Stability of conoidal waves in media with positive and negative dispersion

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The stability of conoidal waves in media with positive dispersion is investigated. The stability of conoidal waves described by the Kadomtsev-Petviashvili equation with negative dispersion is rigorously proven, and the completeness of the corresponding set of eigenfunctions is established. For the anisotropic model [V. E. Zakharov and E. A. Kuznetsov, Sov. Phys. JETP **39**, 285 (1974). A. B. Mikhaĭlovskiĭ, S. V. Makurin, and A. I. Smolyakov, *ibid.* **62**, 928 (1985)], it is shown that conoidal waves are unstable regardless of the sign of the dispersion. The instability limits and the maximum growth rates are determined in the extreme case of strongly and weakly linear waves.

§1. INTRODUCTION

Wave propagation with account taken of nonlinearity and of weak dispersion is described for many media by universal evolutional equations, the best known of which is that of Korteveg–de Vries (KdV). A multidimensional generalization of the KdV equation is that of Kadomtsev and Petviashvili (KP)[1]

$$\frac{\partial}{\partial x}(u_t+6uu_x+u_{xxx})=-3\beta^2\Delta_\perp u. \qquad (1.1)$$

The function $u(\mathbf{r}, t)$ describes here the perturbation in the medium, say the velocity, $\Delta_\perp = \partial^2/\partial y^2 + \partial^2/\partial z^2$, and the coefficient β^2 determines whether the medium as positive ($\beta^2(0)$ or negative ($\beta^2>0$) dispersion. Examples of waves with weak dispersion, which can be described by the KP equation, are ion-acoustic, magnetosonic, high-frequency drift waves in a plasma and gravitation-capillary waves on shallow water.

In the case of substantially anisotropic media one encounters another two-dimensional generalization of the KdV equations:

$$u_t+6uu_x+u_{xxx}=\sigma\Delta_\perp u_x, \qquad (1.2)$$

obtained by Zakharov and Kuznetsov[2] to describe magnetized ion sound ($\sigma<0$).

The simplest and most thoroughly investigated one-dimensional solutions of Eqs. (1.1) and (1.2) are solitary waves (solitons) and periodic ones (conoidal). These are the main subjects of the theory of nonlinear waves in dispersive media.

Soliton stability was investigated in detail in many studies (see the review by Zakharov *et al.*[3] and the references therein). It was established that in the KP model the solitons are stable in the case of a negative dispersion ($\beta^2>0$) and unstable for a positive ($\beta^2<0$) one.[1,4] In the isotropic model (1.2), the soliton stability is also determined by the sign of the function σ: in the physically important case of positive dispersion ($\sigma<0$) the solitons are thus unstable.[5]

The stability of conoidal waves was first considered on the basis of the KdV equation by Witham,[6] who obtained the frequencies of three stable acoustic modes. The stability of nonlinear waves, without assuming the perturbation to be small, was investigated (likewise for the KdV equation) by Kuznetsov and Mikhaĭlov.[7] In the one-dimensional case, conoidal waves are stable also to finite perturbations.

Perturbations with transverse modulation superimposed on periodic waves (within the framework of the KP equation) were first considered by Infeld,[8,9] who obtained a dispersion equation cubic in frequency. It was concluded correctly in Refs. 8 and 9 that in the long-wave limit the conoidal waves are stable or unstable whenever the solitons are stable or unstable.

The papers cited, except Refs. 4 and 7, were based on different perturbation-theory schemes (adiabtic approximation[7] or direct expansion in powers of small quasimoments and small transverse wave numbers[8,9]). For nonlinear equations, however, which lend themselves to the Lax commutation representation (particularly KdV and KP), the inverse scattering problem method (ISPM)[10] leads to substantial progress in the investigation of the stability of the exact solutions to arbitrary small perturbations. This circumstance was used by Zakharov in Rev. 4. The connection between the inverse scattering problem and the problem of the stability of solutions in integrable models was subsequently analyzed and recast in a much simpler form by Kuznetsov, Fál'kovich, and the author.[11] By way of example we determined the stability of conoidal waves in the KP model to perturbations with arbitrary quasimoments q and transverse wave numbers k. Exact expressions were obtained in Ref. 11 for the perturbations, and algebraic relations that describe implicitly the dispersion relation were obtained (see §2). It was established, in particular, that in the case of positive dispersion the conoidal waves are unstable in the transverse wave-number interval $k<k_{cr}$, and the threshold value of k_{cr} was obtained. The frequencies of the constructed perturbations were found to be pure real for negative frequency dispersion.

Our conclusion that nonlinear periodic waves are stable in media with negative dispersion (within the framework of the KP model) were criticized in a recent paper by Mikhaĭlovskiĭ, Makurin, and Smolyakov,[12] who considered also other multidimensional generalizations of the KdV equation (the anisotropic, hybrid, and vector models). The question of the stability of conoidal waves was solved again in Ref. 12 in the long-wave limit, in analogy with the procedure used in Refs. 8 and 9. Through failure to track the transition of this limit in the general expressions of Ref. 11 and through errors in the solution of the resultant dispersion equation (see §4). The authors of Ref. 12 "found" unstable perturbation of highly nonlinear waves in the KP model with negative dispersion. Similar errors are contained in the analysis of the hybrid and anisotropic (1.2) models.

The connection between the stability of nonlinear waves and the sign of the dispersion of the medium is thus of principal importance. Returning to this question, we consider only the KP equation (1.1) and the anisotropic model (1.2), which are the real cases of physical importance.

We show initially in §3 that the general relations (§2) lead in the long-wave limit to Witham's results[6] on the stability of one-dimensional perturbations.

We analyze next in detail in §§ 4 and 5 the stability of conoidal waves in the KP model. In the case of negative dispersion we prove rigorously the stability of these waves, establishing that the constructed perturbations with real frequencies constitute the complete set of eigenfunctions of the linearized KP equation.

Knowledge of the complete set for any equation (say KdV or KP) permits construction of a rather compact perturbation theory, compared with the standard scheme, for the equation with small increments. This is demonstrated in §6 where we consider the stability of conoidal waves in the anisotropic model (1.2). For perturbations with large-scale transverse corrugation (small wave numbers k) the right-hand side of (1.2) can be regarded as a small addition to the KdV equation, and the solution of the linearized equation (1.2) can be sought in the form of an expansion in the eigenfunctions of the linearized KdV equation. The resultant dispersion equation is analyzed in detail in limiting case of strongly nonlinear (close to solitons) and weakly linear (close to sine) waves. In the anisotropic mode (1.2), the conoidal waves are unstable at all signs of σ. We determine the instability boundaries and the maximum values of the growth rate.

§2. GENERAL SOLUTIONS OF THE LINEARIZED KdV EQUATION

The KdV equation and its non-one-dimensional generalizations (1.1) and (1.2) have solutions in the form of a plasma periodic stationary wave

$$u_0(x-Vt)={}^1/_6V-2\wp(x-Vt+i\omega'|\omega,\omega'), \qquad (2.1)$$

where $\mathscr{P}(z|\omega,\omega')$ is a Weirestrass elliptic function with real (2ω) and imaginary $(2i\omega')$ periods ($\mathscr{P}(x+i\omega')$ is real and bounded for real x).

