation (15) is a transcendental equation for $\Delta\omega$ in this case. If $q\ll(v_T/c)^2$, then we can ignore $\Delta\omega/kv_T\ll 1$ in (15) and we can set $z=(\omega-\omega_c)c/2^{1/2}\omega v_T$. Equation (15) then determines $\Delta\omega$ explicitly. In particular, the damping rate is $\gamma/\omega=(\pi/8)^{1/2}q\times(c/v_T)\exp(-z^2)$. If, on the other hand, we have $q\sim(v_T/c)^2$, then we have $\gamma/\omega\sim|\mathrm{Re}\Delta\omega/\omega|\sim qc/v_T$ in order of magnitude. In addition to this solution, which corresponds to the usual mode, Eq. (15) has solutions which correspond to K modes. To find these solutions under the condition $q\ll v_T/c$, we use the asymptotic expression (8). Introducing $z_l=x_l-iy_l$ $(|x_l|<y_l)$, we find

$$\begin{gathered}\omega_l=\omega_c(1+2^{1/2}k_\parallel v_T z_l/\omega_c),\quad x_l=(\pi/2+l\pi)(2y_l)^{-1},\\ y_l^2={}^1/_2\{\ln(-1)^{l+1}A+[(\ln(-1)^{l+1}A)^2+(\pi/2+l\pi)^2]^{1/2}\},\\ A=N_c(N_c^2-1)[(2\pi)^{1/2}q(c/v_T)]^{-1},\quad N_c=kc/\omega_c,\end{gathered} \tag{16}$$

where for $N_c>1$ we have $l+1=2m$, while for $N_c<1$ we have $l=2m$, $m=0,\pm 1,\ldots$. Equations (16) become inapplicable at $N_c\sim 1$, i.e., in the region where the usual mode and the K mode "intersect." In this case we have, in place of (4),

$$-az=i\pi^{1/2}w(z),\quad a=4v_T^2/qc^2\sim 1, \tag{17}$$

whose asymptotic analytic solutions are

$$\begin{gathered}\omega_l=\omega_c(1+2^{1/2}k_\parallel v_T z_l/\omega_c),\quad z_l=x_l-iy_l,\quad x_l=\pm(y_l+\Delta_l),\\ y_l=\left\{\frac{1}{2}\ln\left(\frac{a\exp(2l\pi+\pi/4)}{(2\pi)^{1/2}}\left[\frac{1}{2}\ln\frac{a\exp(2l\pi+\pi/4)}{(2\pi)^{1/2}}\right]\right)\right\}^{1/2}\\ \Delta_l=(2l\pi+\pi/4-2y_l^2)(2y_l)^{-1}\quad l=0, 1, 2,\ldots\end{gathered} \tag{18}$$

The difference $\omega-\omega_c$ and the value of γ found from (16) and (18) are on the order of the values in (1). During longitudinal propagation, in the case of either the anomalous skin effect or a low-density plasma, Eq. (4) thus contains, in addition to the known solutions, a family of new solutions which lie on side of the electron cyclotron frequency.

b) Oblique propagation $(k_\perp\neq 0)$. Assuming $k_\perp\sim k_\parallel$ for the estimates below, we find in the case of a dense plasma $(q\gg v_T/c)$ that the dispersion relation (3) contains solutions of K modes of two types. For one type, $|\mathrm{Re}z|$ is approximately equal to $|\mathrm{Im}z|$ and depends weakly on $k_\perp$; in the limit $\vartheta\to 0$, this type converts into (9)–(11). The other solution depends strongly on $k_\perp$; for it we have $|\mathrm{Re}z|<|\mathrm{Im}z|$. For the first solution in (3), the first term is the largest. Making use of this circumstance, we can put (3) in the form $w(z)\Lambda_0=0$; i.e., $w(z)=0$ or $\Lambda_0=0$. For the solutions $w(z)=0$ we find expressions (9)–(11), in which we need to make the substitution $k\to k_\parallel$. [The numerical solutions in Ref. 3 for "oblique" propagation are approximately the same as the solutions z_1^-, z_2^-, and z_3^-, found from Eqs. (9)–(11) and Table I.] At fixed values of ω and ϑ, the equation $\Lambda_0=0$ yields known expressions[6] for the refractive indices for ordinary and extraordinary waves. If we now seek solutions of this equation for ω for given $k_\perp$ and $k_\parallel$ we can set $\omega=\omega_c+\Delta\omega$ $(|\Delta\omega|\ll\omega_c)$, and we can expand Λ_0 in a series in $\Delta\omega$:

$$\Lambda_0(\omega, k_\perp, k_\parallel)=\Lambda_0(\omega_c, k_\perp, k_\parallel)+(\partial\Lambda_0/\partial\omega)_{\omega=\omega_c}\Delta\omega=0. \tag{19}$$

