Macroscopic manifestations of a chiral anomaly in a gravitational field

A. D. Dolgov, V. I. Zakharov, and I. B. Khriplovich

Institute of Nuclear Physics, Siberian Branch, Academy of Sciences of the USSR

(Submitted 13 August 1987)

Zh. Eksp. Teor. Fiz. **94**, 45–49 (February 1988)

A chiral anomaly is shown to lead to several macroscopic effects for a rotating gravitating object. In particular, a "vacuum condensate" $\langle \mathbf{EH} \rangle$ arises, where **E** and **H** are the electric and magnetic fields. The magnitude of this condensate falls off as a power of the distance from the object. The corresponding vacuum currents are also found. Arguments are presented to support the assertion that for rotating black holes the anomaly leads to the emission of massless particles: gravitons, photons, and neutrinos. As a result of this process, the black hole loses its angular momentum. Parametrically, the effect is comparable to the Hawking effect.

The axial current a_μ of massless Weyl fermions is not conserved in an external gravitational field[1,2]:

$$\partial_\mu a^\mu = -\frac{1}{384\pi^2} R\tilde{R}, \tag{1}$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, and $\tilde{R}_{\mu\nu\alpha\beta} = \varepsilon_{\mu\nu\lambda\sigma} R^{\lambda\sigma}_{\ \ \alpha\beta}/2$. The simplest physical entity for which this anomaly is nonzero, when even classical fields are substituted in, is a rotating gravitating object. It is thus natural to ask what the macroscopic consequences of a chiral anomaly would be. We will discuss three such effects here: 1) the appearance of an $F\tilde{F}$ condensate near even electrically neutral rotating objects ($F_{\mu\nu}$ is the electromagnetic stress tensor, and $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}/2$; 2) vacuum currents of a dipole type; and 3) the emission of massless particles, which results in a loss of angular momentum from rotating black holes.

1. The existence of an $F\tilde{F}$ condensate follows from the analog of Eq. (1) for photons[3]:

$$\partial_\mu K^\mu = -\frac{1}{96\pi^2} R\tilde{R}, \tag{2}$$

where the current $K^\mu = -\varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta$ is an analog of the axial current a^μ in the sense that the expectation value over the single-particle state of the operator, $\int d^3x K_0$ is equal to $+1$ or -1 for a left-hand or a right-hand photon (A_ν is its vector potential). By virtue of the operator identity $\partial_\mu K^\mu = F\tilde{F}/2$ we find from (2)

$$F\tilde{F} = \frac{1}{48\pi^2} R\tilde{R}. \tag{3}$$

For a uniform rotating object of radius r_0 we find, in the weak-field approximation,

$$R\tilde{R} = \begin{cases} 0, & r<r_0 \\ 36 r_g^2 (\mathbf{ra}) r^{-8}, & r>r_0 \end{cases}. \tag{4}$$

Here r_g is the gravitational radius, and $\mathbf{a} = \mathbf{M}/m$, where **M** is the angular momentum of the object, and m is its mass. From the quantitative standpoint, of course, relation (3) could be important only for black holes of small radius, $r_g \lesssim 10^{-13}$ cm.

A relation analogous to (3) holds for a gluon condensate, leading to an induced θ term. However, an external gravitational field with $R\tilde{R} \neq 0$ by itself leads to significant CP-violation effects.

2. We move on to the next step: We calculate not the divergence but the vacuum current itself. We first note that Eqs. (1) and (2) can be written in a common form:

$$\partial_\mu j^\mu_{(s)} = A_{(s)}(\mathbf{r}), \tag{5}$$

where the anomalous term $A_{(s)}(r)$ has the form $A_{(s)} = -s^2 R\tilde{R}/96\pi^2$, for neutrinos and photons, and s is the helicity of the particle. (This relation may not hold for other spins.) In the static case, Eq. (5), $\operatorname{div} \mathbf{j}_{(s)} = \mathbf{A}_{(s)}(\mathbf{r})$, has precisely the same form as the electrostatic equation $\operatorname{div} \mathbf{E} = \rho$.