The KP equation (1.1) linearized against the background of the stationary solution (2.1) takes the form

$$\frac{\partial}{\partial x}\{-i\Omega u+[(6u_0-V)u+u_{xx}]_x\}=3\beta^2k^2u. \qquad (2.2)$$

We have transformed to a reference frame that moves with the velocity V of the initial wave, using the substitution $x-Vt\to x$. In view of the homogeneity in the transverse coordinates and time, the small perturbation δu was chosen in standard form:

$$\delta u(x,\mathbf{r}_\perp,t)=u(x)\exp(-i\Omega t+i\mathbf{k}\mathbf{r}_\perp). \qquad (2.3)$$

The eigenfunctions of the KP equation for small perturbations (2.2) were obtained in Ref. 11:

$$u(x)=\frac{\partial}{\partial x}\left\{\frac{\sigma(x+i\omega'+a)\sigma(x+i\omega'+b)}{\sigma^2(x+i\omega')\sigma(a)\sigma(b)}\times\exp[-(x+i\omega')(\zeta(a)+\zeta(b))]\right\}, \qquad (2.4)$$

where $\sigma(z)$ and $\zeta(z)$ are Weierstress functions. The perturbation $u(x)$ have the Bloch form, $u(x+2\omega)/u(x)=e^{2iq\omega}$, with quasimomentum

$$q=i[\zeta(a)+\zeta(b)-\zeta(\omega)(a+b)/\omega]. \qquad (2.5)$$

The relation between the spectral parameters a and b is

$$\wp(a)-\wp(b)=-i\beta k, \qquad (2.6)$$

the oscillation frequency is expressed in their term as follows:

$$\Omega=-2i[\wp'(a)+\wp'(b)]. \qquad (2.7)$$

The velocity of (2.4)–(2.7) can be verified directly by substituting the solutions (2.4) in Eq. (2.2).

Note that the solution $u(x)$ (2.4) and the frequency Ω (2.7) are doubly periodic functions of the parameters a and b, so that we can confine ourselves everywhere to consideration of the value of a and b inside the rectangle of the periods.

The requirements that the solutions be bounded in space necessitates in turn that the quasimomentum be real, $\operatorname{Im} q=0$, or

$$\operatorname{Re}[\zeta(a)+\zeta(b)-\zeta(\omega)(a+b)/\omega]=0. \qquad (2.8)$$

The two complex parameters a and b are subject to real (2.8) and complex (2.6) constraints, which are equivalent to three real conditions. The parameters a and b vary therefore on the complex plane along those curves on which the frequency Ω (2.7) is specified.

Equations (2.5)–(2.7) specify thus in implicit form the dispersion relations—the dependences of the frequency Ω on the quasimomentum q and on the transverse wave number k.

§3. ONE-DIMENSIONAL PERTURBATIONS OF NONLINEAR WAVES

For one-dimensional oscillations, $k=0$, it follows from the spectral relation (2.6) that $b=\pm a$. If $b=-a$, the frequency Ω and the quasimomentum q are zero, and the solution $u(x)$ is a neutral stable shear mode: $u(x)=\mathscr{P}'(x+i\omega')\sim u_0'(x)$. The parameter a is arbitrary in this case.

If $b=a$ the frequency and quasimomentum are respectively

$$\Omega=-4i\wp'(a), \qquad (3.1)$$

$$q=i[\zeta(a)-\zeta(\omega)a/\omega]. \qquad (3.2)$$

The parameter a must now be specified on the vertical lines $\operatorname{Re} a=n\omega$, and only on these lines is the quasimomentum q (3.2) real. But the derivative $\mathscr{P}'(a)$ is pure imaginary on these lines, and it follows from (3.1) that the oscillation frequency Ω is pure real.

It must be emphasized that the quasimomentum q specified by Eq. (3.2) or (2.5) is defined in a system of irreducible Brillouin zones and can assume arbitrary real values. Thus, when the parameter a is varied along the vertical sides of a rectangle with corners at the points $(-i\omega')$, $(\omega-i\omega')$, $(\omega+i\omega')$, $(i\omega')$ the quasimomentum q (3.2) runs through all values from $-\infty$ to $+\infty$, while q' remains continuous when the parameter a changes jumpwise:

$$q(\omega+i\omega')=q(i\omega')=\pi/\omega,\quad q(\omega-i\omega')=q(-i\omega')=-\pi/\omega.$$

At these points, the frequency Ω (3.1) is also continuous (and equal to zero, since $\mathscr{P}'(a)=0$ precisely at $a=\pm i\omega'$, $\omega\pm i\omega'$, and also at $a=\omega$, where $q=0$.

The foregoing means that a plot of the function $\Omega\,(q)$ defined by Eqs. (3.1) and (3.2) is continuous, but has kinks at the points $q=\pm\pi/\omega$ (see Fig. 1). In this case $\Omega(0)=\Omega(\pm\pi/\omega)=0$. It remains to note that at the kink points $|q|$ is precisely equal to the reciprocal-lattice period. Three acoustic oscillation modes are produced therefore in the reduced Brillouin zones, and their frequencies can be easily obtained by expanding the relations (3.1) and (3.2) near the corresponding values of the parameter a. Let

$$a_l=\omega_l+\delta a_l,\quad l=1,2,3,\quad \omega_1=\omega,\quad \omega_2=\omega+i\omega',\quad \omega_3=i\omega' \quad (|\delta a_l|\ll|\omega_l|); \tag{3.3}$$

It follows then from (3.1) and (3.2) that

$$\Omega_l=-4i\wp''(\omega_l)\delta a_l,\qquad q_l\equiv q(a_l)=q(\omega_l)+\delta q, \tag{3.4}$$

where $q(\omega_1)=0$, $q(\omega_{2,3})=\pm\pi/\omega$, and δq is the reduced quasimomentum:

$$\delta q=\frac{\partial q}{\partial a}\bigg|_{\omega_l}\delta a_l=-2i\left(e_l+\frac{\zeta(\omega)}{\omega}\right)\delta a_l\qquad\left(|\delta q|\ll\frac{\pi}{\omega}\right). \tag{3.5}$$

Eliminating δa_l from (3.4) and (3.5), we obtained the sought-for frequencies

$$\Omega_l=2\,\frac{\wp''(\omega_l)}{e_l-\overline{\wp}}\,\delta q.$$

Here and elsewhere $e_l=\mathscr{P}(\omega_l)$, $\overline{\mathscr{P}}=\langle\mathscr{P}(x+i\omega')\rangle=-\zeta(\omega)/\omega$. It is convenient to express finally the dispersion relations in terms of Jacobi elliptic functions with parameters connected with the values of e_l and $\zeta(\omega)$ as follows[13]:

$$\nu^2=e_1-e_3,\quad s^2\nu^2=e_2-e_3,\quad e_1+\zeta(\omega)/\omega=\nu^2\lambda,$$

where s and s' are the moduli of the elliptic functions $(s^2+s'^2=1)$, $\lambda=E(s)/K(s)$ is the ratio of the complete elliptic integrals of the first and second kind; we have $\lambda\approx 1-s^2/2-s^4/16$, at $s\ll 1$ and $\lambda\approx 2[\ln(16/s'^2)]^{-1}$ at $s'\ll 1$. The derivatives $\mathscr{P}''(\omega_l)$ are

$$\wp''(\omega)=2\nu^4s'^2,\quad \wp''(\omega+i\omega')=-2\nu^4s^2s'^2,\quad \wp''(i\omega')=2\nu^4s^2. \tag{3.6}$$

In the upshot we obtain the following expressions for the frequencies of the one-dimensional long-wave oscillations:

$$\Omega_1=4\nu^2\frac{s'^2}{\lambda}\delta q,\quad \Omega_2=-4\nu^2\frac{s^2s'^2}{\lambda-s^2}\delta q,\quad \Omega_3=-4\nu^2\frac{s^2}{1-\lambda}\delta q, \tag{3.7}$$

which agree fully with the results of Refs. 6 and 12.