The resulting expression for $\Delta\omega$ holds only for those values of $k_\perp$ and $k_\parallel$ for which $\Lambda_0(\omega_c, k_\perp, k_\parallel)$ is close to zero. Assuming that $k_\perp$ is given, we find the following expression for the values $k_{\parallel 0}=k_{\parallel 0}(k_\perp)$ which cause $\Lambda_0(\omega_c, k_\perp, k_\parallel)$ to vanish:

$$k_{\parallel 0}^2=-\frac{\omega_c^2}{c^2}\frac{(N_\perp^2-N_1^2)(N_\perp^2-N_2^2)}{N_\perp^2-2N_1^2}, \tag{20}$$

$$N_\perp^2=\sin^2\vartheta N^2,\quad N_1^2=1-q,\quad N_2^2=2-q,\quad q=\omega_p^2/\omega_c^2.$$

Expanding $\Lambda_0(\omega_c, k_\perp, k_\parallel)$ around $k_{\parallel 0}$, we find from (19)

$$\Delta\omega=-\left(\frac{\partial\Lambda_0}{\partial k_\parallel}\bigg/\frac{\partial\Lambda_0}{\partial\omega}\right)_{\omega=\omega_c, k_\parallel=k_{\parallel 0}}\Delta k_\parallel,\quad \Delta k_\parallel=k_\parallel-k_{\parallel 0}. \tag{21}$$

For the solutions of the second type, the first and last terms on the left side of Eq. (3) are the largest. Correspondingly, Eq. (3) becomes

$$\Lambda_0+i\pi^{1/2}qz_0w(\mu^2/4)(1-q-\sin^2\vartheta N^2)=0. \tag{22}$$

The solution of this equation, $z_l=x_l-iy_l$ $(|x_l|<y_l)$, takes the following form when we use the asymptotic expression (8):

$$\begin{gathered}\omega_l=\omega_c(1+2^{1/2}k_\parallel v_T z_l/\omega_c),\quad x_l=(\pi/2+l\pi)(2y_l)^{-1},\\ y_l^2={}^1/_2\{\ln(-1)^l B+[(\ln(-1)^l B)^2+(\pi/2+l\pi)^2]^{1/2}\},\\ B=\Lambda_0[q(\pi/8)^{1/2}\mu^{3/2}(1-q-\sin^2\vartheta N^2)\,\mathrm{tg}\,\vartheta]^{-1}.\end{gathered} \tag{23}$$

For $B>0$ we have $l=2m$ here, while for $B<0$ we have $l=2m-1$, $m=0,\pm 1,\ldots$. Adopting the parameter values[3] $\mu^{1/2}=0.03$, $k_\parallel v_T/\omega_c=0.01$, $v_T/c=0.0442$, and $q=2.25$ for definiteness, we find from (23) $z_0=-0.26-2.99i$, $z_1=-1.22-3.22i$ and $z_1=0.77-3.07i$. (These solutions were not given in Ref. 3.) Comparison of these solutions with solutions (9)–(11) shows that the frequencies of the solutions of the second type lie considerably closer to ω_c, while their damping is slightly greater than the damping of the corresponding solutions of the first type.

The dispersion relations $w(z)\Lambda_0 = 0$ and (22) and, correspondingly, their solutions are not valid at $\Delta k_\parallel / k_\parallel \ll v_T/c$. In this case we find from Eq. (3)

$$i\pi^{1/2}w(z)\left[\frac{q}{2}z_0\frac{\partial\Lambda_0}{\partial k_\parallel}\Delta k_\parallel + z\left(\Lambda_2+\frac{q}{2}\omega_c\frac{\partial\Lambda_0}{\partial\omega}\right)\right] +\Lambda_1+q^2(i\pi^{1/2}z_0w)^2(\mu^2/8)(1-q-\sin^2\vartheta N^2)=0. \quad (24)$$

Equation (24) also has solutions of two types, for which analytic expressions can be found in the limit $\Delta k_\parallel / k_\parallel \ll v_T/c$. In place of (7) we then have[6]

$$i\pi^{1/2}zw(z)=C\equiv-\Lambda_1\left(\Lambda_2+\frac{q}{2}\omega_c\frac{\partial\Lambda_0}{\partial\omega}\right)^{-1} \quad (25)$$

