Frequency-anticorrelated quantum states

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The use of quantum nondemolition measurements of the energy of a group of N photons in a waveguide makes it possible to create a quantum state in which a minimum energy uncertainty $(\omega\Delta\tau N)^{-1}$ is achieved. The coordinate uncertainty of such a group is equal to $c\Delta\tau$. This state of a group of photons in a traveling wave has properties characteristic of a single photon. Doppler measurements with such states make it possible to overcome the standard quantum limit of the sensitivity of gravitational free-mass antennas. The influence of dissipation on the resolution in such measurements is estimated.

1. INTRODUCTION

Various quantum states of an electromagnetic field and the related methods of quantum nondemolition measurements (QNDM) have been intensively investigated during the last few years. Few advances were made in this field, particularly in QNDM. Yet it is clear by now that the use of nonclassical states of a field can become quite useful. For example, the use of phase-squeezed states in laser gravitational antennas[1–3] permits a substantial decrease of the pump power.[4] States with a specified number of photons can provide a substantial energy gain in information transmission.[5]

The following terms were coined following the research by many workers: phase-squeezed quantum state of a radiation field (phase bunching: $\Delta\varphi < 1/N^{1/2}$) and amplitude-squeezed quantum state (debunching: $\Delta N < N^{1/2}$). Yet the analyzed states comprise only a small fraction of all possible quantum states of a radiation field. Neither adequate QNDM methods nor their possible use in experiment are known for the majority of quantum states.

It is known that if the experimenter has a self-excited generator of cw coherent radiation, the relative deviation of its frequency cannot be less than

$$\Delta\omega/\omega \approx 1/2QN^{1/2}, \quad N = W\tau/\hbar\omega, \tag{1}$$

where W is the power, Q the quality factor, τ the averaging time, and N the number of photons. In this equation (usually called the Schawlow-Townes formula) N has a characteristic upper bound that can be reached in practice only at sufficiently large Q.[6,7] The use of coherent quantum radiation to measure the velocity of a macroscopic body by means of the Doppler effect leads to a standard quantum limit for the velocity-measurement error:[8]

$$\Delta V_{SQL} \approx (\hbar/2m\tau^{1/2}), \tag{2}$$

where m is the mass of the body. This limit can be reached at an optimal N_{opt}. If phase-squeezed rather than coherent radiation is used, the limit (2) remains in force, but N_{opt} decreases faster the stronger the squeezing.

Von Neumann[9] proved a nontrivial assertion: the velocity of a body can be measured with the aid of *one* photon, using the Doppler effect, with an error

$$\Delta V \approx c/\omega\tau, \tag{3}$$

where c is the speed of light and τ is the averaging time. It is natural to search for the cause of the qualitative difference between Eqs. (2) and (3), when a large group of photons is used instead of one solitary one, in the method of preparing the group and in the recording methods. The present paper is devoted to an analysis of this problem, to the problem of the minimum error of the average frequency of a group of photons, and also to some related consequences for experimental programs. We show that properties typical of a single photon can be obtained under certain conditions by QNDM of the energy of a group of photons in a traveling wave.

2. PREPARATION OF FREQUENCY-ANTICORRELATED STATES

It was shown in Refs. 10 and 11 that by using a nonlinear (quadratic in amplitude) interaction of an electromagnetic cavity with a measuring instrument it is possible to measure the total number of photons in a mode exactly, and the energy E with small error. Such a measurement is usually called a quantum nonperturbing measurement of energy, since the number of photons is unchanged by the measurement, and the instrument produces strong fluctuations only in the phase, which is not recorded. In this case

$$\Delta E\Delta\varphi \gtrsim \hbar\omega/2, \quad \Delta E \ll \hbar\omega, \quad \Delta\varphi \gg 1. \tag{4}$$

Naturally, such a measurement is possible also in a traveling wave. It was found convenient to implement it in practice by using the idea of the interaction between the signal and measuring waves via the cubic nonlinearity of the dielectric.[12] This principle is the basis of a recent experiment[13] demonstrating quantum nondemolition detection of optical quadrature amplitudes.