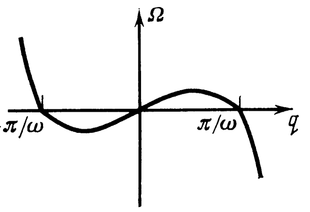

FIG. 1. Frequency of one-dimensional oscillations vs the wave number (in the irreducible band model).

§4. LONG-WAVE OSCILLATIONS IN THE KP MODEL

It follows from the spectral relation (2.6) that at small wave numbers k the parameter b is close to a or to $-a$ ($b=\pm a$ at $k=0$). The situation with a small quasimomentum (2.5) is covered by the second case. We put

$$a=a_0+\Delta,\quad b=-a_0+\Delta\quad(|\Delta|\ll|a_0|).$$

In the leading order we have [see (2.6)]

$$\Delta=-i\beta k/2\wp'(a_0). \tag{4.1}$$

The quasimomentum q (2.5) and the frequency Ω (2.7) are respectively

$$q=-2i[\wp(a_0)-\overline{\wp}]\Delta, \tag{4.2}$$

$$\Omega=-4i\wp''(a_0)\Delta. \tag{4.3}$$

Eliminating Δ from relations (4.1)-(4.3), we obtain the connection between the frequency Ω and the parameter a_0:

$$\Omega=2\,\frac{\wp''(a_0)}{\wp(a_0)-\overline{\wp}}\,q \tag{4.4}$$

and an equation that defines a_0:

$$\wp'(a_0)=-\frac{\beta k}{q}[\wp(a_0)-\overline{\wp}].$$

Squaring both halves of the last equation, we arrive at an equation cubic in $\mathscr{P}=\mathscr{P}(a_0)$:

$$(\wp-e_1)(\wp-e_2)(\wp-e_3)=(\beta k/2q)^2(\wp-\overline{\wp})^2. \tag{4.5}$$

Each of its three roots specifies, in accordance with Eq. (4.4), one of the frequencies of the small oscillations.

We plot the right- and left-hand sides of (4.5) (Fig. 2) and take it into account that $\overline{\mathscr{P}(x+i\omega')}$ is contained in the interval (e_3,e_2). It becomes obvious from Fig. 2 that in the case of negative dispersion $(\beta^2>0)$ Eq. (4.5) always has three real roots. The corresponding frequencies (4.4), are also real and attest to the stability of the long-wave perturbations.

For positive dispersion $(\beta^2<0)$, when the branches of the parabola in Fig. 2 are directed downward, two roots of Eq. (4.5) acquire imaginary increments if the ratio $(k/q)^2$ exceeds the critical value (see below). The frequencies Ω (4.4) also becomes complex (instability of periodic wave in the case of positive dispersion).

We show now that the system (4.4) and (4.5) is equivalent to the dispersion equation obtained in Ref. 12. It is convenient to introduce in place of $\mathscr{P}(a_0)$ the quantity x given by $\mathscr{P}(a_0)-\overline{\mathscr{P}}=\nu^2x$. Equation (4.5) takes then the form

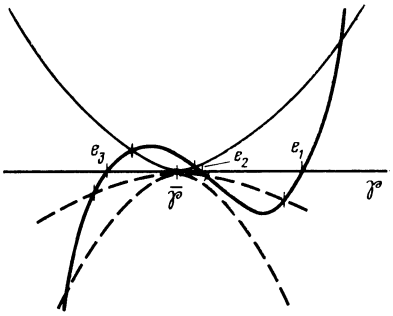

FIG. 2. Graphic solution of the dispersion equation (4.5) for small oscillations in the KP model.

$$(x-\lambda)(x-\lambda+s'^2)(x+1-\lambda)=rx^2 \qquad (4.6)$$

or $x^3+c_2x^2+c_1x+c_0=0$. We introduced here the notation $r=(\beta k/2qv)^2$, and the coefficients c_i are given respectively by

$$c_0=\lambda(\lambda-s'^2)(1-\lambda)>0, \quad c_1=3\lambda^2-2\lambda(1+s^2)+s'^2<0,$$

$$c_2=\tilde{c}_2-r, \quad \tilde{c}_2=1-3\lambda+s'^2. \qquad (4.7)$$

The frequency Ω (4.4) takes the form

$$\Omega=4qv^2(3x+2\tilde{c}_2+c_1/x). \qquad (4.8)$$

To obtain a closed equation for the frequency, we must solve Eq. (4.8) which is quadratic in x, substitute the expression for x in (4.6), and eliminate the irrationalities. We obtain the equation

$$c_0y^3+{}^1/_2(3c_0c_2+c_1^2)y^2+c_1^2c_2y+{}^3/_8(3c_0-c_1c_2)^2+{}^1/_2c_1^3=0, \qquad (4.9)$$

where $y=\Omega/8qv^2-\tilde{c}_2$. Apart from the notation, Eq. (4.9) coincides, coefficient by coefficient, with that obtained in Ref. 12 for the KP model. The generalized relations obtained by us in Ref. 11 duplicate thus in the long-wave limit the dispersion equations obtained in Refs. 8 and 12 by perturbation theory.

In the case of negative dispersion, all three roots of (4.9) are real, refuting the erroneous statement in Ref. 12. Recall that all the roots of a cubic equation with real coefficients, reduced to the standard form $y^3+py+q=0$, are real if its discriminant $(q/2)^2+(p/3)^3$ is negative. The discriminant of (4.6) or (4.5) is negative, as follows from the graphic solution (Fig. 2). On the other hand, the discriminant of (4.9) can be represented as a product of two factors, one of which is the discriminant (4.6) and the other is the square of a real quantity.