Solution (25) at $|z|\gg 1$ and $|\mathrm{Im} z|\gg 1$ can be put in the following form with the help of the asymptotic expression (8):

$$\omega_l=\omega_c(1+2^{1/2}k_\parallel v_T z_l/\omega_c),$$
$$z_l=x_l-iy_l,\quad x_l=\pm(y_l+\Delta_l),$$
$$y_l=\left\{\frac{1}{2}\ln\left(\exp\left(l\pi-\frac{\pi}{4}\right)(C+1)(-1)^l\left\{2(2\pi)^{1/2}\left[\frac{1}{2}\cdot\ln\frac{\exp(l\pi-\pi/4)(C+1)(-1)^l}{2(2\pi)^{1/2}}\right]^{1/2}\right\}^{-1}\right)\right\}^{1/2}, \quad (26)$$
$$\Delta_l=(l\pi-2y_l^2-\pi/4)(2y_l)^{-1}.$$

With $C+1>0$ we have $l=2m$, $m=1,2,\ldots$; with $C+1<0$ we have $l=2m+1$, $m=0,1,\ldots$. As an example we consider the case $q=1$, $k_\perp = k_\parallel$, $N=1$. We then have $C=0.5$, and from (26) we find $z_1 = 1.97-1.4i$ and $z_2=2.67-2.21i$. A numerical solution of Eq. (25) yields similar values: $z_1=1.92-1.41i$ and $z_2=2.65-2.22i$. With increasing index m, the deviation of the analytic solution from the numerical solution becomes even smaller. Equation (25) has yet another solution with $|z|<1$. For this particular example, an analytic calculation using the expansion $w(z) = 1+2i\pi^{-1/2}z - z^2$ leads to $z = -0.23i$, while the numerical solution of (25) yields $z = -0.22i$. This solution is the continuation of solution (21) into the region $\Delta k_\parallel / k_\parallel \ll v_T/c$, and for the continuation of solution (23) into the same region

$$\omega_l=\omega_c(1+2^{1/2}k_\parallel v_T z_l/\omega_c),\quad z_l=x_l-iy_l,$$
$$x_l=l\pi/2y_l,\quad y_l=\{{}^1/_2\ln(-1)^l a[{}^1/_2\ln(-1)^l a]^{1/2}\}^{1/2}, \quad (27)$$
$$a=\left(\Lambda_2+\frac{q}{2}\omega_c\frac{\partial\Lambda_0}{\partial\omega}\right)\left[q^2\left(\frac{\pi}{8}\right)^{1/2}\mu(1-q-N_\perp^2)\mathrm{tg}^2\vartheta\right]^{-1}$$

With $a>0$ we have $l=2m$, while with $a<0$ we have $l=2m+1$, $m=0, \pm 1, \pm 2, \ldots$.

We turn now to the case of a low-density plasma: $q \ll v_T/c$. In this case, Eq. (3) splits into two equations, the first of which,

$$N^2=1+q(2-N^2\sin^2\vartheta)i\pi^{1/2}z_0w(z)/2,$$

becomes the same as (4) if we make the replacement $q\to q(2-N^2\sin\vartheta)/2$ and if we replace k by $k_\parallel$ in the argument of w. Consequently, the solutions of this equation are described by Eqs. (15)–(18), in which we need to make the same replacements. The second equation which follows from (3) is of the form of (22), where $\Lambda_0 = (N^2-1)(N^2\sin^2\vartheta - 1)$. For $\omega = kc+\Delta\omega$ and $|\mathrm{Im} z|\ll 1$ it describes an ordinary wave. In the case $|\mathrm{Im} z|\sim 1$ this equation has solutions in the form of K modes as in (23), with

$$B=\Lambda_0[q(\pi/8)^{1/2}\mu^{3/2}(1-N^2\sin^2\vartheta)\mathrm{tg}\,\vartheta]^{-1}.$$

