The Hilbert space of supersymmetric quantum systems

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It is shown that in a number of models the requirements of supersymmetry are fulfilled only when certain additional conditions are imposed on the wave functions of the states. If this circumstance is disregarded, paradoxes can arise. In particular, the question of the fermion condensate in the supersymmetric σ-model and in supersymmetric gauge theories is analyzed.

1. INTRODUCTION

A striking feature of supersymmetric theories is the fact that certain complex and highly nontrivial properties (the structure of the vacuum, condensates, etc.) can be established reliably by considering a simplified variant of the theory in which a weak-coupling regime is realized. One of the first examples of this type was the counting of the number of vacuum states,[1] which was carried out in a finite (small) volume. In the process, complicated field-theory models are reduced to quantum mechanics. An extremely elegant phenomenon was discovered in Ref. 2. It was found that the introduction of matter fields into supersymmetric gauge theories can lead to spontaneous breaking of the gauge symmetry and to an increase in mass of the gauge bosons. Here, a weak-coupling regime, which can be fully monitored theoretically, arises automatically. On this path, models have been found[3,4] in which nonperturbative dynamics leads to spontaneous supersymmetry breaking. Finally, a very recent example is the calculation of gluino condensates $\langle \mathrm{Tr}\lambda\lambda \rangle$ in supersymnetric gluodynamics.[5] The fact of the gluino condensation ($\langle \mathrm{Tr}\lambda\lambda \rangle \neq 0$), as will be clarified below, poses an extremely interesting question concerning the relationship between the complete Hilbert space of the system and the supersymmetry. In the present paper we shall give an exhaustive solution of the question for a number of quantum-mechanical problems, We shall show that the general supersymmetry properties (SUSY)—the supersymmetry of the spectrum, the annihilation of the vacuum state by the action of the supercharge, etc.—can be fulfilled only in the case when certain additional conditions are imposed on the wavefunctions of the states. Although we do not go beyond the framework of the analysis of supersymmetric quantum-mechanical models, it is clear that the qualitative conclusions will be valid in field theory as well.

The starting observation that initiated the present work is an apparent paradox that arises in supersymmetric gauge theories and σ-models. Its essence is as follows. We consider a supersymmetric Yang-Mills (SYM) theory without matter, with the group SU(N). The supersymmetry here remains unbroken (see, e.g., Ref. 1), but a no less interesting phenomenon occurs: The discrete $Z_2 \times Z_N$ symmetry that remains from the classical chiral U(1) invariance when the anomaly is taken into account is spontaneously broken to Z_2, this being manifested in the formation of a gluino condensate $\langle \mathrm{Tr}\lambda_\alpha \lambda^\alpha \rangle$ (Ref. 5). It is here that we encounter the paradox. The operator $\lambda^2(x)$ can be represented as the average component of the superfield $V(x, \theta, \bar\theta)\, W_\alpha(x, \theta, \bar\theta)$, i.e., in the form of the following anticommutator containing the supercharge Q:

$$\lambda_\alpha{}^a \lambda^{\alpha a}(x) \sim \{Q_\alpha,\ \lambda^{a\beta}(\sigma_\mu)_\beta{}^\alpha A_\mu{}^a(x)\}. \qquad (1)$$

If, as is customary, the action of Q on the supersymmetric vacuum gives zero, the right-hand side of (1) cannot have a nonzero vacuum average, in explicit contradiction to the results of Ref. 5.

A similar situation obtains in supersymmetric Kahler σ-models. In CP(N) models, a fermion condensate $\langle \bar\psi(1 \pm \gamma_5)\psi \rangle$ is also generated,[6,7] as can be verified by expanding in $1/N$. On the other hand, the operators $\bar\psi(1 \pm \gamma_5)\psi$ are the average components of the Kahler superpotential $V(\Phi, \bar\Phi)$ and can be represented in the form

$$\bar\psi(1\pm\gamma_5)\psi \sim \{\bar Q^\alpha,\ \psi_\beta \bar\varphi\}(1\pm\gamma_5)_\alpha{}^\beta. \qquad (2)$$

Once again, it would appear that the representation (2) contradicts the nonzero vacuum average of these operators, since the supersymmetry is not broken in the given theory.