We consider next the perturbations of strongly linear waves, $s'^2 \ll \lambda \ll 1$. We note here that in Ref. 12 in the limit $s'^2 \ll 1$ the equation analogous to (4.9) retains the terms $\sim(s'^2/\lambda)^2$, but leaves out the larger terms $\sim\lambda$, and this is the cause of the erroneous conclusions. We start out from the system (4.6), (4.8), which is easier to analyze than Eq. (4.9). If the parameter r is not too small, we can neglect in (4.6) the quantity s'^2 compared with λ. The roots of (4.6) are then quantities of first or zeroth order of smallness in λ (if r is not close to unity, see below). In the leading orders we obtain

$$x_{1,2}=\lambda(1\pm r^{1/2})^{-1}, \quad x_3=r-1. \qquad (4.10)$$

The values of the frequencies, in the initial variables, are

$$\Omega_{1,2}^2=16\beta^2k^2v^2, \quad \Omega_3=-4qv^2+3\beta^2k^2/q. \qquad (4.11)$$

It should be noted that the frequencies $\Omega_{1,2}$ coincide with the soliton oscillation frequencies, and the frequency Ω_3 is equal to the natural frequency for the linear KP equation ($u_0=0$). This is easily understood by recalling that the conoidal wave (2.1) can be represented as a sum of soliton solutions:

$$\wp(x+i\omega')=C-\left(\frac{\pi}{2\omega'}\right)^2\sum_n \mathrm{ch}^{-2}\left[\frac{\pi}{2\omega'}(x-2n\omega)\right].$$

At $s'^2 \ll 1$ the real period of the wave 2ω (the distance between the soliton crests) is much larger than the imaginary one $2\omega'$ (the characteristic scale of variation of the solution near the crests), meaning that the solitons do not overlap. The perturbations with frequencies $\Omega_{1,2}$ are then localized in the vicinities of the soliton crests, while those with frequencies Ω_3 are localized in the extended regions of the constant background.

Unlike the corresponding "one-dimensional" expressions (3.7), Eqs. (4.11) take into account the transverse dispersion of the oscillations. Comparison of the expressions for the frequencies $\Omega_{1,2}$ in (4.22) and (3.7) shows that for very small wave numbers (when $r\sim(s'^2/\lambda)^2$, account must be taken of the terms $\propto s'^2$ in Eq. (4.5) whose roots must be sought in this case in the form $x=\lambda+Bs'^2$. Putting $r=C(s'^2/\lambda)^2$, we obtain $2B=-1\pm(1+4C)^{1/2}$. Equation (4.8) yields now the following frequency values:

$$\Omega_{1,2}^2=(4qv^2s'^2/\lambda)^2+16\beta^2k^2v^2 \qquad (4.12)$$

(expression (4.11) for the frequency Ω_3 remains in force). Expression (4.12) shows the manner in which a transition takes place from the one-dimensional oscillations (3.7) to the transverse ones (4.11) with increase of k. For a medium with positive dispersion ($\beta^2<0$) we get from (4.12) the threshold value

$$|\beta|^2(k/q)_{cr}^2=v^2(s'^2/\lambda)^2,$$

above which the conoidal waves become unstable if $s'^2 \ll 1$ (the same value was obtained in Ref. 12).

Finally, we obtain the frequency separation near the degeneracy point. It follows from Eqs. (4.10) for the roots of (4.5) that they no longer hold if the parameter r is close to unity, $r-1\sim\lambda^{1/2}$. We emphasize that this is possible only for a medium with negative dispersion, $\beta^2>0$. The two roots are now of order $\lambda^{1/2}$ (the third root $x=\lambda(1+r^{1/2})^{-1}$ and the frequency $\Omega=-8qv^2$ remain the same as before). Putting $r-1=2C\lambda^{1/2}$ and substituting $x=A\lambda^{1/2}$ in (4.5) we obtain $A=C\pm(C^2+2)^{1/2}$. Of course, the roots remain real in this case, too:

$$\Omega=4qv^2\{2r\pm[(r-1)^2+8\lambda]^{1/2}\}.$$

Conoidal waves in the KP model with negative dispersion are thus stable to long-wave perturbations. We shall extend this statement in §5 to include oscillations with arbitrary quasimomenta and wave numbers.

§5. ARBITRARILY SMALL PERTURBATIONS IN THE KP MODEL

We consider in this section only media with negative disperison, $\beta^2>0$, and put henceforth $\beta=1$. We ascertain first, for small k, the course of curves a and b on which the frequency is specified. As already noted, Eq. (2.6) has at small k two types of solution ($|\Delta|\ll|a_0|$):

1) $a=a_0+\Delta, \quad b=a_0-\Delta,$
2) $a=a_0+\Delta, \quad b=-a_0+\Delta,$

with Δ given by Eq. (4.1) for both cases. For the first solution, the quasimomentum q [Eq. (2.5)] has the same form (3.2) as in the one-dimensional case (with a replaced by a_0). It is real if a_0 lies on the vertical segments: Re $a_0=0, \pm\omega$. The derivative $\mathscr{P}'(a_0)$ is then pure imaginary and, as follows from (4.1), Δ is real. The parameter a_0 varies therefore along a curve close to the indicated vertical segments.

For the second solutions, as seen from Fig. 2, the quantities $\mathscr{P}(a_0)$ [the roots of (4.5)] belong to the interval $(e_1, +\infty)$ or to the segment (e_1, e_2). This means that the param-

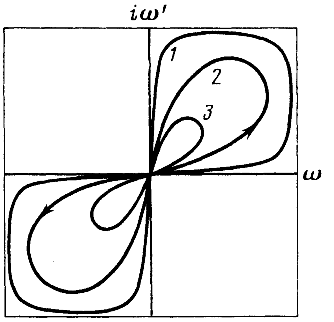

FIG. 3. Level lines Im $\mathscr{P}(a) = -k/2$ on which the frequency Ω (5.4) is specified. Curves 1, 2, 3: $k_1 < k_2 < k_3$.

eter a_0 lies now on one of the horizontal segments $\mathrm{Im} a_0 = 0, \pm \omega'$, so that Δ (4.1) is pure imaginary, and the curve a passes near these segments (Fig. 3)

Note that in the case of positive dispersion the curves a and b have a qualitatively different behavior; this case was studied in detail in Ref. 11.

The expansion (4.1) is no longer valid near the corners of the rectangles made up the indicated segments, where $\mathscr{P}'(a_0) = 0$, and the next terms of the expansion must be taken into account. Putting $a = \omega_l + \delta a$, $b = \omega_l + \delta b$, we find from (2.6) and (2.8) that individual sections of the curve a are joined near the corners by smooth transitions (Fig. 3):

$$\mathrm{Im}(\delta a)\,\mathrm{Re}(\delta a) = -k/\wp''(\omega_l).$$

We consider finally the curve a near zero. Recognizing that $\mathscr{P}(a) \approx a^{-2}$ and $\zeta(a) \approx a^{-1}$ for small a, we find from (2.6) and (2.8) that the "horizontal" and "vertical" sections of curve a reach zero along two branches:

$$\mathrm{Im}\, a = k(\mathrm{Re}\, a)^3/4, \quad \mathrm{Re}\, a = k(\mathrm{Im}\, a)^3/4.$$

It remains to note that in all cases the parameter b is either equal to $(-a^*)$ or differs from it by a period). In view of the periodic dependences of the perturbations (2.4) and of the frequency (2.7) on a and b, we can put everywhere

$$b = -a^*, \tag{5.1}$$

and then the entire a curve is given by a single formula (Fig. 3)

$$\mathrm{Im}\, \wp(a) = -k/2. \tag{5.2}$$

The quasimomentum (2.5) is then automatically real:

$$q = -2\, \mathrm{Im}[\zeta(a) - \zeta(\omega) a/\omega], \tag{5.3}$$

as are also the eigenfrequencies

$$\Omega = 4\, \mathrm{Im}\, \wp'(a) \tag{5.4}$$

[cf. (3.1), (3.2) for the one-dimensional case.]