3. WAVES NEAR HARMONICS OF THE ELECTRON CYCLOTRON FREQUENCY ($n\geqslant 2$) IN A PLASMA OF ARBITRARY DENSITY

We write the dielectric tensor in the form

$$\varepsilon_{11}=\varepsilon_1+q\frac{n^2e^{-\mu}I_n}{\mu}\varphi_n,$$
$$\varepsilon_{22}=\varepsilon_1+qe^{-\mu}\left[\frac{n^2I_n}{\mu}+2\mu(I_n-I_n')\right]\varphi_n,$$
$$\varepsilon_{12}=i\varepsilon_2-iqne^{-\mu}(I_n-I_n')\varphi_n,$$
$$\varepsilon_{13}=q\frac{k_\perp}{k_\parallel}\frac{ne^{-\mu}I_n}{\mu}(x-n)\varphi_n, \quad (28)$$
$$\varepsilon_{23}=iq\frac{k_\perp}{k_\parallel}e^{-\mu}(I_n-I_n')\varphi_n,$$
$$\varepsilon_{33}=\varepsilon_3+q\frac{k_\perp^2}{k_\parallel^2}\frac{e^{-\mu}I_n}{\mu}(x-n)^2\varphi_n,$$
$$\varepsilon_1=1-\frac{\omega_p^2}{\omega^2-\omega_c^2},\quad \varepsilon_2=\frac{\omega_p^2\omega_c}{\omega(\omega^2-\omega_c^2)},\quad \varepsilon_3=1-\frac{\omega_p^2}{\omega^2},$$
$$q=\frac{\omega_p^2}{\omega^2},\quad \varphi_n=2i\pi^{1/2}z_0\exp(-z_n^2),$$

where $|z_n|\gg 1$ and $|\mathrm{Im}\, z_n|\gg 1$. For the usual ordinary and extraordinary waves with frequencies close to $n\omega_c$, the damping is small, $|\mathrm{Im} z_n|\ll 1$, and the tensor ε_{ij} is again given by (28) if we make the replacement $\phi_n\to\phi_n/2$.

In this case the dispersion relation for electromagnetic cyclotron K waves splits into two equations:

$$\varphi_n^{(1)}=-D_0/D_1\equiv\xi_{1n}, \quad (29)$$

$$\varphi_n^{(2)}=-D_1/D_2\equiv\xi_{2n}. \quad (30)$$

Here

$$D_0=A_0N^4+B_0N^2+C_0,$$
$$A_0=\varepsilon_1\sin^2\vartheta+\varepsilon_3\cos^2\vartheta,$$
$$B_0=-\varepsilon_1\varepsilon_3(1+\cos^2\vartheta)-(\varepsilon_1^2-\varepsilon_2^2)\sin^2\vartheta,$$
$$C_0=\varepsilon_3(\varepsilon_1^2-\varepsilon_2^2),$$
$$D_1=q(n^2\mu^{n-1}/2^n n!)[N^4\sin^2\vartheta-\varepsilon_3(1+\cos^2\vartheta)N^2+2(\varepsilon_1-\varepsilon_2)(\varepsilon_3-N^2\sin^2\vartheta)],$$
$$D_2=q^2[\mu^{2n}n^2/2^{2n}(n!)^2(n+1)](\varepsilon_3-N^2\sin^2\vartheta).$$

The solutions of Eqs. (29) and (30) can be written in the form

$$\omega^{(i)}=n\omega_c(1+2^{1/2}k_\parallel v_T z_n^{(i)}/\omega_c),$$
$$z_n^{(i)}=x_n^{(i)}-iy_n^{(i)},$$
$$x_n^{(i)}=(\pi/2+l\pi)(2y_n^{(i)})^{-1},\quad i=1,2,$$
$$y_n^{(i)2}=\frac{1}{2}\left\{\ln\frac{(-1)^{l+1}\xi_{in}}{2\pi^{1/2}z_0}+\left[\left(\ln\frac{(-1)^{l+1}\xi_{in}}{2\pi^{1/2}z_0}\right)^2+\left(\frac{\pi}{2}+l\pi\right)^2\right]^{1/2}\right\}. \quad (31)$$

For $\xi_{in}<0$ we have $l=2m$, while for $\xi_{in}>0$ we have $l=2m+1$, $m=0,\pm 1,\pm 2,\ldots$.

It can be seen from (31) that near each harmonic of the electron cyclotron frequency ($n\geqslant 2$), and also near the fundamental electron cyclotron frequency, there are two sets of K modes, described by Eqs. (31) with 1, 2. One of them

($i = 1$) borders on electrostatic waves at $kc \gg \omega$; the second corresponds to definitely nonelectrostatic waves. Jain and Christiansen[5] have numerically found the solution $\omega \approx 3\omega_c$ at $m = -1$ in the particular case of electrostatic waves and for a set of parameter values under which the condition $\xi_{1n} > 0$ holds. In the case of nonelectrostatic waves, Matsuda[4] has found a solution of the equation near the second harmonic for a set of parameter values under which the condition $\xi_{1n} < 0$ holds. That solution leads to numerical results which agree with (31): $\omega^{(1)} \approx 2\omega_c$ for $m = -3, -4, -5$.

