Quantum mechanics: problems intermediate between exactly solvable and completely unsolvable

A. V. Turbiner

Institute of Theoretical and Experimental Physics
(Submitted 20 July 1987)
Zh. Eksp. Teor. Fiz. **94**, 33–44 (February 1988)

For certain exactly solvable quantum mechanical problems, we have found associated quasisolvable problems for which the first N states are known explicitly. The corresponding potentials are nontrivial and contain several free parameters, the known eigenfunctions and energy eigenvalues being related to one another by analytic continuation. A known exactly solvable problem emerges as $N \to \infty$. We note a finite radius of convergence for the strong-coupling expansion of the spectrum. A quasisolvable problem has been found for the class of periodic potentials which go into a Mathieu potential in the limit $N \to \infty$.

1. INTRODUCTION

The importance of the existence of exactly solvable problems[1] in quantum mechanics is undisputed. A variety of physical processes are modeled after these problems, but the models are usually fairly gross ones which do not reflect many of the important properties of the phenomena being studied. This makes it necessary to consider all manner of perturbations of the exactly solvable problems and construct an appropriate perturbation theory, which has its drawbacks. This is one of the reasons for the ongoing search for new problems which can be solved exactly. It is worth pointing out that searching for exactly solvable problems is tantamount to constructing orthogonal polynomials in certain given coordinates. Each successive polynomial is usually of higher degree than its predecessor. In the present paper, we describe an unorthodox approach to this problem: we search for all orthogonal polynomials in the class of polynomials of fixed degree. These will clearly differ in the number of real roots within the region pertinent to the problem at hand, and the number of such polynomials will be finite. We thus obtain potentials for which a portion of the spectrum is known explicitly.

One important property of all known exactly solvable problems involving smooth potentials is that they all emerge from the factorization method of Infeld and Hull,[1] and admit of a simple supersymmetric extension.[2] Their spectra and wave functions are known explicitly.[1)]

Recent years have witnessed the extensive development (see Refs. 3, 4, 5–7, and citations therein) of a novel approach to finding exact solutions of the Schroedinger equation, based on the simple observation that any normalized function $\psi(x)$ can be considered the wave function for some state in a certain potential,

$$V-E=\Delta\psi/\psi, \tag{1}$$

where the constant E has the sense of an energy. In the one-dimensional case, if $\psi(x)$ has n nodes, it is the wave function for the nth excited state. Furthermore, $\psi(x)$ may be unnormalizable, in which case we use the Wronskian to find a second linearly independent solution, and can sometimes construct a normalizable function from these. We can thus find a set of potentials in which one state is known exactly. This was sufficient to construct a perturbation theory, for example, by the method of nonlinearization.[8] One important advance in the development of this approach has been the construction of a nontrivial potential in which two levels are known.[5]

In the present paper, we present two types of multiparameter problems, which we refer to as quasisolvable: firstly, those for which we have complete information on the first N levels ($N = 1, 2,\ldots$), which are related to one another by analytic continuation, and secondly, those for which we have N potentials of a certain type which differ only in the value of one parameter, with a certain single energy corresponding to the ith state of the ith potential ($N = 1, 2,\ldots$). In the limit $N \to \infty$, these become exactly solvable by the factorization method described in Ref. 1. All such problems can be used to model physical phenomena for which a knowledge of the entire spectrum is not needed.

Since these problems have nontrivial analytic properties as functions of their parameters, we briefly recall what is known of the analytic properties of eigenfunctions and eigenvalues. It was shown in Ref. 9, in the quasiclassical approximation for the potential $V = \gamma x^2 + x^4$, that 1) as a function of γ, the eigenvalues have an infinity of complex-conjugate pairs of branch points, which coalesce toward $|\gamma| = \infty$ along the ray with $\arg \gamma = \pi$, and every singularity corresponds to an intersection of eigenvalues; 2) eigenvalues of a given parity form a single Riemann sheet; 3) the integral of the square of the wave function goes to zero at a singularity. Several rigorous results were obtained in Ref. 10 along these lines, and the positions of the first branches were recently found numerically.[11] Note that the existence of singularities for finite γ will lead to a finite radius of convergence in the strong-coupling expansion for the eigenvalues and eigenfunctions. It has also been shown that analogous analytic properties obtain for other quantum mechanical problems, such as the Mathieu equation,[12,13] the potential $V = \varepsilon|x| + x^2$ (Ref. 14), certain solid-state energy band problems,[15] and the two-center Coulomb problem.[16–18]

This paper is structured as follows. In Sec. 2, we discuss one-dimensional, and in Sec. 3, two-dimensional spherically symmetric quasisolvable problems. Quasisolvable problems for periodic potentials are described in Sec. 4. Section 5 contains our hypothesis on a hierarchy of spectral problems, and our conclusions.