By virtue of the continuous dependence on the transverse wave number, it is natural to assume that relations (5.1)–(5.4) remain valid also for arbitrary k. We shall make this statement rigorous by showing that perturbation of type (2.4) constitute a complete set for the indicated choice of the parameters a and b.

In investigations of orthogonality and completeness properties it is convenient to deal not with the functions $u(x)$ (2.4) themselves, but their antiderivatives:

$$\varphi_q(x) = \frac{\sigma(x+i\omega'+a)\,\sigma(x+i\omega'-a^*)}{\sigma^2(x+i\omega')\,\sigma(a)\,\sigma(a^*)} \times \exp[-(x+i\omega')(\zeta(a)-\zeta(a^*))], \tag{5.5}$$

which satisfy the equation

$$\frac{\partial}{\partial x}\{-i\Omega\varphi + (6u_0 - V)\varphi_x + \varphi_{xxx}\} = 3k^2\varphi. \tag{5.6}$$

The conjugate equation differs from (5.6) only in the sign of Ω. The eigenfunctions ψ of the conjugate problem are therefore obtained from (5.5) by reversing the sign of a: $\psi_q(x) = \varphi_{-q}(x)$.

Note that by virtue of the homogeneity in transverse directions, the planar Fourier harmonics $\exp(i\mathbf{k}\cdot\mathbf{r}_\perp)$ form a complete set with respect to the transverse coordinates, so that it suffices to consider the functions ψ and φ for a fixed value of k.

In accordance with the form of Eq. (5.6) and its conjugate, it is natural to specify the scalar product of the functions of the direct and conjugate spaces as follows:

$$\langle\langle \psi | \varphi \rangle\rangle = \frac{i}{2} \int_{-\infty}^{+\infty} dx \left[\psi(x) \frac{\partial \varphi(x)}{\partial x} - \frac{\partial \psi(x)}{\partial x} \varphi(x) \right]. \tag{5.7}$$

Of course, any two Bloch functions with different reduced quasimomenta are orthogonal. Also orthogonal are the eigenfunctions ψ and φ corresponding to different values of Ω. Thus, the eigenfunctions $\psi_{q'}(x), \varphi_q(x)$ are not orthogonal only if the following conditions are simultaneously met:

$$\Omega(q) = \Omega(q'), \quad q - q' = l\pi/\omega \tag{5.8}$$

(the presence of a part with an arbitrary integer l in the right-hand side of the last equality is due to the fact that the quasimomenta q are specified in an irreduced-band scheme). It can be shown that the conditions (5.8) are compatible only at $l = 0$, i.e., when the nonreduced quasimomenta or, equivalently, the parameters a and a' coincide. The final form of the orthogonality relation is

$$\langle\langle \psi_{q'} | \varphi_q \rangle\rangle = 2\pi\rho(q)\,\delta(q-q'), \quad \rho(q) = \frac{i}{2}\langle \varphi_{-q}\varphi_q' - \varphi_{-q}'\varphi_q \rangle. \tag{5.9}$$

(Here and elsewhere, angle brackets denote averaging over the period 2ω). Using expression (5.5) for the functions φ_q and the addition formulas for elliptic functions, we get

$$\rho(q) = \mathrm{Im}\,\{\wp'(a^*)[\wp(a) - \overline{\wp}]\}. \tag{5.10}$$

If the functions φ_k (5.5) form a complete set, the completeness condition should, in accordance with the definition (5.7) of the scalar product and with the orthogonality relation (5.9), should be of the form

$$2\pi\delta(x-x') = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) i \,⨍ \frac{dq}{2\rho(q)} \varphi_{-q}(x')\varphi_q(x) \times \left(\equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) I \right) \tag{5.11}$$

(the integral is taken in the sense of principal value).

We prove this relation by direct calculation of the integral I which it contains. We change in this integral to integration along the contour (5.2) in the complex a plane. From expression (5.3) for the quasimomentum it follows that

$$dq=-i(\wp(a)-\overline{\wp})da+i(\wp(a^*)-\overline{\wp})da^*,$$

and the differentials da and da^* are connected in turn on the curve (5.2) by the relation

$$\wp'(a)da=\wp'(a^*)da^*,$$

from which, taking (5.10) into account, we get

$$dq/2\rho(q)=-da/\wp'(-a^*).$$

Next, we gather into separate groups in the product $\varphi_{-q}(x')\varphi_q(x)$ the factors that depend on a and a^*. The sought integral takes ultimately the form

$$I=-i\int\frac{da}{\wp'(-a^*)}f(a)f(-a^*), \tag{5.12}$$

where

$$f(a)=\frac{\sigma(x+i\omega'+a)\sigma(x'+i\omega'-a)}{\sigma(x+i\omega')\sigma(x'+i\omega')\sigma^2(a)}\exp(-\Delta x\zeta(a)) \tag{5.13}$$

is a doubly periodic function of its argument a with an essential singularity at zero; the quantities x and x' are now parameters, $\Delta x = x - x'$; the integration along the contour C (5.2) in the direction marked by the arrow in Fig. 3; at zero, the integral is understood in the sense of the principal value.

The purpose of calculating the integral (5.12) is to find a closed (i.e., dependent only on a) differential form $F(a)da$ that coincides on the integration contour (5.2) with the differential form in (5.12) (which depends on both a and a^*), and use next the methods of the theory of functions of complex variable. This program can be implemented and it is found that $I = \pi[\text{sign}(x - x')]/2$. Calculation of the completeness integral (5.12), which is most complicated technically, is relegated to the Appendix. The subsequent differentiation $(\partial/\partial x - \partial/\partial x')I$ yields the desired completeness relation (5.11).

Thus, solutions (2.4) with spectral parameters of the form (5.1) and (5.2) and with real frequencies (5.4) constitute all the eigenfunction of Eq. (2.2) for the perturbations. This disposes finally of the question of instability of conoidal waves in the KP model with negative dispersion.

§6. STABILITY OF CONOIDAL WAVES IN THE ANISOTROPIC MODEL

6.1. Dispersion equation

We consider the anisotropic-model equation linearized against the background of the stationary solution u_0 (2.1):

$$-i\Omega u+[(6u_0-V)u+u_{xx}]_x=-\sigma k^2u_x. \tag{6.1}$$

The perturbations are chosen here in the same form (2.3) as in §2.

Equation (1.2) does not have a Lax commutation representation, so that the method used by us in Ref. 11 for the KP equation cannot yield solutions of the linearized equation (6.1). We confine ourselves therefore to long-wave oscillations. We regard, for small k, the right-hand side of (6.1) as a perturbation and seek the solution in the form of an expansion in the eigenfunctions of the linearized KdV equation.