The resolution of the paradox lies in the fact that the action of the operators $\mathscr{O}_\alpha$ ($\mathscr{O}_\alpha = \psi_\alpha \varphi$ in the σ-model, and $\mathscr{O}^\alpha = \lambda^{a\beta}(\sigma_\mu)_\beta{}^\alpha A_\mu{}^a$ in the SYM model)on the vacuum wavefunction gives a function with "bad" properties: In the supersymmetric σ-model the latter function is not normalizable, while in the SYM model it is gauge-noninvariant. The supersymmetry does not act in the space of the bad functions. In particular, the supercharges Q_α and $\bar Q^\alpha$ cease to be Hermitian conjugates, and the reasoning that leads to the relation $\langle \{Q_\alpha \mathscr{O}^\alpha\} \rangle = 0$ is not valid. Thus, we are concerned with the following phenomenon: The entire Hilbert space of the state vectors is divided into two sectors—the "physical" sector, which can be fixed by a definite additional condition on the states, plus the rest of the space (the "nonphysical" sector). In the physical sector all the consequences of SUSY (boson-fermion degeneracy, zero energy of the lowest state, etc.) are fulfilled. Entering the nonphysical sector can lead to the result that the formal conclusions of SUSY do not hold.

In Sec. 2 we shall analyze a very simple one-dimensional example in which the supersymmetry algebra is realized not on all states but only on states satisfying a certain additional requirement. In Sec. 3 we analyze the quantum mechanics obtained by reduction of the supersymmetric σ-model. Section 4 is devoted to the analogous problem arising from the SYM model. Section 5 presents the conclusion and the main results.

2. ONE-DIMENSIONAL QUANTUM-MECHANICAL MODEL

The supersymmetric quantum-mechanical Hamiltonian of Witten[8] is well known:

$$H = \{p^2 + W^2(x) + [\bar\psi,\ \psi]\, W'(x)\}/2, \qquad (3)$$

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where x ia a bosonic variable, ψ is a complex Grassmann variable,[1)] $p = -i\partial/\partial x$, and $\bar{\psi} = \partial/\partial\psi$. The supercharges corresponding to the Hamiltonian (3) are

$$Q=\psi[p+iW(x)], \quad \bar{Q}=\bar{\psi}[p-iW(x)], \tag{4}$$

so that

$$Q^2=\bar{Q}^2=0, \quad \{Q, \bar{Q}\}=2H. \tag{5}$$

In Ref. 8, polynomial functions $W(x)$ were considered. In this case, the Hamiltonian (3) and supercharges (4) do not have singularities in the entire region $-\infty < x < \infty$, and the supersymmetry algebra is realized on all normalizable eigenfunctions of the operator H. The entire Hilbert space decomposes in this case into two regions: bosonic states with wavefunction $\Psi_B(x, \psi) = a_B(x)$, and fermionic states with $\Psi_F(x, \psi) = \psi a_F(x)$. For $E \neq 0$ the fermionic and bosonic states are paired:They have the same energy and are obtained from each other by the action of the operators Q and $\bar{Q}$. If the leading power in the polynomial $W(x)$ is odd, the lowest state has zero energy and its wavefunction is proportional to

$$\exp\left(\pm\int_0^x W(y)\,dy\right),$$

where the sign $+$ or $-$ is chosen so as to ensure normalizability. The choice of the sign determines which state (the fermionic or the bosonic) is the lowest (vacuum) state.

We now consider a function $W(x)$ of the form

$$W(x)=-\omega x+1/x. \tag{6}$$

In the bosonic sector the Hamiltonian has the form

$$H_B=\frac{p^2}{2}+\frac{\omega^2x^2}{2}-\frac{3\omega}{2} \tag{7a}$$

while in the fermionic sector,

$$H_F=\frac{p^2}{2}+\frac{\omega^2x^2}{2}+\frac{1}{x^2}-\frac{\omega}{2}. \tag{7b}$$