 0038-5646/88/020230-07$04.00

2. ONE-DIMENSIONAL CASE

We consider the one-dimensional Schroedinger equation

$$H\psi=E\psi,\quad H=-d^2/dx^2+V(x) \tag{2}$$

and perform the wave-function transformation[8]

$$\psi(x)=p(x)\exp\{-\varphi(x)\}, \tag{3}$$

where $p(x)$ is a function containing information about the zeroes of $\psi(x)$ in minimal fashion (for example, in the simplest case, $p(x)$ is an nth degree polynomial with n real roots, where we seek the $(n+1)$th eigenfunction). Taking $y=\varphi'$, Eqs. (2) and (3) give

$$y'-y^2-p^{-1}[p''-2yp']=E-V(x). \tag{4}$$

Our objective is to find a complete reduction of the fraction of the left-hand side of (4), and moreover, expressible in two terms in some variables. We emphasize that this reduction results in a one-term expression in exactly solvable problems.

We now consider specific cases.

Generalized Morse potential. We take

$$y_1=-ae^{-\alpha x}+b+ce^{\alpha x},\quad a\geqslant 0,\quad \alpha>0,\quad c\geqslant 0, \tag{5}$$

and let $p=1$. Substituting (5) into (4), we obtain

$$V_0=a^2e^{-2\alpha x}-a(\alpha+2b)e^{-\alpha x}+c(2b-\alpha)e^{\alpha x}+c^2e^{2\alpha x},\quad E_0=2ac-b^2. \tag{6}$$

For the potential (5), which depends on the parameters a, b, c, and α, we therefore know one eigenvalue, which is a unique analytic function of these parameters and is isolated from the rest of the spectrum. With the chosen parameter values, we obtain a potential which increases as $|x|\to\infty$ in which we know the ground state, since the wave function falls off and is positive. We now take $p=e^{-\alpha x}+A$, and determine A from the requirement that the potential have no singularities [see (4)] for real x. Equation (4) then gives rise to an addition to the potential (5),

$$V_1=-2\alpha ae^{-\alpha x}, \tag{7}$$

where

$$A_\pm=\{-\alpha-2b\pm[(\alpha+2b)^2+16ac]^{1/2}\}/4a,$$
$$E_{0,1}=2ac-b^2-\alpha\{\alpha+2b\pm[(\alpha+2b)^2+16ac]^{1/2}\}/2,$$

where the plus sign refers to the ground state (positive eigenfunction) and the minus sign to the first excited state in the potential (6), (7). Note that the levels and wave functions become interleaved, forming a two-sheeted Riemann surface with branch points at $(\alpha+2b)=\pm 4i(ac)^{1/2}$. It can be shown that when

$$p=A_0e^{-\alpha(N-1)x}+A_1e^{-\alpha(N-2)x}+\ldots+A_{N-1}, \tag{8}$$

the reducibility of (4) results in the addition of $V_{N-1}=-2\alpha a(N-1)e^{-\alpha x}$ to the potential (5), while the result of reduction in (4) is $V_{N-1}+\Delta E$, where ΔE is some unknown constant with the connotation of an energy correction. In solving for the coefficients A_i, we then obtain a set of N linear homogeneous equations with Jacobi matrix elements $\tilde{J}_{ik}$. The constant ΔE enters linearly with unit coefficient in the diagonal matrix elements: $\tilde{J}_{ik}=J_{ik}-\Delta E\delta_{ik}$. This set of linear equations can be viewed as the problem of determining the eigenvalues and eigenfunctions of the Jacobi matrix J_{ik}, with ΔE being an eigenvalue of J_{ik}. If the Jacobi matrix is cast in canonical form, with $J_{ii}=a_i$, $J_{i,i+1}=-b_i$, and $J_{i+1,i}=-c_i$ (see p. 28 of Ref. 19), its matrix elements in the present case will be

$$a_i=(N-i)[2b+\alpha(N-i)],\quad b_i=-2ai,$$
$$c_i=-2c(N-i+1),\quad i=1,\,2,\ldots,N. \tag{9}$$

Inasmuch as $b_ic_i>0$, all of the ΔE_i are real, as expected from the fact that the original Schroedinger operator is Hermitian. Since J is an $N\times N$ matrix, we have an Nth degree equation for ΔE with real roots. For $N=2$ and 3, the explicit form of this equation is

$$(\Delta E)^2-(2b+\alpha)\Delta E-4ac=0\quad(N=2),$$
$$(\Delta E)^3-(6b+5\alpha)(\Delta E)^2+4[(b+\alpha)(2b+\alpha)-4ac]\Delta E$$
$$+32ac(b+\alpha)=0\quad(N=3). \tag{10}$$