Let us examine in greater detail the procedure of QNDM of the energy of a group of photons propagating in a waveguide. Assume that initially, using QNDM in the cavity mode, we have measured the number N of the photons in the cavity, and next, connecting the cavity to a waveguide, obtained a train (group) of traveling photons. Obviously, by varying the coupling of the waveguide to the cavity, we get *a priori* information on the length $\Delta x_{gr} - v\tau_{gr}$ of a train containing all N photons (τ_{gr} is longer than the time τ^* of photon emission from the cavity, v is the photon propagation velocity). It is also obvious that the relative energy uncertainty of each photon is of the order of $(\omega\tau^*)^{-1}$, and the relative uncertainty of the total energy of the group is

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$$\Delta E/E \approx 1/\omega\tau^* N^{1/2} \tag{5}$$

($E = N\hbar\omega$ is the energy of the train). If several independent photon groups are prepared in this manner, the spread of their relative energies is also given by Eq. (5).

Assume now that a waveguide section of length $l > v\tau_{gr}$ constitutes the device for the QNDM of the energy. In this setup it is necessary that the value of v over this section of the waveguide depend on the generalized coordinate q of the measuring instrument:

$$v = v_0(1-q/d), \tag{6}$$

where $1/d$ is the coupling constant. As an illustrative example of such an instrument one can imagine a waveguide section in which one of the walls can be displacement in the transverse direction by the pressure of the electromagnetic field. In this case q is the wall displacement and d is of the order of the waveguide diameter.

The generalized force acting on the instrument is equal to $F = E/d$, and the generalized momentum Φ of the instrument changes in the chosen measurement time $\tau_m < 1/v$ by an amount $E\tau_m/d$. The error in the measurement of E is determined by the momentum uncertainty $\Delta\Phi$

$$\Delta E = \frac{d}{\tau_m}\Delta\Phi. \tag{7}$$

In view of the uncertainty Δq of the coordinate of the instrument, the propagation velocity also has an uncertainty $\Delta v = v_0\Delta q/d$. The train is therefore displaced by the measurement a random distance (along the waveguide)

$$\Delta x = \tau_m\,\Delta v = v_0\tau_m\,\Delta q/d, \tag{8}$$

which corresponds to an uncertainty of its time of arrival at the chosen waveguide point behind the measuring unit,

$$\Delta\tau = \Delta x/v_0 = \tau_m\,\Delta q/d. \tag{9}$$

Since $\Delta\Phi\Delta q \geqslant \hbar/2$, the product of Eqs. (7) and (9) yields the following analog of Eq. (4) for a group of photons traveling along the waveguide

$$\Delta E\Delta\tau \geqslant \hbar/2. \tag{10}$$

It follows from this simple relation that after the measurement the relative error of the photon-group energy is

$$\Delta E/E \geqslant \hbar/2\Delta\tau E = 1/2\omega\Delta\tau N. \tag{11}$$

We emphasize that the uncertainty $\Delta\tau$ is a parameter that depends on the experimenter.

If QNDM is effected in such a way that $\Delta\tau \approx \tau^*$, a comparison of (5) and (11) shows that the gain equals to $N^{1/2}$. The average energy of the photon group is determined with a much lower error than that of independently emitted photons:

$$\Delta\omega/\omega \approx \Delta E/E = 1/2\omega\Delta\tau N. \tag{12}$$

This means that the frequencies of the individual photons in the group have become anticorrelated with respect to the measured average frequency (see the Appendix). It was this feature of the quantum state which motivated the title of the paper. We emphasize that the cost paid for the exact knowledge of the average frequency (and energy) in the group is the uncertainty $\Delta\tau$ (see Fig. 1).