It can be seen that in the bosonic sector the Hamiltonian does not have singularities at $x = 0$, and represents an ordinary harmonic oscillator. If the problem (7a) were considered separately, with no connection with the supersymmetric system (3), we would say immediately that the wavefunction of the lowest state is

$$\Psi=\text{const}\exp(-\omega x^2/2), \tag{8}$$

and the energy $E = -\omega$, i.e., is negative! How can one reconcile this situation with supersymmetry, which requires $E \geqslant 0$? The point is that the Hilbert space of the system described by the Hamiltonian (3) should be narrowed in comparison with the Hilbert space of the bosonic system (7a). Some of the states that are admissible from the point of view of (7a) fall in the nonphysical sector for the Hamiltonian (3). Specifically, the supersymmetry is realized only on states satisfying the additional condition $\Psi(x, \psi) = 0$ for $x = 0$. In fact, the action of the supercharge Q on the function (8) (and on all even levels of the Hamiltonian (7a)) gives a non-normalizable state with a singularity $\mathcal{O}(1/x)$, in view of the singularity of $W(x)$ at $x = 0$. In other words, the pathological state (8) does not have a superpartner.

The condition $\Psi(0, \psi) = 0$ selects from the spectrum of the Hamiltonian H_B half of the states—those that are odd under the replacement $x \to -x$. Their energies are $0, 2\omega, 4\omega, \ldots$. For $E \neq 0$ these states have fermionic superpartners. (It is easy to convince oneself that $a_B = \mathcal{O}(x)$ and $a_F = \mathcal{O}(x^2)$ for $x \to 0$.) With the example of (6) we have demonstrated how a situation can arise in which the consequences of SUSY are valid only for some of the states in the Hilbert space. The necessity of imposing additional conditions on the wavefunctions in the given space is connected with the fact that the superpotential has a singularity at $x = 0$. We note that the Hamiltonian (3) with the superpotential (6) has been considered previously in Refs. 9. In these papers the existence of the state (8) with negative energy was interpreted as breaking of the supersymmetry. It seems to us that a different interpretation is preferable—the supersymmetry is realized, but on a narrowed class of states.

$$W(x)=-f(x)+1/x,$$

$$f(x)=\mathcal{O}(x) \quad \text{for} \quad x\to 0, \; f(\pm\infty)=\pm\infty.$$

The presence of the term $1/x$ leads to the result that the wavefunction

$$\sim\exp\left(\int_0^x W(y)\,dy\right)$$

with $E = 0$ has a zero at $x = 0$. By virtue of the oscillator theorem this implies that in the (nonsingular) potential $V = (W^2 + W')/2$ a level with $E < 0$ should exist. If $f(x) \equiv 0$, we obtain superconformal quantum mechanics with $W(x) = 1/x$, in which a very similar situation is also realized.[10] The choice (6) makes the spectrum discrete, and makes the phenomenon clearer.

3. THE SUPERSYMMETRIC σ-MODEL

The supersymmetric CP(1) σ-model is described indetail in the review Ref. 7. In standard superfield notation the action of the model has the form

$$S={}^1/_4\int d^2x\, d^2\theta\, d^2\bar{\theta}\, V(\Phi, \bar{\Phi}),$$

where Φ is a complex chiral superfield, and V is the Kähler potential

$$V(\Phi, \bar{\Phi})=\ln(1+\Phi\bar{\Phi}). \tag{9a}$$

Corresponding to the potential (9a) is the metric

$$h(\Phi, \bar{\Phi})=\partial\bar{\partial}V=\frac{1}{(1+\Phi\bar{\Phi})^2}. \tag{9b}$$

As is well known, the manifold of (9) is a sphere with radius $1/\sqrt{2}$. In the following, however, we shall find it convenient to change the normalization of the fermionic component of the superfield Φ. In this connection, we give the Hamiltonian of the model in component form:

$$H=\int dx\Big\{\Big(\pi-\frac{i\bar{\psi}\psi}{2h}\partial h\Big)h^{-1}\Big(\bar{\pi}+\frac{i\bar{\psi}\psi}{2h}\bar{\partial}h\Big) + \frac{R}{4}\bar{\psi}^2\psi^2+\frac{i}{2}[\bar{\psi}\sigma_3\partial_x\psi - (\partial_x\bar{\psi})\sigma_3\psi]+\frac{i\bar{\psi}\sigma_3\psi}{2h}[\partial_x\varphi\,\partial h-(\partial_x\bar{\varphi})\bar{\partial}h]+h\partial_x\varphi\partial_x\bar{\varphi}\Big\}. \tag{10}$$