For any $N>1$, the roots ΔE_i are interleaved, forming an N-sheet Riemann surface, with $E_i=E_0+\Delta E_i$. For a suitable N-dependence as $N\to\infty$, we have $c\to 0$ and $c_i\to 0$. The spectrum is then interleaved, and we wind up with the well-known exactly solvable Morse potential

$$V=A\{\exp[-2\alpha(x-x_0)]-2\exp[-\alpha(x-x_0)]\}.$$

Another family of quasisolvable problems associated with the Morse potential is generated by

$$y_2=-ce^{-2\alpha x}+ae^{-\alpha x}+b,\quad c\geqslant 0,\quad \alpha>0,\quad b>0. \tag{11}$$

For $p=1$, we have the potential

$$V_0=c^2e^{-4\alpha x}-2ace^{-3\alpha x}+[a^2-2c(b+\alpha)]e^{-2\alpha x}+a(2b+\alpha)e^{-\alpha x},$$
$$E_0=-b^2, \tag{12}$$

in which the lowest eigenvalue is known, and is isolated from the rest of the spectrum. Taking the expression in (8) for $p(x)$, it can be shown that the reduction of (4) leads to

$$V_{N-1}=2c\alpha(N-1)e^{-2\alpha x}+\alpha\lambda e^{-\alpha x}. \tag{13}$$

The parameter λ is analogous to ΔE in the preceding example, and comes from the solution of the Nth degree characteristic equation for the Jacobian with

$$a_i=2a(N-i),\quad b_i=-2ci,\quad c_i=-(N-i+1)[2b+\alpha(N-i+1)],$$
$$i=1,\,2,\ldots,N, \tag{14}$$

Thus, N eigenvalues and eigenfunctions, all with the same energy $E=-b^2$ and specifying the ith state of the ith potential of the class $V=V_0+V_{N-1}$ [see (12)–(13)], become interleaved and form an N-sheet Riemann surface. As $N\to\infty$, $c\to 0$ and $b_i\to 0$ and the interleaving of the Riemann surface disappears, giving rise to the Morse potential.

There is one more family generated,

$$y_3=ce^{2\alpha x}-b+ae^{\alpha x},\quad c\geqslant 0,\quad \alpha>0,\quad b>0. \tag{15}$$

For $p=1$, we have the potential (12) with the replacements

$\alpha \to -\alpha$, $c \to -c$, and $b \to -b$. Making use of (8) for the factor preceding the exponential, we obtain by reducing the fraction in (4)

$$V_{N-1}=\alpha(\alpha+2b)(N-1)+\alpha\lambda e^{\alpha x}. \tag{16}$$

We add the constant term in (16) to the energy, obtaining

$$E_N=-b^2-\alpha(\alpha+2b)(N-1), \tag{17}$$

while λ has the same meaning as in the previous example. Here the Jacobi matrix is

$$a_i=2c(N-i), \qquad b_i=-[\alpha(N-i-1)(N-i-2)-(\alpha+2b)i],$$

$$c_i=-2c(N-i+1), \qquad i=1,\ 2,\dots N. \tag{18}$$

We find that $c \to 0$ and $c_i \to 0$ as $N \to \infty$, resulting in the Morse potential.

Generalized Peshlya-Teller potential. We take

$$y_1=a\ \mathrm{th}\ \alpha x+c\ \mathrm{sh}^2\ \alpha x,\quad c\geqslant 0,\quad \alpha>0,\quad a>0. \tag{19}$$

Substituting (19) into (4) with $p = 1$, we obtain

$$V_0=-a(a+\alpha)\mathrm{ch}^{-2}\ \alpha x-c(c+2\alpha-2a)\mathrm{ch}^2\ \alpha x+c^2\ \mathrm{ch}^4\ \alpha x,$$
$$E_0=2ac-a^2-\alpha c, \tag{20}$$

and therefore the ground state is known for the potential (20), separately from the rest of the spectrum. When $p = \mathrm{th}\alpha x$, the potential (20) is supplemented by

$$V_1=-2\alpha(a+\alpha)\mathrm{ch}^{-2}\ \alpha x,\quad E_1=2ac-a^2+\alpha c, \tag{21}$$

where E_1 is the first level in the potential (20), (21). Note that since the resulting potentials are even, the Riemann surfaces of the even and odd states are distinct. When $p = \mathrm{th}^2\alpha x + A$, the term added to (20) is $V_2 = -2\alpha(2a + 3\alpha)\mathrm{ch}^{-2}\alpha x$, and

$$A_\pm=\{-(a+c+2\alpha)\pm[(a+c+2\alpha)^2+2c(2a+3\alpha)]^{1/2}\}/(2a+3\alpha),$$

$$E_{0,2}=2ac-a^2-\alpha\{2a-c+2\alpha\pm2[(a+c+2\alpha)^2+2c(2a+3\alpha)]^{1/2}\}. \tag{22}$$