3. DOPPLER MEASUREMENTS WITH FREQUENCY-ANTICORRELATED PHOTON GROUPS

The energy of a group of photons reflected from a body moving with initial velocity V is

$$E' = E\frac{1-V/c}{1+V/c+(2E/mc^2)(1-V^2/c^2)^{1/2}}, \tag{13}$$

where E is the energy prior to the reflection, c the speed of light, and m the mass of the body. Note that the operators of all the quantities in (13) commute with one another. The equation has therefore the same form in the quantum case, except that the classical variables are replaced by the corresponding operators.

If we now measure twice the group energy (before and after the reflection) by the described method, we can determine V from the difference $E' - E$. Obviously the error ΔV is determined by the value of ΔE. In the nonrelativistic case ($V \ll c$ and $E \ll mc^2$) we obtain from (11) and (13)

$$\Delta V = c/2\omega\Delta\tau N. \tag{14}$$

This equation is a generalization of (3) to the case of an arbitrary number N of photons. Thus, a group of photons in a frequency-anticorrelated state behaves as one photon with total energy $h\,N$.

Measurement of the body velocity should be accompanied, in accord with the uncertainty relation, by a perturbation Δx_m of its coordinate. The mechanism of this perturbation was indicated in Ref. 9. It stems from the fact that the body acquires by reflection a known momentum $p = 2E/c$ at the random instant of reflection of the photon group from the body. As shown in Ref. 9,

$$\Delta x_m = p\Delta\tau/m = E\tau/mc. \tag{15}$$

The product of the uncertainties (14) and (15) is equal to $\hbar/2m$, as expected.

Let us dwell on the influence of dissipation on the accuracy of the Doppler measurement of the velocity. Let R be the probability of the photon reaching the receiver without being absorbed either by the transmission line or through

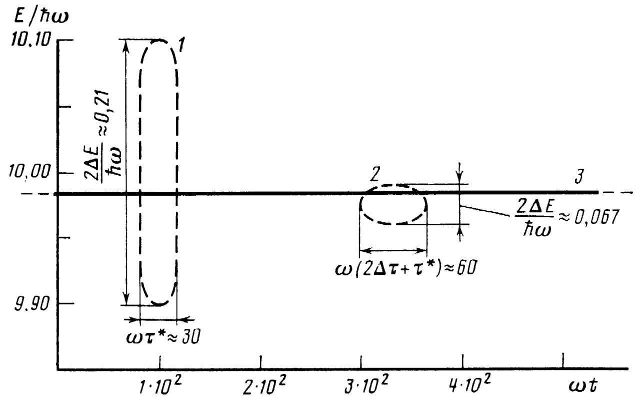

FIG. 1. 1—Initial state: $\Delta E\tau^* \approx \hbar N^{1/2}$ for $N = 10$ and $\omega\tau^* = 30$; 2—frequency-anticorrelated state $\Delta E\Delta\tau = \hbar/2$ for $2\omega\Delta\tau = 30$; 3—state with given energy: $\Delta E \to 0, \Delta\tau \to \infty$.

reflection from the body. Dissociation increases the energy uncertainty of a group of photons via two mechanisms. First, the number of absorbed photons is a random quantity with a variance

$$(\Delta_R N)^2 = NR(1-R).$$

Second, the energy of each of the $N(1-R)$ photons absorbed on the average has an uncertainty $\hbar/2\tau^*$. Consequently, the energy uncertainty due to dissipation is

$$(\Delta_R E)^2 = (\hbar\omega\Delta_R N)^2 + N(1-R)(\hbar/\tau^*)^2$$

$$= \hbar^2 N(1-R)(\omega^2 R + 1/(\tau^*)^2). \quad (16)$$

Inasmuch as $\omega\tau^* \gg 1$ always, the main contribution to (16) is made by the first term.