Here ψ_α is a two-component complex Grassmann spinor,

$$\bar{\psi}^\alpha=\partial/\partial\psi_\alpha,\quad \pi=-i\partial/\partial\varphi,\quad \bar{\pi}=-i\partial/\partial\bar{\varphi},\quad \bar{\psi}\psi=\bar{\psi}^\alpha\psi_\alpha=\bar{\psi}^1\psi_1+\bar{\psi}^2\psi_2,\quad \psi^2=\psi_\alpha\psi^\alpha=\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta\quad(\varepsilon^{12}=1),\quad \bar{\psi}^2=\bar{\psi}^\alpha\bar{\psi}_\alpha,$$

and R is the scalar curvature

$$R=\frac{1}{h^2}\left(-\partial\bar{\partial}h+\frac{\partial h\,\bar{\partial}h}{h}\right)=2. \tag{11}$$

The normalization of the field ψ differs from that used in Refs. 6 and 7 by the factor $(h(\varphi,\bar{\varphi}))^{1/2}$ (the ψ used here is obtained from the field of Refs. 6 and 7 by multiplying by $h^{1/2}$), so that the variables ψ_α and $\bar{\psi}^\alpha$ become canonically conjugate.

The order of the operators in (10) is important and is strictly defined, if we wish to preserve the supersymmetry algebra at the quantum level.[11,12] This algebra has the form

$$\{Q_\alpha,\bar{Q}^\beta\}=\delta_\alpha{}^\beta H-(\sigma_3)_\alpha{}^\beta P, \tag{12}$$

where P is the momentum operator, and the supercharges Q_α, $\bar{Q}^\beta$ are determined as follows:

$$Q_\alpha=\int dx\left\{\left[\pi-\frac{i\bar{\psi}\psi}{2h}\partial h\right]\frac{\psi_\alpha}{h^{1/2}}+h^{1/2}(\partial_x\bar{\varphi})(\sigma_3)_\alpha{}^\gamma\psi_\gamma\right\},$$
$$\bar{Q}^\beta=\int dx\left\{\frac{\bar{\psi}^\beta}{h^{1/2}}\left[\bar{\pi}+\frac{i\bar{\psi}\psi}{2h}\bar{\partial}h\right]+h^{1/2}(\partial_x\varphi)\bar{\psi}^\gamma(\sigma_3)_\gamma{}^\beta\right\}. \tag{13}$$

The field Hamiltonian (10) contains an infinite number of degrees of freedom. Moreover, the σ-model is a theory with strong coupling: At large distances the effective charge increases. In the present paper, our problem does not involve the dynamical investigation of the σ-model as such. We wish to illustrate only one aspect of the model, and to this end we shall reduce the theory to a quantum-mechanical model. By this reduction we mean the following. We consider the theory (10) in a small "box", impose periodic boundary conditions on all the fields, expand them in modes, and then throw away all nonzero modes. The zero modes correspond to fields that do not depend on x, i.e., to constant fields. The Hamiltonian describing the interaction of the zero modes is obtained from (10) if we assume that all the fields appearing in (10) are constant. Below, these quantum-mechanical degrees of freedom, which do not depend on x, will be denoted by the same symbols as those used for the corresponding fields. It should be noted that, in principle, the nonzero modes generate corrections that change the interaction of the zero modes. It is possible to obtain an effective Hamiltonian for the zero modes in which, instead of simply throwing away the nonzero modes, we take their contribution into account in the Born-Oppenheimer approximation. The corresponding procedure is discussed in detail in Ref. 12. Since our aim is illustrative, and the Born-Oppenheimer approximation is certainly not valid, e.g., in $\mathcal{O}(N)$ gluodynamics (see Sec. 4), we shall not concern ourselves with this question, but shall concentrate on the quantum mechanics as such.