As indicated in §3, in the one-dimensional case ($k = 0$) the eigenfunctions are determined by Eq. (2.4) under the condition $b = a$:

$$u_q=\frac{\partial}{\partial x}\varphi(x,a),\quad \varphi(x,a)=\frac{\sigma^2(x+i\omega'+a)}{\sigma^2(x+i\omega')\sigma^2(a)}\exp[-2x\zeta(a)]. \tag{6.2}$$

Their (non-reduced) quasimomentum q and frequency Ω are given by Eqs. (3.1) and (3.2). The eigenfunctions u of the conjugate problem, just as in §5, are given by the condition $\bar{u} = \varphi(x, -a)$. It is now natural to define the scalar product as $\int\bar{u}udx$, so that the orthogonality condition takes the form

$$\int\bar{u}_{q'}(x)u_q(x)dx=2\pi\delta(x-x')\wp'(a)[\wp(a)-\overline{\wp}]$$
$$=2\pi\delta(q-q')\rho(a). \tag{6.3}$$

The eigenfunctions (6.2) form a complete set, with the completeness condition in fact a particular case of the one obtained in §5 for the KP equation.

Since Eq. (6.1) has a Bloch-type solution, it follows that for a fixed reduced quasimomentum q the solution should be a discrete sum $u = \Sigma\, c_l u_{ql}$ of functions (6.2) whose quasimomenta q differ from q by the reciprocal lattice vector: $q_l = q + l\pi/\omega$. Substituting this sum in (6.1) and using the orthogonality relation (6.3) we obtain an infinite system of linear equations for the amplitudes c_l:

$$-ic_l(\Omega-\Omega_l)\rho(a_l)=-\sigma k^2\sum_{l_1}c_{l_1}\langle\bar{u}_lu_{l_1}'\rangle. \tag{6.4}$$

In the general situation, when the quantities $\rho(a_l)$ (6.3) are not small, only one of the coefficients c_l differs from zero in the zeroth approximation, i.e., the solution $u(x)$ is simply one of the functions (6.2). It can be shown that in this case the correction to the frequency (3.1) is real.

Of greater interest is the degenerate situation in the presence of small $\rho(a_l)$ (6.3) ($\mathscr{P}'(a_l)\sim k$). The derivative $\mathscr{P}'(a)$ vanishes at three non-equivalent points $a = \omega_l$ (3.3). At the same points, according to (3.2)–(3.5), the reduced quasimomentum of the functions q (6.2) also vanishes. Degeneracy sets is therefore at small values $q\sim k$ (here and elsewhere q denotes the reduced quasimomentum). As noted in §3, at small q there are three acoustic modes corresponding to the values of the parameter a near the three degeneracy points ω_l.

For small q, the system (6.4) reduces thus in the principal order to three equations. In the left hand sides of these equations, the eigenfrequency Ω, the characteristic frequencies Ω_l (3.7), and the normalization parameters ρ_l are quantities of first order of smallness in q, while the quantities ρ_l (6.3) are easily expressed in terms of q with the aid of Eqs. (3.3)–(3.5): $\rho_l = \frac{1}{2}iq\mathscr{P}''(\omega_l)$. The right-hand sides of the system (6.4) however, contain already the small factor k^2, so that it suffices to calculate the mean values $\langle\bar{u}_lu_{l_1}'\rangle$ in the zeroth approximation, putting $q = 0$ (or, equivalently, $a_l = \omega_l$). These mean values turn out then to be the same for all $l = 1, 2, 3$:

$$\langle\bar{u}_lu_{l_1}'\rangle=-\langle[\wp'(x+i\omega')]^2\rangle\equiv\langle\wp'^2\rangle.$$

Equations (6.4) for the three amplitudes c_l take ultimately the form

$$c_l=\frac{2\sigma k^2}{q}\langle\wp'^2\rangle\frac{1}{\wp''(\omega_l)}\frac{1}{\Omega-\Omega_l}\sum_1^3c_n.$$

Adding them and cancelling out the quantity $\Sigma c_n \neq 0$, we find the following dispersion equation, which is equivalent to the one obtained in Ref. 12 for the anisotropic model:

$$\frac{4q^2v^6}{\sigma k^2\langle\wp'^2\rangle}=\frac{1}{s'^2}\frac{1}{y-s'^2/\lambda}-\frac{1}{s^2s'^2}\frac{1}{y+s^2s'^2/(\lambda-s'^2)}+\frac{1}{s^2}\frac{1}{y+s^2/(1-\lambda)}\equiv F(y). \qquad (6.5)$$

We have used here expressions (3.6) and (3.7) for the quantities Ω_l and $\mathcal{P}''(\omega_l)$ and introduced in place of the frequency Ω the variable $y=\Omega/4qv^2$. Leaving out the general expression, we present only the limiting values of $\langle\mathcal{P}'^2\rangle$, viz, $\langle\mathcal{P}'^2\rangle\approx v^6s^4/2$, for $s^2\ll 1$ and $\langle\mathcal{P}'^2\rangle\approx(8/15)v^6\lambda$ for $s'^2\ll 1$.

It is obvious from the plot of the function $F(y)$ (Fig. 4) that in the case of positive disperison, $\sigma=-1$, Eq. (6.5) has complex roots if the ratio $(k/q)^2$ exceeds a certain critical value $\varkappa_1^2$ (s). In the case $\sigma=1$ the oblique perturbations are unstable in a certain angle interval $\varkappa_2^2<(k/q)^2<\varkappa_3^2$. The critical values $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are easiest to obtain by determining the extrema of the function $F(y)$. Omitting the calculations, we present below the values of $\varkappa_l$ in the limiting cases of strongly and weakly nonlinear waves.

We present one more dispersion relation for purely transverse perturbations, $q=0$, which can be easily obtained from (6.5) by expanding the right-hand side for large $y=\Omega/4qv^2$ ($|y|\gg 1$):

$$\Omega^2=-\frac{4\sigma k^2}{v^2}\langle\wp'^2\rangle\frac{c_1}{c_0}.$$

The values of c_1 and c_0 are given by expressions (4.7), and their ratio c_1/c_0 is always negative, since[13] $(-c_1)=3\langle[\mathcal{P}(x+i\omega')-\overline{\mathcal{P}}]^2\rangle/2v^4$.

The stability of transverse perturbations of conoidal waves (just as that of solitons) is thus determined in the anisotropic model by the sign of the dispersion σ.[5,12]

6.2 Oblique perturbations of strongly nonlinear waves ($s'^2\ll\lambda\ll 1$)

The critical values of $\varkappa_l$ are

$$\varkappa_1^2={}^{15}/_4(s'^2/\lambda)^2,\quad \varkappa_{2,3}^2={}^{15}/_4(1\pm 2\lambda^{1/2}). \qquad (6.6)$$

For positive dispersion, $\sigma<0$, almost all the oblique perturbations are unstable, except the almost-longitudinal ones $((k/q)^2<\varkappa_1^2\ll 1)$. For negative perturbations, on the contrary, instability sets in only in a very narrow angle interval near $(k/q)^2=15/4$.