Direct conversion of the uncertainty (16) into the velocity-measurement error leads to the expression

$$\Delta_R V = \frac{c\Delta_R E}{2RE} = \frac{c}{2}\left(\frac{1-R}{NR}\right)^{1/2}. \quad (17)$$

It follows therefore that to obtain a measurement accuracy exceeding the limit (2) the energy of the probing pulse must be exceedingly high:

$$E > (1-R)mc^2/R. \quad (18)$$

By changing somewhat the measurement scheme, however, we can improve the situation substantially. The energy uncertainty that leads to the appearance of the restriction (17) is due to a change in the number of photons, while the signal-induced (Doppler) change of energy is due to a change of their frequency. If we monitor the initial and final values of not only the energy but also the number of photons in the group, we can determine the number of absorbed photons and eliminate by the same token the first term in (16). The velocity-measurement error is then

$$\Delta_R V \approx \frac{\hbar c}{RE\tau^*}[N(1-R)]^{1/2} = \frac{c(1-R)^{1/2}}{R\omega\tau^* N^{1/2}}. \quad (19)$$

In sum, we can state that the von Neumann case is realized with a group of photons only at very low dissipation ($N(1-R) < 1$). If, however, the photon absorption is appreciable, the use of frequency-anticorrelated states leads nonetheless to a resolution gain of the order of $(1-R)^{1/2}$.

4. USE OF FREQUENCY-ANTICORRELATED STATES IN GRAVITATIONAL ANTENNAS

A gravitational wave is known to produce a field of transverse acceleration gradients that can be conveniently described by the dimensionless amplitude of the variation of the metric $h \approx 2\Delta L/L$ (L is the distance between the free test masses). If h is determined by coordinate measurements with the aid of coherent or phase-squeezed electromagnetic radiation, the standard quantum limit for h is

$$h_{\rm SQL} \approx \frac{1}{L}\left(\frac{2\hbar\tau_{\rm grav}}{m}\right)^{1/2}$$

$$\approx 3\cdot 10^{-23}\left(\frac{L}{4.10^5\ \text{cm}}\right)^{-1}\left(\frac{\tau_{\rm grav}}{10^{-3}\ \text{s}}\right)^{1/2}\left(\frac{10^4\ \text{g}}{m}\right)^{1/2}. \quad (20)$$

We use in this equation the parameters of the laser gravitational antenna of the LIGO program,[1] viz., $\tau_{\rm gr}$ is the duration of the gravitational radiation spike, m the value of one of the masses, and L is the distance between the masses.

Another experimental possibility is to use a group of photons in a frequency-anticorrelated state and record the variations of the trial-mass velocity difference $\Delta V_{\rm grav} = \frac{1}{2}hL\omega_{\rm grav}$ ($\omega_{\rm grav}$ is the frequency of the gravitational radiation). We emphasize that in the second procedure the preparation of the state of the photon group is such that the observer obtains the distance between the masses with a large error, with an ensuing resolution gain in the velocity measurement. If ΔV of Eq. (19) is compared with $\Delta V_{\rm grav}$, putting $\tau_{\rm grav} = \Delta\tau$ and $\omega_{\rm grav}\tau_{\rm grav} \approx 2\pi$, then

$$h_{min} \approx \frac{c(1-R)^{1/2}}{\pi\omega L N^{1/2}} \approx 2\cdot 10^{-23}\left(\frac{10^{-7}}{1-R}\right)^{-1/2}\left(\frac{\omega}{4\cdot 10^{15}\,\text{s}^{-1}}\right)^{-1}$$

$$\times\left(\frac{L}{4\cdot 10^5\,\text{cm}}\right)^{-1}\left(\frac{N}{10^{16}}\right)^{-1/2}. \quad (21)$$

It can be seen from the foregoing example that the limit $h_{\rm SQL}$ can be exceeded only if the rather stringent requirements $E \approx 4\cdot 10^4$ erg, $\Delta E/E \approx 2\cdot 10^{-23}$, and $1 - R \approx 1\cdot 10^{-7}$ are met. It must be noted, however, that the dynamic range of the meter must not be too large (approximately seven decades) and that the requirements imposed on $\Delta E/E$ and $1-R$ refer to the value of h, which is five orders smaller than the presently attained sensitivity.