The general solution of Eq. (5) is

$$\mathbf{j}_{(s)}(\mathbf{r}) = \frac{1}{4\pi} \nabla \int \frac{d^3r' A_{(s)}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{j}_0(\mathbf{r}), \tag{6}$$

where $\mathbf{j}_0$ is the conserved current, and $\operatorname{div} \mathbf{j}_0 = 0$. Ignoring a possible nonzero current $\mathbf{j}_0$ for the time being, we find, in the case of the source (4),

$$\mathbf{j}_{(s)}(\mathbf{r}) = \frac{s^2 r_g^2}{48\pi^2} \begin{cases} \mathbf{a} r_0^{-6}, & r<r_0 \\ \frac{2}{r_0^3} \frac{r^2\mathbf{a} - 3\mathbf{r}(\mathbf{ra})}{r^5} - \frac{r^2\mathbf{a} - 6\mathbf{r}(\mathbf{ra})}{r^8}, & r>r_0. \end{cases} \tag{7}$$

Vacuum currents thus arise around a rotating gravitating object and decay as r^{-3} with distance. We wish to stress, however, that this analogy with electrostatics is not complete, since there is no condition $\operatorname{curl} \mathbf{j} = 0$. In particular, there exists a solution of Eq. (2) which falls off as r^{-2} at large r:

$$\mathbf{j}_{(s)}(\mathbf{r}) = -\frac{3s^2}{32\pi^2} r_g^2 \frac{\mathbf{r}(\mathbf{ra})}{r^4} \left(\frac{1}{r_0^4} - \frac{1}{r^4} \right). \tag{8}$$

This solution describes the emission of particles. Since $\mathbf{j}_{(s)}$ is the difference between the currents of the left-hand and right-hand particles, more left-hand particles are emitted into the lower hemisphere ($\mathbf{ra} < 0$), and more right-hand particles are emitted into the upper hemisphere ($\mathbf{ra} > 0$).

We thus see that Eq. (5) by itself, without any additional physical considerations, cannot answer the question about the emission of particles. In the case of a weak external field, there is apparently no such emission. The situation here is largely similar to pair creation in a Coulomb field. The corresponding instability arises in the solution of the Dirac equation only if $Z\alpha > 1$, regardless of the mass of the particles which are created. The physical meaning of this condition is simple: The energy $Z\alpha/\lambda$, which is the energy of the interaction of a wave packet localized near a Coulomb center in a region with a minimum size λ, must be greater than the energy of a particle at infinity, $\omega \approx 1/\lambda$. In precisely the same way, the energy of the gravitational interaction of the spin of a particle with the rotation of a star, $r_g a/r^3$, must be greater than $1/\lambda$ here. In the case of a weak field, with $r \gg r_g$, this condition clearly does not hold.

3. We thus arrive at the problem of the physical manifestations of the anomaly in the field of a rotating black hole, with a length scale r_0 which is equal to r_g. We need to examine problems involving the formulation of correct boundary conditions at $r = r_g$ and the more technical problem of dealing with the curvature of space-time in Eq. (5). We instead consider the following simplified model. We return to expression (7) for the vacuum current. This expression essentially corresponds to the situation in which at each point in which the anomaly $A(\mathbf{r})$ is nonzero there is an isotropic current source of intensity $A(\mathbf{r})$. A vector addition of the currents flowing away from various points with $A(\mathbf{r}) \sim (\mathbf{ar})$ leads to the dipole asymptotic behavior (7) or the absence of emission.