For $p = (A + \mathrm{th}^2\alpha x)\mathrm{th}\alpha x$, the addition to (20) is $V_3 = -6\alpha(a + 2\alpha)\mathrm{ch}^{-2}\alpha x$, and

$$A_\pm=\{-(a+c+4\alpha)\pm[(a+c+4\alpha)^2-3\alpha(2a+5\alpha)]^{1/2}\}/(2a+5\alpha),$$

$$E_{1,3}=2ac-a^2-\alpha\{2a-5c+2\alpha\pm[(a+c+4\alpha)^2-3\alpha(2a+5\alpha)]^{1/2}\}. \tag{23}$$

This differs from the situation for the generalized Morse potential, in that levels of the same parity form a two-sheeted surface. When

$$p=A_0\ \mathrm{th}^k\ \alpha x+A_1\ \mathrm{th}^{k-1}\ \alpha x+\dots+A_k, \tag{24}$$

the addition to (20) is $V_k = -\alpha k(\alpha k + \alpha + 2a)\mathrm{ch}^{-2}\alpha x$, and the first $N = [k/2] + 1$ states of parity $(-1)^k$ are known.

Corrections to E_0 or E_1 come from solving an Nth degree equation derived from the Jacobi matrix

$$a_i=-2\{\alpha[k^2-k(2i+1)+i^2]-ai+c(k-i)\},$$
$$b_i=-\{\alpha[k^2-k(2i+5)+(i+1)(i+2)]-2a(i+2)\},$$
$$c_i=-\alpha(k-i+1)(k-i+2),$$

where if k is even, i is odd, and vice versa. These are interleaved, forming a surface of N sheets. When $k \to \infty$, they become disentangled and $c = 0$, and we have the exactly solvable Peshlya-Teller potential $V \propto \mathrm{ch}^{-2}\alpha x$.

Another family is generated by

$$y_2=b\ \mathrm{th}^3\ \alpha x+a\ \mathrm{th}\ \alpha x,\ a>b,\ \alpha>0. \tag{25}$$

For $p = 1$, we have a potential with a known ground state:

$$V_0=-b^2\ \mathrm{ch}^{-6}\ \alpha x+b(2a+3b+3\alpha)\mathrm{ch}^{-4}\ \alpha x$$
$$-(a+3b)(a+b+\alpha)\mathrm{ch}^{-2}\ \alpha x,$$

$$E_0=-(a+b)^2. \tag{26}$$

Since (25) is an odd potential, all potentials generated will be even, and the Riemann surfaces for odd and even states will be distinct. The result of the reduction in (4) is

$$V_k=-2\alpha bk\ \mathrm{ch}^{-4}\ \alpha x+2\alpha(bk+\lambda)\mathrm{ch}^{-2}\ \alpha x. \tag{27}$$

The value of the parameter λ is derived from an equation of degree $N = [k/2] + 1$. The result is a problem in which N states of parity $(-1)^k$ with energies (26) are known in potentials like (26) and (27), and are interrelated by analytic continuation in one of the parameters a, b, or α. In the limit $k \to \infty$, we have $b \to 0$ and the problem reverts to a Peshlya-Teller potential.

Generalized harmonic oscillator. Folowing [3–7], we take

$$y=bx^3+ax,\quad b\geqslant 0,\quad a>0. \tag{28}$$

By virtue of the even parity of the resulting potentials, states of differing parity form two distinct Riemann surfaces. For $p = 1$,

$$V_0=b^2x^6+2abx^4+(a^2-3b)x^2,\ E_0=a, \tag{29}$$

and for $p = x$,

$$V_1=b^2x^6+2abx^4+(a^2-5b)x^2,\ E_1=3a. \tag{30}$$

When $p = x^2 + A$, the addition to (29) is $V_2 = -4bx^2$, and

$$A_\pm=[a\pm(a^2+2b)^{1/2}]/2b,\ E_{0,2}=3a\pm2[-(a^2+2b)^{1/2}], \tag{31}$$

while for $p = x(x^2 + A)$, the addition to (29) is $V_3 = -6bx^2$, and

$$A_\pm=[a\pm(a^2\pm6b)^{1/2}]/2b,\ E_{1,3}=5a\pm2[-(a^2+6b)^{1/2}]. \tag{32}$$

$E_{0,2}$ and $E_{1,3}$ form two-sheeted Riemann surfaces respectively, with branch points $a = \pm i(2b)^{1/2}$ and $a = \pm i(6b)^{1/2}$. When $p = A_0x^k + A_1x^{k-1} + \dots + A_k$, the addition to (29) is $V_k = -2kbx^2$; $N = [k/2] + 1$ states of parity $(-1)^k$ are known for this potential. The corrections to the energy ΔE are the roots of an Nth degree equation derived from the Jacobian

$$a_i=-2a(k-i),\ b_i=-2bi,\ c_i=-(k-i+1)(k-i+2), \tag{33}$$

where if k is even, then $i \leqslant k$ is even, and vice versa. The limit $k \to \infty$ gives $b \to 0$; the spectrum becomes disentangled, and we have the potential $V = a^2x^2$.