We note in conclusion that the quantum-measurement procedure considered is an extreme case, in which the almost total lack of information on the coordinate leads to a maximum gain in the velocity resolution. Obviously, there exists a range of measurement procedures intermediate between the one considered here and the traditional one.

We note also that the Doppler measurement of the momentum of a free particle with the aid of radiation in a frequency-anticorrelated state is, in the absence of dissipation, asymptotically nonperturbing, since the momentum perturbation is $\Delta p \approx 2\Delta E/c$, and ΔE can be small because of the increase of $\Delta\tau$.

The limit (2) can be exceeded even if the error in the measurement of the instrument momentum Φ is bounded by its standard quantum limit. All that is needed for this purpose is a proper choice of E and $1/d$.

The dynamic range of the measurements can be considerably decreased by measuring not E and E' separately, but only the difference $E - E'$. In this case the instrument must have a memory, and the signs of the generalized force should be opposite in the forward and backward passages of the photons.

APPENDIX A

Statistical properties of a wave packet with a given number of photons

We consider a one-dimensional transmission line—an infinite long line. This eliminates the technical difficulties connected with considering a vector electromagnetic field in three-dimensional space, but preserves all the crucial aspects of measurement theory.

Let $\hat{a}(\omega)$ and $\hat{a}+(\omega)$ be the annihilation and creation operators of photons moving along the x axis. These operators satisfy the commutation relations for a continuous mode spectrum $[\hat{a}(\omega),\hat{a}(\omega')] = \delta(\omega-\omega')$.

The state of a line on which a specified number N of photons travel in the x direction is described by the wave vector

$$|N\rangle=\frac{1}{(N!)^{1/2}}\int_0^\infty \psi(\omega_1,\ldots,\omega_N)$$

$$\times \hat{a}^+(\omega_1)\ldots\hat{a}^+(\omega_N)\,d\omega_1\ldots d\omega_N|0\rangle, \qquad (A.1)$$

where $\psi(\ldots)$ is a wave function normalized to unity and symmetric under permutation of the arguments. Let us find the possible minimal energy uncertainty of a wave packet in state (A.1)

$$\hat{E}=\int_0^\infty \hbar\omega\hat{a}^+(\omega)\hat{a}(\omega)\,d\omega \qquad (A.2)$$

if its length is given. The energy variance is

$$(\Delta E)^2=\int_0^\infty (\hbar\omega_1+\ldots+\hbar\omega_N)^2|\psi(\omega_1,\ldots,\omega_N)|^2 d\omega_1\ldots d\omega_N$$

$$-\left(\int_0^\infty N\hbar\omega_1|\psi(\omega_1,\ldots,\omega_N)|^2 d\omega_1\ldots d\omega_N\right)^2. \qquad (A.3)$$

The spatial distribution of the energy is characterized by the function

$$W(x)=\frac{\langle\hat{\rho}(x)\rangle}{\langle\hat{E}\rangle}=\frac{\hbar N}{2\pi\langle E\rangle v}\int_0^\infty (\omega\omega')^{1/2}\exp\left\{i(\omega-\omega')\frac{x}{v}\right\}$$

$$\times\psi^*(\omega_1,\ldots,\omega_{N-1},\omega)\psi(\omega_1,\ldots,\omega_{N-1},\omega')\,d\omega_1\ldots$$

$$\times d\omega_{N-1}\,d\omega\,d\omega', \qquad (A.4)$$

where $\hat{\rho}$ is the energy density operator. The coefficient in (A.4) is chosen such that $W(x)$ be normalized as the probability distribution over x. The variance $(v\Delta\tau)^2$ of this distribution specifies the length of the wave packet.