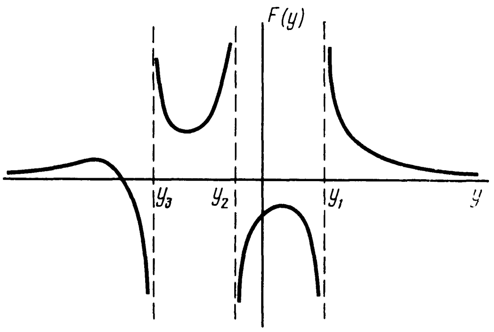

FIG. 4. Plot of the right-hand side $F(y)$ of the dispersion equation (6.5) for small oscillations in the anisotropic model $\Omega_i=4qv^2y_i$ are the frequencies (3.7) of the one-dimensional oscillations.

We obtain now the roots of Eq. (6.5) for almost longitudinal waves $(k/q)^2\sim\varkappa_1^2$ (6.6). The roots y are of the order of s'^2/λ, and it follows directly from (6.5) that

$$\Omega_{1,2}^2=(4qv^2)^2\{(s'^2/\lambda)^2+4\sigma k^2/15q^2\} \qquad (6.7)$$

in accordance with the one-dimensional expressions (3.7) and the critical value of $\varkappa_1^2$ (6.6) (at $\sigma=-1$). The third frequency is $\Omega_3=-4qv^2$.

If, however, $(k/q)^2\gg\varkappa_1^2$ and is of zeroth order, Eq. (6.5) becomes

$$\frac{15}{2}\frac{q}{\sigma k^2\lambda}=\frac{2}{\lambda y^2}-\frac{1}{y}+\frac{1}{y+1}. \qquad (6.8)$$

If two roots are given as before by expression (6.7), in which the first term can be neglected; this coincides with the dispersion relation for soliton perturbations.[5] The third root must be sought in the form $y+1=A\lambda$, hence

$$\Omega_3=4qv^2\{-1+{}^2/_{15}\sigma k^2\lambda(q^2-{}^4/_{15}\sigma k^2)^{-1}\}. \qquad (6.9)$$

Expressions (6.7) and (6.9) are valid everywhere except in a narrow degeneracy region near $\sigma(k/q)^2=15/4$ (6.6) (negative dispersion, $\sigma=1$), where the frequency Ω_3 is close to one of the frequencies $\Omega_{1,2}$. We put in this case $(q/k)^2=(4/15)(1+2C\lambda^{1/2})$. The roots of Eq. (6.8) should be sought in the form $y=-1+(1/2)B\lambda^{1/2}$. It follows then from (6.8), in the leading order, that

$$B+1/B=2C \quad\text{or}\quad B=C\pm(C^2-1)^{1/2}.$$

Instability takes place at $C^2<1$. The maximum of the growth rate is reached at $C=0((k/q)^2=4/15)$ and is equal to

$$\Gamma_{max}=2qv^2\lambda^{1/2}$$

in contrast to the result of Ref. 12.

6.3 Oblique perturbations of weakly nonlinear waves, $s^2\ll 1$

The critical values of $\varkappa_1$ are in this case

$$\varkappa_1^2=24(5\sqrt{5}-11)s^{-4}\approx 4{,}33s^{-4},\quad \varkappa_2^2=3,$$

$$\varkappa_3^2=24(5\sqrt{5}+11)s^{-4}\approx 532/s^4. \qquad (6.10)$$

For positive dispersion ($\sigma=-1$) only the almost transverse perturbations ($(k/q)^2>\varkappa_1^2$) are unstable, and for negative dispersion ($\sigma=1$) the instability region extends from almost transverse perturbations $(k/q)^2<\varkappa_3^2$ to values $(k/q)^2=3$ (cf. the conclusions for perturbations of strongly nonlinear waves). Equation (6.5) is written, with the required accuracy, in the form

$$\frac{8q^2}{\sigma k^2s^4}=\frac{1}{y-1+s^2/2}-\frac{1+s^2}{s^2}\frac{1}{y+2-7s^2/4}+\frac{1}{s^2}\frac{1}{y+2-s^2/4}. \qquad (6.11)$$

In the case $(k/q)^2\sim 1$, its roots are close to $y_1=1$ or to $y_{2,3}=-2$. With the corrections taken into account, we obtain the following expressions for the frequencies:

$$\Omega_1=4qv^2\left(1-\frac{s^2}{2}+\frac{6k^2}{8q^2}s^4\right),$$

$$\Omega_{2,3}=4qv^2\left\{-2+s^2\pm\frac{3}{4}s^2\left(1-\frac{\sigma k^2}{3q^2}\right)^{1/2}\right\}.$$

The last expression, in particular, specifies the stability boundary $\varkappa_2^2=3$ (6.10) for negative dispersion ($\sigma=1$).

The expressions obtained are not valid for almost-transverse waves, when $(k/q)^2 \sim s^{-4}$. In this case the quantities $y-1$ and $y+2$ are not small and Eq. (6.11) with $s^2 \ll 1$ is transformed into

$$\frac{1}{2r}=\frac{1}{y-1}-\frac{1}{y+2}-\frac{3}{2}\frac{1}{(y+2)^2} \qquad (6.12)$$

or

$$y^3+3y^2-4-3r(y+5)=0, \qquad (6.13)$$

where we have introduced the symbol $r=(\sigma k^2 s^4/16q^2) \sim 1$. The extrema of the right-hand side of (6.12) indeed determine the critical values $\varkappa_1^2$ and $\varkappa_3^2$ (6.10).

There are no more compact expressions for the solutions of (6.13) than the general Cardano equations for the roots of a cubic equation. It is possible, however, to obtain the exact maximum of the instability growth rate (at $\sigma = 1$). Let $y_1(r)$ and $y_2(r)$ be the real and imaginary parts of the root of (6.13), $y = y_1 + iy_2$. Let furthermore $y_2(r)$ be a maximum at the point r_0, $y_2'(r_0) = 0$. The derivative $y'(r_0) = y_1'(r_0)$ is then pure real. Differentiating (6.13) with respect to r we get $y' = (y+5)/(y^2+2y-r)$. The condition $\operatorname{Im} y'(r_0) = 0$ leads then to the relation

$$y_1^2+y_2^2+10y_1+r_0+10=0.$$

Two more real relations are obtained by writing down separately the imaginary and real parts of the initial equation (6.13)

$$3y_1^2-y_2^2+6y_1-3r_0=0,$$

$$y_1^3-3y_1y_2^2+3(y_1^2-y_2^2)-3r_0(y_1+5)-4=0.$$

Expressing in the last equation the values of y_2^2 and r_0 in terms of y_1 with the aid of the first two relations, we obtain for y_1 the cubic equation

$$y_1^3+3y_1^2-9y_1-{}^{17}/_2=0.$$

All its roots are real, but only one of them, $y_1 = -4.56$, yields the necessary (positive) values for y_2^2 and r_0. The instability growth rate Γ has thus a maximum (different from the one obtained in Ref. 12)

$$\Gamma_{max}/4qv^2=2.16,$$

which is reached for almost transverse perturbations s: $(k/q)^2 = 162/s^4$. We emphasize that purely transverse perturbations are stable ($\sigma = 1$) and differ from obtained ones by the critical value of $\varkappa_3^2$ (6.13).