3. THE d-DIMENSIONAL SPHERICALLY SYMMETRIC CASE

If we treat the radial part of the d-dimensional Schroedinger operator with a spherically symmetric $V(r)$ in a similar manner, we obtain

$$y'+r^{-1}Dy-y^2-p^{-1}\{p''+[Dr^{-1}-2y]p'\}=E-V(r), \quad (34)$$

where $D=d+2l-1$; l is the orbital quantum number (the problem is formulated over a semiaxis). As in the one-dimensional case, our objective is a complete reduction of the fraction on the left-hand side of (34).

Generalized harmonic oscillator. The analog of (28) is

$$y=br^3+ar+cr^{-1},\ b\geqslant 0,\ c<(D+1)/2. \quad (35)$$

For $p=1$,

$$V_0=b^2r^6+2abr^4+[a^2-b(D+3-2c)]r^2+c(c-D+1)r^{-2},$$
$$E_0=a(D+1-2c). \quad (36)$$

For $p=r^2+A$, the reducibility requirement leads to two solutions: $A>0$ (ground state for fixed l) and $A<0$ (first excited state). The potential (36) is supplemented by $V_1=-4br^2$, and the corresponding states have energies

$$E_{0,1}=a(D-1-2c)\pm[4a^2+8b(D+1-2c)]^{1/2}, \quad (37)$$

where the lower sign refers to the ground state and the upper to the excited state. Taking

$$p=A_0(r^2)^{N-1}+A_1(r^2)^{N-2}+\ldots+A_{N-1} \quad (38)$$

for the factor preceding the exponential, the addition to the potential (36) that we obtain by reduction is $V_{N-1}=-4b(N-1)r^2$. Additions ΔE to the energy (36) are given by the roots of the characteristic equation for the Jacobi matrix

$$a_i=-4a(N-i),\ b_i=-4bi,$$
$$c_i=-2(N-i+1)(2N-2i+D+1-2c),$$
$$i=1,\ 2,\ldots,N. \quad (39)$$

These levels are interleaved, forming a Riemann surface of N sheets. In the limit $N\to\infty$, $b\to 0$ and $b_i\to 0$, the spectrum is no longer interleaved, yielding a known, exactly solvable problem (for example, see p. 158 of Ref. 20). The latter is a generalization of the harmonic oscillator in spherical coordinates to nonintegral angular momentum.

The generalized Coulomb problem. We now take

$$y_1=a+cr^{-1}+br,\quad b\geqslant 0,\quad c<(D+1)/2. \quad (40)$$

With $p=1$ (ground state), we have the potential

$$V_0=b^2r^2+2abr-a(D-2c)r^{-1}+c(c-D+1)r^{-2},$$
$$E_0=b(D+1-2c)-a^2, \quad (41)$$

in which the lowest state is isolated from the rest of the spectrum. For $p=r+A$, the energy is $E_1=b(D+3-2c)-a^2$. The addition to the potential (41) is

$$V_1=-\{a\pm[a^2+2b(D-2c)]^{1/2}\}r^{-1}, \quad (42)$$

where the plus sign corresponds to the ground state, the minus sign to the first excited state. Taking

$$p=A_0r^{N-1}+A_1r^{N-2}+\ldots+A_{N-1} \quad (43)$$

as the factor preceding the exponential, it can be shown that reducing the fraction in (34) yields

$$V_{N-1}=-2b(N-1)+\lambda r^{-1}. \quad (44)$$

The constant term in (44) can be ascribed to the energy

$$E_N=-a^2+b(D-1+2N-2c), \quad (45)$$

while the parameter λ can be found by solving the Nth degree characteristic equation for the Jacobi matrix

$$a_i=-2a(N-i),\quad b_i=-2bi,\quad c_i=-(N-i+1)(N-i+D-2c),$$
$$i=1,\ 2,\ldots,N. \quad (46)$$

The resulting problem is that of N potentials coupled by analytic continuation, for which the ith state of the ith potential corresponds to the energy (45). For $N\to\infty$, $b\to 0$ and $b_i\to 0$, the potentials are no longer interleaved, giving the known, exactly solvable Kratzer potential (see p. 157 of Ref. 20), which generalizes the Coulomb potential to nonintegral angular momenta.