Our simplified model of the black hole can be summarized by saying that we require total absorption of the current at the surface $r = r_g$. In other words, in the integral in expression (6) we eliminate not only $r < r_g$ but also the shadow region (Fig. 1). Furthermore, at $r > r_g$ we ignore the difference between the metric and a plane metric, incorporating the curvature only in the anomaly itself. The exact expression for it in the Kerr metric is

$$\frac{1}{2}\varepsilon^{\varkappa\lambda\rho\sigma}\frac{1}{(-g)^{1/2}}R_{\rho\sigma}{}^{\mu\nu}R_{\mu\nu\varkappa\lambda} = 12 r_g^2 a r\cos\theta \times \frac{(3r^2-a^2\cos^2\theta)(r^2-3a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^6}. \quad (9)$$

(In deriving this equation it is convenient to use the expression from Ref. 4 for the components of the Riemann tensor in the tetrad formalism.) For simplicity, we also discard terms of higher order in a; i.e., we actually use expression (4) at $r > r_g$. The asymptotic current is then

$$j_{(s)} = \frac{1}{4\pi}\int d\mathbf{r}'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}A_{(s)}(\mathbf{r}'). \quad (10)$$

This expression tends asymptotically toward the following expression at large r:

$$\mathbf{j}_{(s)} = -\frac{3s^2}{128\pi^2}\frac{1}{r_g^2}\frac{\mathbf{r}(\mathbf{ra})}{r^4}, \quad (11)$$

where we have retained only the terms which fall off as r^{-2}. This model thus has a flux of neutrinos and photons at infinity. The particles of right-hand helicity are emitted preferentially along the angular momentum of the black hole, and those of left-hand helicity in the opposite direction. Clearly, the same effect occurs for gravitons, although the coefficient of the corresponding anomaly has yet to be calculated explicitly by anybody.

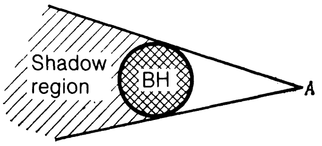

FIG. 1. Range of integration in expression (6) for the case of a black hole. A is the observation point.

An important point is that both the left-hand particles emitted opposite the angular momentum of the hole and the right-hand particles emitted parallel to the angular momentum carry off spin angular momentum of the same sign. As a result of this emission, the rotation of the star slows down.

The loss of angular momentum from a rotating black hole due to the emission of particles was studied in Refs. 5 and 6. The phenomenon which we are analyzing here leads to an effect on the same order of magnitude, and the emission itself is parametrically the same as Hawking radiation. Nevertheless, it may be that these phenomena are physically different. At least in the approximation used here, the intensity at which the particles are created depends on their spin trivially, only through the coefficient s^2 in expression (5). In the Hawking mechanism, the spin dependence is completely different.[7] The creation of particles due to the anomaly is also distinct from the known process of superradiation, since the latter does not occur for neutrinos.[4,8] It should be noted in this connection that according to a numerical integration of the Dirac equation in the field of a Kerr black hole[7] the emission of neutrinos does not vanish even in the limiting case $a = r_g/2$, in which the surface temperature of the black hole goes to zero, and the Hawking radiation correspondingly vanishes.

We believe that these results are a manifestation of the anomalous creation of neutrinos which we have been discussing here.

The relationship between a chiral anomaly in an electromagnetic field and the emission of fermions was studied in Refs. 9 and 10 for the case of a dyon with an electric charge Q and a magnetic charge g. As a result of the emission, the dyon loses its electric charge, converting into a magnetic monopole. Analogously, in the model discussed above angular momentum is emitted by a black hole. Let us pursue this analogy a bit further.

An important consideration for the emission by a dyon is the circumstance that both the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}$ are nonzero, and the $\mathbf{EH}$ anomaly is nonzero. The interaction of a magnetic monopole with a fermion is strong because the magnetic charge g is large. For this reason, as we know, the s wave may contain only outgoing particles or only incoming particles, depending on the orientation of the magnetic moment of the fermion with respect to the magnetic field of the dyon. There is no dependence on the electric charge, since particles with opposite charges and helicities have an identical magnetic moment. This degeneracy is lifted by the electric charge of the dyon. Correspondingly, the Coulomb energy of the dyon falls off because of the emission of charged particles. A strong magnetic field is important for this process, since it is specifically the magnetic interaction which couples the particle with the corresponding direction of the magnetic moment near the monopole.