We have proved that conoidal waves are always stable in the KP model with negative dispersion. Thus, the stability of periodic nonlinear waves described by the KP equation (1.1) (just as the stability of solitons) is uniquely determined by the sign of the dispersion of the medium. In the isotropic model (1.2), on the contrary, unstable oblique perturbations exist for any sign of the dispersion.

In conclusion, the author thanks A. E. Kuznetsov and B. E. Fal'kovich for numerous helpful discussions.

APPENDIX

Calculation of the completeness integral (5.12)

We return in the integral (5.12) to the notation $-a^* = b$:

$$I=-i\int_c \frac{da}{\wp'(b)} f(a)f(b), \qquad (A.1)$$

but regard now b to be a function of a, implicitly given by Eq. (2.6).

We must separate in the function $b(a)$ a branch that assumes on the integration contour (5.2) ($\operatorname{Im}\mathscr{P}(a) = -k/2$) the necessary value $b = -a^*$ (recall that the roots of (2.6) on the line of Eq. (5.2) are $b = \pm a^*$).

The function $b(a)$ (2.6) has branch cuts at points $\dot a$ for which

$$\wp(a)=e_l-ik, \quad l=1,\,2,\,3$$

[in which case $\mathscr{P}(b) = e_l$ or $\mathscr{P}'(b) = 0$]. Each rectangle of the periods on the a plane has, with allowance for the parity of the $\mathscr{P}$-function, six branch points. All lie on the level line $\operatorname{Im}\mathscr{P}(a) = -k$. We draw the necessary cuts along this line, as shown in Fig. 5 (they are the originals of the transforms of the standard cuts on the $w = \mathscr{P}(b)$ plane), viz., along the real axis from $(-\infty)$ e_3 and from e_2 to e_1.

The complete Riemann surface on which the function

$$F(a)=-i\frac{f(a)f(b)}{\wp'(b)} \qquad (A.2)$$

is obtained by adding the second sheet and superimposing the opposite edges of the cuts of the different sheets. The function $F(a_\pm)$ is unique and doubly periodic on the constructed two-sheet surface. It follows also from the form of $f(a)$ (5.13) that $F(a)$ is analytic everywhere except at the two points corresponding to the value $a = 0$ (we call them 0_1 and 0_2). The point 0_1 (0_2) lies on the left (right) edge of the central cut of the upper sheet. At small values of a, as follows from (2.6), b takes on values close to $\pm a$. We therefore fix finally the function $f(a)$ (A.2), assuming that $b \approx a$ ($b \approx -a$) in the vicinity of the point 0_1 (0_2). The function $f(a)$ has then at the point 0_1 an essential singularity of the form $\exp(-2\Delta x/a)$, and at the point 0_2 a simple pole with a residue equal to $-i/2$ (see (A.1) and (5.13)).

The level lines (5.1) can be drawn on both sheets of the Riemann surface. The integration paths in (A.1) must be chosen to be those sections of these curves, on which b takes on the necessary value $-a^*$ out of the two possible ($\pm a^*$). We agree to designate as the right-hand (left-hand) loop that part of the curve (5.2) which lies in the first (third) quadrant of the rectangle of the periods. We consider on the

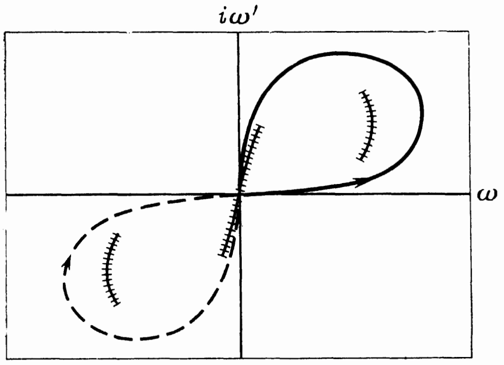

FIG. 5. Branch cuts and integration path (5.2) for the completeness integral (A.1). Solid and dashed lines—parts of path on the upper and lower sheets of the Riemann surface.

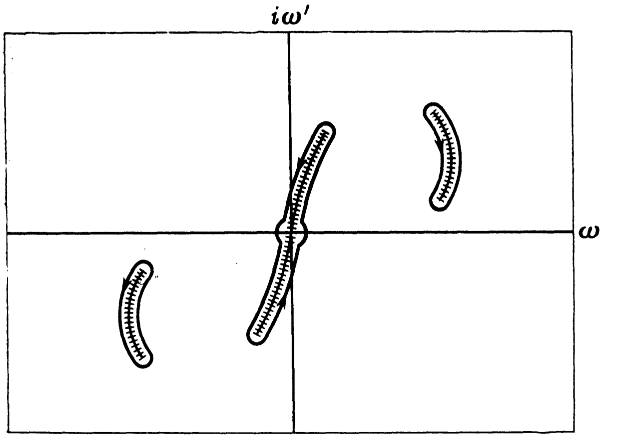

FIG. 6. Integration path shifted to the edges of the cuts, raised to the upper sheet, and closed near zero by small semicircles.

upper sheet a small section of the right-hand loop in the vicinity of the point 0_1 (it bears against the imaginary axis). On the one hand, for a complex number near the imaginary axis we have $-a^* \approx a$, and on the other hand we have $b \approx a$ near the point 0_1. The necessary equality $b = -a^*$ is therefore satisfied on the indicated section, meaning also along the entire right-hand loop on the upper sheet (the solid line in Fig. 5). It can be shown in exactly the same manner that the left-hand loop of the curve (5.2) should lie on the lower sheet (dashed line in Fig. 5). The integral of the closed differential form $F(a)da$ (A.2) coincides then with the initial (5.12). We emphasize that since the integral is taken in the sense of the principal value, the integration path is not closed and has free ends near zero.

We now deform the integration path in such a way that it passes along the edges of the cuts. The difference between the integrals along the new contour and the initial ones is $-(\pi/2)\operatorname{res} F(a) = -\pi/4$ on the upper sheet and $+\pi/4$ on the lower. These increments cancel each other.

Next, in view of the identity of the edges of the cuts, we can move the entire integration path to one of the sheets, say the upper, and close it by two small semicircles, as shown in Fig. 6. The left-hand semicircle bypasses (on the upper sheet) the essential singular point 0_1, and the integral along it tends to zero only if $\Delta x < 0$ (the condition for the damping of $\exp[-2\Delta x \zeta(a)]$ at $\operatorname{Re} a < 0$). The integral along the right-hand semicircle, which bypasses a simple pole, is equal to $\pi/2$. We add also the integrations over the boundaries of the rectangle of the periods (Fig. 6). We obtain in sum an integral along a closed contour that encloses a region where the function $F(a)$ is analytic; the integral is therefore zero. At the same time, the integrals over the opposite sides of the rectangles cancel each other by virtue of the periodicity of $F(a)$.

The sought integral is thus equal to $-\pi/2$ at $\Delta x < 0$. Similarly, transferring the integration path to the lower sheet, we can find the value of the integral at $\Delta x > 0$, namely $I = \pi/2$. In combined notation we obtain for the completeness integral (5.12) the expression

$$I = \pi[\operatorname{sign}(x - x')]/2.$$

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