Another quasisolvable Coulomb problem arises when

$$y_2=a+cr^{-1}-br^{-2},\ b\geqslant 0,\ a>0. \quad (47)$$

When $p=1$,

$$V_0=b^2r^{-4}-b(2c-D+2)r^{-3}$$
$$+[c(c-D+1)-2ab]r^{-2}-a(D-2c)r^{-1}$$
$$E_0=-a^2. \quad (48)$$

If we take the expression in (43) for $p(r)$, it can then be shown that reduction of the fraction in (34) yields

$$V_{N-1}=-2a(N-1)r^{-1}+\lambda r^{-2}. \quad (49)$$

The parameter λ makes its appearance as one of the roots of the characteristic equation for the Jacob matrix

$$a_i=(N-i)(N-i+D-1-2c),\quad b_i=-2ai,\quad c_i=-2b(N-i+1). \quad (50)$$

This is analogous to the situation in the previous problem. As $N\to\infty$, b$\to 0$ and $c_i\to 0$, and we obain the exactly solvable Kratzer potential. Note that this problem was investigated by Korol'[21] for $a=0$.

4. QUASISOLVABLE PROBLEMS AMONG PERIODIC POTENTIALS

We have thus described quasisolvable problems on the line $(-\infty,+\infty)$ and the semi-infinite line segment $[0,\infty)$ in Secs. 2 and 3. We now move on to periodic potentials. Let

$$y=a\sin 2\alpha x. \quad (51)$$

It is well known (see Refs. 12 and 13, for example) that there are four possible forms for the factor preceding the exponential in (3):

$$p_1=A_0\cos^N 2\alpha x+A_1\cos^{N-1}2\alpha x+\ldots+A_N, \quad (52)$$

$$p_2=\sin 2\alpha x(A_0\cos^{N-1}2\alpha x+A_1\cos^{N-2}2\alpha x+\ldots+A_{N-1}), \quad (53)$$

$$p_3=\cos\alpha x(A_0\cos^{N-1}2\alpha x+A_1\cos^{N-2}2\alpha x+\ldots+A_{N-1}), \quad (54)$$

$$p_4=\sin\alpha x(A_0\cos^{N-1}2\alpha x+A_1\cos^{N-2}2\alpha x+\ldots+A_{N-1}). \quad (55)$$

Consider states of the first type, as in (52). Letting $p=1$ and substituting (51) into (4), we obtain

$$V_0=-a^2\cos^2 2\alpha x-2a\alpha\cos 2\alpha x,\quad E_0=-a^2, \quad (56)$$

and we thereby find the eigenfunction and the energy of the lowest state of this type in the periodic potential (56). Now let the factor before the exponential be $p=\cos 2\alpha x+A$. Reducing the fraction in (4) gives the condition

$$A_\pm=[\alpha\pm(\alpha^2+4a^2)^{1/2}]/2a, \quad (57)$$

with the plus sign corresponding to the ground state ($|A_+|>1$), and the minus sign to excited states ($|A_-|<1$). The potential (56) is supplemented by

$$\Delta V_1=-4a\alpha\cos 2\alpha x, \quad (58)$$

while the corresponding energies are

$$E_\pm=-a^2+2\alpha^2\mp 4\alpha(\alpha^2+4a^2)^{1/2}. \quad (59)$$

In the general case of (52), the addition to the potential (56) is $\Delta V_N=-4a\alpha N\cos 2\alpha x$, giving rise to the potential

$$V_{N+1}=-a^2\cos^2 2\alpha x-2a\alpha(2N+1)\cos 2\alpha x, \quad (60)$$

in which $N+1$ states are known, being related to one another by analytic continuation in the parameters a and α, and forming an $(N+1)$-sheet Riemann surface; the wave functions are of the form

$$\psi=p\exp\{-(a/2\alpha)\cos 2\alpha x\}. \quad (61)$$

We next move to a consideration of states of the second type, as in (53). Letting $p=\sin 2\alpha x$, the addition to (56) is the same as (58), and we thus know one state of the second type in this potential, with energy

$$E_1=-a^2+4\alpha^2, \quad (62)$$

as well as two states of the first type with energies (59). In the general case of (53), we obtain the potential (60), in which we know N states of the second type, forming an N-sheet Riemann surface in a or α. Furthermore, in this potential, we known the $(N+1)$th state of the first type.

The simplest state of the third type, as in (54), is $p=\cos\alpha x$. The addition to the potential is

$$\Delta V_1=-2a\alpha\cos 2\alpha x, \quad (63)$$

and as a result, we know the lowest-lying state of the third type in the potential

$$V_1=-a^2\cos^2 2\alpha x-4a\alpha\cos 2\alpha x,\quad E_0=-a^2-a\alpha+\alpha^2/4, \quad (64)$$

where E is the energy. The potential

$$V_N=-a^2\cos^2 2\alpha x-4a\alpha N\cos 2\alpha x, \quad (65)$$

[compare (60)] with N interleaved states of the third type, is obtained when $p=p_3$ [see (54)]. Exactly the same potential arises in the problem of N states of the fourth type.