The approximate dispersion relation (29) becomes inapplicable if the frequency of the K mode, $\omega \approx 2\omega_c$, is close to the frequency of longitudinal plasma waves, ω_+, for which we have $A_0 = 0$, since in this case the condition $|z_2| \gg 1$ is violated. Assuming $|N| \gg 1$, we then find from Ref. 6 the following equation, in place of (29):

$$\left[A_0 + i(2\pi)^{1/2} \frac{v_T}{c} N \frac{\sin^4\vartheta}{\cos\vartheta} qw(z) \right] N^2 + B_0 = 0, \quad (32)$$

$$A_0 = (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)[\omega^2(\omega^2 - \omega_c^2)]^{-1},$$
$$B_0 = (\omega^2 - \omega_p^2)(2\omega_p^2 + \omega_c^2 - \omega^2)[\omega^2(\omega^2 - \omega_c^2)]^{-1},$$
$$\omega_\pm^2 = {}^1/_2(\omega_p^2 + \omega_c^2) \pm {}^1/_2[(\omega_p^2 + \omega_c^2)^2 - 4\omega_p^2\omega_c^2\cos^2\vartheta]^{1/2},$$
$$z = (\omega - 2\omega_c)(2^{1/2} k_\| v_T)^{-1}.$$

If $A_0 \ll v_T N/c$, then Eq. (32) has the known solution[6] $N = (i \pm 3^{1/2})|a|^{1/3}/2$ in the region $|z| < 1$, while in the region with $|z| \gg 1$ and $|\mathrm{Im} z| \gg 1$ an asymptotic solution of (32) corresponding to a K mode is possible:

$$\omega_l = 2\omega_c(1 + 2^{1/2} k_\| v_T z_l/\omega_c),$$
$$z_l = x_l - iy_l, \quad x_l = (\pi/2 + l\pi)(2y_l)^{-1},$$
$$y_l^2 = \frac{1}{2}\left\{ \ln\frac{(-1)^l a}{2\pi^{1/2}} + \left[\left(\ln\frac{(-1)^l a}{2\pi^{1/2}} \right)^2 + \left(\frac{\pi}{2} + l\pi \right)^2 \right]^{1/2} \right\},$$
$$a = B_0 \left[2^{1/2} q \frac{v_T}{c} N^3 \frac{\sin^4\vartheta}{\cos\vartheta} \right]^{-1} \quad (33)$$

For $a > 0$ we have $l = 2m$, while for $a < 0$ we have $l = 2m + 1$, $m = 0, \pm 1, \pm 2, \ldots$.

With $2\omega_c \approx \omega_+$ and $A_0 \sim v_T N/c$, Eq. (32) again has solutions in the form of K modes, but the relation $|N| \gg 1$ holds for them, so the depth to which they penetrate into the plasma is smaller than given by (29) and (30), for which we have $|N| \sim 1$.

4. CONCLUSION

This study of the general dispersion relation for electromagnetic waves in a plasma in a magnetic field in the long-wave limit ($k_\perp \rho_L \ll 1$), shows that in addition to the usual wave branches corresponding to ordinary, extraordinary, and plasma waves there are two other groups of wave branches: K modes, with frequencies which lie close to $n\omega_c$. Simple asymptotic expressions have been derived for the frequencies and damping rates of these waves with high indices. Numerical solutions which have been carried out show that these asymptotic expressions also apply to low-index K modes. Analytic results derived in particular cases correspond to the results of previous numerical calculations.[1–5] The frequencies and damping rates are given in order of magnitude by Eqs. (1).

The origin of the electron cyclotron K modes is quite different from that for the weakly damped electron cyclotron modes, which stem from finite-Larmor-radius effects, and the damping is exponentially weak. For K modes, the governing factor is the interaction of the waves with resonant particles. Finite-Larmor-radius effects are important for K modes with $\omega \approx n\omega_c$ ($n \geqslant 2$) and for one type of K modes, with $\omega \approx \omega_c$, which are similar to magnetostatic waves. For K modes with $\omega \approx \omega_c$, which are approximately electrostatic waves, the finite-Larmor-radius effects are inconsequential.

The K modes may prove important in the excitation of waves by external sources with frequencies $\omega \approx n\omega_c$. For example, the penetration of an electromagnetic wave into a dense plasma is determined by a cyclotron K mode.

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