We consider the case of greatest practical interest, when the frequency bandwidth of the wave packet is much smaller than its average frequency $\bar{\omega}$. In this approximation $\Delta E\Delta\tau$ has a minimum if the wave function is Gaussian. It is easy to show in this case that

$$(\Delta E\,\Delta\tau)^2=\frac{\hbar^2 N}{4}\frac{(N-2)\beta+1}{1-\beta}, \qquad (A.5)$$

where β is the correlation coefficient of the photon frequencies. The quantity $\Delta E\Delta\tau$ decreases with decrease of β. At the minimum value $(N-1)^{-1}$ of β we get

$$\Delta E\Delta\tau=\hbar/2. \qquad (A.6)$$

We note for comparison that if the photons are not correlated, $\beta=0$, then

$$\Delta E\Delta\tau=\hbar N^{1/2}/2. \qquad (A.7)$$

APPENDIX B

Influence of zero line fluctuations on the energy-measurement accuracy

No account was taken in the derivation of Eq. (10) that the energy of the zero-point oscillations on a finite section of an infinite line fluctuates and makes therefore an additional contribution to the wave-packet-energy measurement error. This contribution is equal to the uncertainty of the readings of the measuring instrument when the transmission line is not excited:

$$(\Delta_0 E)^2=\left(\frac{d}{\tau_m}\right)^2\int_0^{\tau_m}\langle 0|\hat{F}(t)\hat{F}(t')|0\rangle dt\,dt', \qquad (B.1)$$

where

$$\hat{F}(t)=\int_0^\infty \alpha(x)\hat{\rho}(x,t)\,dx$$

is the operator of the force acting on the instrument, $\hat{\rho}(x,t)$ is the energy-density operator, and $\alpha(x)$ characterizes the coupling of the instrument to the line in the section x. The coupling is substantially different from zero over an interaction interval of length $v\tau_x$, and satisfies the normalization condition

$$\int_{-\infty}^{\infty}\alpha(x)\,dx=\frac{v\tau_x}{d}. \qquad (B.2)$$

Omitting the cumbersome but in principle straightforward calculations, we present the calculated (B.1):

$$(\Delta_0 E)^2\doteq\frac{1}{3}\left(\frac{\hbar\tau_x d}{2\pi\tau_{\text{изм}}}\right)^2\int_0^\infty \mathrm{Re}(A^2(\omega\tau_x))$$

$$\times\cos\omega\tau_m\,(1-\cos\omega\tau_m)\,\omega\,d\omega, \qquad (B.3)$$

where

$$A(\eta)=\int_{-\infty}^{\infty}\alpha(x)\exp\left(-\frac{i\eta x}{v\tau_x}\right)\frac{dx}{v\tau_x}$$

is the Fourier transform of $\alpha(x)$. The function $A(\eta)$ differs substantially from zero at $\eta\lesssim 1$; from the condition (B.2) it follows that $A(0)=1/d$.

If the investigated wave packet is to remain in the interaction interval during the entire measurement time, the following condition must be met

$$\Delta\tau+\tau_m<\tau_x. \qquad (B.4)$$

The integrand in (B3) differs substantially from zero at $\omega\tau_x\lesssim 1$, so that the inequality $\omega\tau_m<1$ certainly holds. We can therefore put

$$\cos\omega\tau_m\,(1-\cos\tau_m)\approx(\omega\tau_m)^2/2.$$

The expression for $\Delta_0 E$ takes then the form

$$\Delta_0 E=k\hbar/\tau_x, \qquad (B.5)$$

where

$$k=\frac{d}{2\pi}\left(\frac{1}{6}\int_0^\infty \mathrm{Re}\,A^2(\eta)\eta^3\,d\eta\right)^{1/2} \qquad (B.6)$$

is a numerical coefficient of order unity. In particular, for the Gaussian function

$$\alpha(x)=\frac{1}{d}\exp\left\{-\frac{\pi x^2}{(v\tau_x)^2}\right\} \qquad (B.7)$$

we have $k=1\sqrt{3}$.

Owing to the inequality (B.4), the value of $\Delta_0 E$ is always less than the measurement error given by Eq. (10). It can thus be concluded that the zero-point fluctuations of the transmission line do not increase substantially the energy-measurement error.

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