We thus see that $2N+1$ states are known in the potential (60), with $N+1$ of these ascribed to the first type, and the rest to the second. In the potential (65), we know $2N$ states, of which N are of the third type and N of the fourth. Only the states of a given type interleave, forming a Riemann surface with a finite number of sheets. We point out that in all cases, the polynomial coefficients A_i [see (52)–(55)] in the set of linear equations are of the same form as in type one and type two solutions (see Secs. 2 and 3), although the matrix for this system ceases to be Jacobian, and instead consists of a four-wide diagonal band.

In the limit $N\to\infty$, we obtain the same Mathieu potential

$$V_\infty=-4a\alpha\cos 2\alpha x \quad (66)$$

as for both (60) and (65), and this is in fact not exactly solvable. In contrast to the foregoing problems of Secs. 2 and 3, therefore, the states remain interleaved; this has been confirmed by numerical calculations.[12,13] The matrix of the system determining the A_i is Jacobian; the exponential factor in the wave functions (61) vanishes, and we have the standard representation for the Mathieu functions.

We have unfortunately not been able to generalize our results to arbitrary quasimomentum and derive energy band structures.

5. CONCLUSION

In Secs. 2 and 3, we have thus constructed quasisolvable problems of two kinds. We first recount problems of the first kind, for which the first N eigenfunctions and eigenvalues are known, and for which information about the remaining states can be obtained by approximate methods:

$$V=a^2e^{-2\alpha x}-a[2b+\alpha(2N-1)]e^{-\alpha x}+c(2b-\alpha)e^{\alpha x}+c^2e^{2\alpha x},$$
$$\psi=p_{N-1}(e^{-\alpha x})\exp\left\{-\frac{a}{\alpha}e^{-\alpha x}-bx-\frac{c}{\alpha}e^{\alpha x}\right\}; \quad (\mathrm{I})$$

$$V=c^2\,\mathrm{ch}^4\,\alpha x-c(c+2\alpha-2a)\,\mathrm{ch}^2\,\alpha x$$
$$-[a(a+\alpha)+\alpha k(\alpha k+\alpha+2a)]\,\mathrm{ch}^{-2}\,\alpha x, \quad (\mathrm{II})$$

$$\psi=p_k(\mathrm{th}\,\alpha x)(\mathrm{ch}\,\alpha x)^{-a/\alpha}\exp\left\{-\frac{c}{4\alpha}\mathrm{ch}\,2\alpha x\right\};$$

$$V=b^2x^6+2abx^4+[a^2-b(2k+3)]x^2,$$
$$\psi=p_k(x)\exp\left\{-\frac{bx^4}{4}-\frac{ax^2}{2}\right\}; \quad (\mathrm{III})$$

$$V=b^2r^6+2abr^4+[a^2-(4N+D-1-2c)b]r^2+c(c-D+1)r^{-2},$$

$$\psi=p_{N-1}(r^2)r^{l-c}\exp\left\{-\frac{br^4}{4}-\frac{ar^2}{2}\right\}. \quad (\mathrm{IV})$$

We have also found quasisolvable problems of a second kind, for which the ith state of the ith potential ($i=1,2,\ldots,N$) always has the same energy E_N:

$$V=b^2r^2+2abr-[a(D-2c)+\lambda]r^{-1}+c(c-D+1)r^{-2},$$

$$\text{(V)}$$

$$E_N=b(2N+D-1-2c)-a^2,\quad \psi=p_{N-1}(r)r^{l-c}\exp\{-ar-br^2/2\};$$

$$V=b^2r^{-4}+b(D-2-2c)r^{-3}+[c(c-D+1)-2ab+\lambda]r^{-2}$$

$$-a(2N+D-2-2c)r^{-1}, \qquad \text{(VI)}$$

$$E_N=-a^2,\quad \psi=p_{N-1}(r)r^{l-c}\exp\{-ar-br^{-1}\};$$

$$V=c^2e^{-4\alpha x}-2ace^{-3\alpha x}+[a^2-2c(b+\alpha N)]e^{-2\alpha x}+(2ab+a\alpha+\lambda)e^{-\alpha x},$$

$$\text{(VII)}$$

$$E_N=-b^2,\quad \psi=p_{N-1}(e^{-\alpha x})\exp\left\{-\frac{c}{2\alpha}e^{-2\alpha x}+\frac{a}{\alpha}e^{-\alpha x}-bx\right\};$$

$$V=c^2e^{4\alpha x}+2ace^{3\alpha x}+[a^2+2c(b+\alpha)]e^{2\alpha x}+(2ab+a\alpha+\lambda\alpha)e^{\alpha x},$$