Looking at this problem from the anomaly point of view, we see that it disappears for the field of a monopole (since $\mathbf{E} = 0$), and it is nonzero and sign-definite for the field of a dyon. In the latter case, particles with a definite chirality are accordingly emitted. The anomaly is quadratic in the charges of the fermions, however, and the assertion that particles with an electric charge of a definite sign are emitted follows from additional considerations.

In the case of a black hole, its strong gravitational field is analogous to the magnetic interaction of a dyon, since it is

TABLE I.

Dyon	Kerr black hole
$F\tilde{F} \neq 0$	$R\tilde{R} \neq 0$
$\int F\tilde{F} d^3r \neq 0$	$\int R\tilde{R} d^3r = 0$
Magnetic interaction	Gravitational interaction
Fall to center	Capture at horizon
Coulomb interaction	Interaction of the spin of the particle with the rotation of the black hole

specifically the gravitational interaction which results in a capture of particles at the Schwarzschild sphere. The relatively weak gravitational interaction of the spin of the particles with the angular momentum of the black hole makes the absorption of particles with a definite sign of the spin projection onto the rotation axis of the black hole preferred from the energy standpoint. For a rotating black hole, the chiral anomaly is nonzero ($R\tilde{R} \neq 0$), but nothing can be concluded about emission solely from information on the anomaly, since the anomaly does not introduce any distinction between the emission and absorption of particles with opposite helicities. If, however, the conclusion that particles are created can be drawn on the basis of some additional physical considerations, then a study of the anomaly might lead to the conclusion that the black hole loses angular momentum. The correspondence between a dyon and a Kerr black hole is illustrated by Table I. We might add that the anomalous nonconservation of the fermion number against a cosmological background with $R\tilde{R} \neq 0$ was studied in Ref. 11.

In addition to current (11), there are vacuum currents which arise in the succeeding orders in r_g/r and which are similar to (7) in that they do not emit particles. The effect is a sort of quantum "hair" on a rotating black hole, which falls off in a power-law fashion with distance. There are, of course, other physical phenomena which lead to a condensate of fields which also falls off as a power of r. In particular, the quantum corrections to the energy-momentum tensor of massless fields fall off as r^{-4} (Ref. 12, for example). The vacuum currents which we have been discussing here fall off as r^{-3}.

We wish to thank M. B. Voloshin, P. V. Vorob'ev, V. N. Gribov, I. Yu. Kobzarev, V. A. Novikov, L. B. Okun', and B. L. Chernyak for extremely valuable discussions.

[1]T. Kimura, Prog. Theor. Phys. **42**, 1191 (1969).
[2]R. Delbourgo and A. Salam, Phys. Lett. **B40**, 381 (1972); T. Eguchi and P. O. Freund, Phys. Rev. Lett. **37**, 1251 (1976).
[3]A. D. Dolgov, V. P. Zakharov, and P. B. Khriplovich, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 511 (1987) [JETP Lett. **45**, 511 (1987)].
[4]S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford Univ. Press, New York, 1983.
[5]D. A. Leahy and W. G. Unruh, Phys. Rev. **D19**, 3509 (1979).
[6]A. Vilenkin, Phys. Rev. **D20**, 1807 (1979).
[7]D. N. Page, Phys. Rev. **D14**, 3260 (1976).
[8]W. G. Unruh, Phys. Rev. Lett. **31**, 1265 (1973); Phys. Rev. **D10**, 3194 (1974).
[9]A. S. Blaer, N. H. Christ, and Ju-Fei Tand, Phys. Rev. Lett. **47**, 1364 (1981); Phys. Rev. **D25**, 2128 (1982).
[10]G. C. Callan, Phys. Rev. **D25**, 2141 (1982).
[11]G. W. Gibbons, Phys. Lett. **B84**, 431 (1979).
[12]V. P. Frolov and A. I. Zel'nikov, in: *Proceedings of the Third Seminar on Quantum Gravity*, Moscow, 1984 (ed. M. A. Markov, V. A. Berezin, and V. P. Frolov), World Sci., 1984, p. 413.

Translated by Dave Parsons