$$\text{(VIII)}$$

$$E_N=-b^2-\alpha(N-1)(\alpha+2b),$$

$$\psi=p_{N-1}(e^{-\alpha x})\exp\left\{-\frac{c}{2\alpha}e^{2\alpha x}-\frac{a}{\alpha}e^{\alpha x}+bx\right\};$$

$$V=-b^2\,\mathrm{ch}^{-6}\,\alpha x+b[2a+3b+\alpha(2k+3)]\,\mathrm{ch}^{-4}\,\alpha x$$
$$-[(a+3b)(a+b+\alpha)+2k\alpha b+\lambda]\,\mathrm{ch}^{-2}\,\alpha x,$$

$$\text{(IX)}$$

$$E_N=-(a+b)^2,$$

$$\psi=p_k(\mathrm{th}\,\alpha x)(\mathrm{ch}\,\alpha x)^{-(a+b)/\alpha}\exp\left(\frac{b}{2\alpha}\mathrm{th}^2\,\alpha x\right),$$

where $N=[k/2]+1$. The energies of the states E_i and the quantities λ_i, $i=1,2,...,N$ for (I)–(IV) and (V)–(IX) respectively are the roots of certain Nth degree algebraic equations, which can be solved explicitly when $N<5$. These equations are the characteristic equations for certain $N\times N$ Jacobi matrices. The potentials (I)–(IX) each have one parameter less than the corresponding potentials in general form.

We wish to emphasize that for the classes of polynomials in $(e^{-\alpha x}, e^{\alpha x})$, $(\mathrm{ch}^2\,\alpha x, \mathrm{ch}^{-2}\,\alpha x)$, x, or (r,r^{-1}), no other potentials exist for which the foregoing approach to the construction of states with $N>2$ is applicable. Attempts to generalize (I)–(IX) [or what amounts to the same thing, to generalize (5), (11), (15), (19), (25), (28), (35), (40), and (47)] while keeping to the representation (3) all violate the requirement that the quotient in (4) and (34) be a two-term expression. Different states then emerge in different potentials, being related to one another by analytic continuation.

An attempt to generalize our method to a multidimensional problem[22] has also failed: it proved to be impossible to construct nontrivial excited states except in those instances where the coordinates were separable.

Notice that our technique for constructing quasisolvable problems is an unorthodox approach to the construction of orthogonal polynomials. Usually, having specified a weighting function, one begins constructing orthogonal polynomials of increasing degree. In the present case, we fix the degree of a certain polynomial, and seek orthogonal polynomials of that fixed degree. These will clearly differ from one another in the number of real roots, and there will be a finite number of them. It is therefore understandable why for the problem over the entire line [see (III), for example] we could only construct even or odd states: a polynomial with real coefficients has an even number of complex roots.

We wish to point out the importance of the potentials (VI), (VII), and (IX) for possible physical applications. The potential (VI) can model atomic interactions where charges interact with each other at small distances, and charged atoms interact at large distances. The potential (VII) can be useful in molecular physics to describe interactions between diatomic molecules.

In closing, we now attempt a classification of spectral problems. As noted in the Introduction, in exactly solvable problems at the differential-equation level, every level crosses every other in the complex parameter plane (except in those cases where they are prevented from doing so by symmetry considerations). For those problems in which the determination of the spectrum reduces to the solution of a set of transcendental equations, eigenvalues exist which are not interleaved (as shown in Ref. 23, for example, where the kth level is interleaved only with levels $k-1$ and $k+1$). For exactly solvable problems, the spectrum is wholly noninterleaved, and there is no crossing of levels. For quasisolvable problems, a part of the spectrum can be identified in which the levels are interleaved among themselves, and these remain isolated from the remainder. We may introduce a symbolic symmetric matrix T in which $T_{ij}\neq 0$ when states i and j are interleaved, and $T_{ij}=0$ otherwise.

We propose a hierarchy among spectral problems: 1) for exactly solvable problems at the differential-equation level, the matrix T has no vanishing elements; 2) for exactly solvable problems reducible to transcendental equations containing only elementary and special functions, there exist $T_{ij}=0$; 3) for quasisolvable problems, T is in block form along the main diagonal; 4) for exactly solvable and nonparametric problems, T is a diagonal matrix.

Note that when the original Hamiltonian belongs to a symmetry group, this hierarchy refers to states of a definite symmetry. Our assertion applies only to the one-dimensional and spherically symmetric problems (4) and (34), but the hypothesis may possibly be valid for multidimensional problems as well.

Further investigations[24] have shown that the quasisolvability of problems (I)–(IX) and (60), (65) is related to the existence of finite-dimensional representations of the group $SL(2,C)$. Any of these problems can be represented as a quantum spinning top in a constant magnetic field.

In closing, we wish to express our deepest gratitude to K. A. Ter-Martirosyan, A. G. Ushveridze, and especially M. A. Shubin for helpful discussions and their interest in this endeavor.

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Translated by M. Damashek