Thus, the quantum-mechanical system with which we shall be concerned is described by the Hamiltonian (the box size $L=1$)

$$H=\left(\pi-\frac{i\bar{\psi}\psi}{2h}\partial h\right)\frac{1}{h}\left(\bar{\pi}+\frac{i\bar{\psi}\psi}{2h}\bar{\partial}h\right)+\frac{1}{2}\bar{\psi}^2\psi^2, \tag{14}$$

$$\bar{Q}^\alpha=\frac{\bar{\psi}^\alpha}{h^{1/2}}\left(\bar{\pi}+\frac{i\bar{\psi}\psi}{2h}\bar{\partial}h\right), \tag{15}$$

$$Q_\alpha=\left(\pi-\frac{i\bar{\psi}\psi}{2h}\partial h\right)\frac{\psi_\alpha}{h^{1/2}}, \tag{16}$$
$$h=(1+\varphi\bar{\varphi})^{-2}.$$

The operators (14) and (15) act on the wavefunctions $\Psi(\varphi,\bar{\varphi},\psi_1,\psi_2)$, which are normal by the condition[2)]

$$\int|\Psi|^2\,d\varphi\,d\bar{\varphi}\prod_{\alpha=1,2}d\psi_\alpha\,d\bar{\psi}^\alpha\exp(\bar{\psi}^\alpha\psi_\alpha)=1. \tag{17}$$

The covariant wavefunctions are normalized with the measure $\sim d\varphi d\bar{\varphi}h(\varphi,\bar{\varphi})$ and are obtained from ours by dividing by $h^{1/2}$.

The Schrödinger equation with Hamiltonian (14) has two solutions with zero energy, in agreement with the value of the Witten index[1]. The explicit expression for the vacuum wavefunctions has the simple form

$$\Psi_{vac\,1}=C\psi_1 h^{1/2},\qquad \Psi_{vac\,2}=C\psi_2 h^{1/2}. \tag{18}$$

It is easily verified directly that

$$Q_\alpha|\mathrm{vac}_{1,2}\rangle=0,\qquad \bar{Q}^\alpha|\mathrm{vac}_{1,2}\rangle=0.$$

(Here it must be taken into account that $\psi_1^2=\psi_2^2=0$ and, consequently, $\bar{\psi}^1\psi_1|\mathrm{vac}_1\rangle=0$, $\bar{\psi}^1\psi_1|\mathrm{vac}_2\rangle=|\mathrm{vac}_2\rangle$, etc.) We shall find the matrix elements of the operators $\bar{\psi}\sigma^\pm\psi$,

$$\bar{\psi}\sigma^+\psi=\bar{\psi}^1\psi_2,\qquad \bar{\psi}\sigma^-\psi=\bar{\psi}^2\psi_1 \tag{19}$$

between the states (18). (These operators coincide with $h\bar{\psi}(1\pm\gamma_5)\psi/2$ in the notation of Refs. 6 and 7.) It is not difficult to obtain

$$\langle\mathrm{vac}_1|\bar{\psi}\sigma^\pm\psi|\mathrm{vac}_1\rangle=\langle\mathrm{vac}_2|\bar{\psi}\sigma^\pm\psi|\mathrm{vac}_2\rangle=0,$$
$$\langle\mathrm{vac}_2|\bar{\psi}\sigma^+\psi|\mathrm{vac}_1\rangle=\langle\mathrm{vac}_1|\bar{\psi}\sigma^-\psi|\mathrm{vac}_2\rangle=-1. \tag{20}$$

The physical vacuum state can be an arbitrary superposition of $|\mathrm{vac}_1\rangle$ and $|\mathrm{vac}_2\rangle$, thereby leading to nonzero averages for the operators $\bar{\psi}\sigma^\pm\psi$. (We note that the instanton arguments of Refs. 6 and 7 do not determine which of the matrix elements—diagonal or transition—are nonzero.)

We are now ready to resolve the paradox formulated in the Introduction. We have

$$\bar{\psi}\sigma^+\psi=i\{\bar{Q}^1,\psi_2\bar{\varphi}\},\qquad \bar{\psi}\sigma^-\psi=i\{\bar{Q}^2,\psi_1\bar{\varphi}\}. \tag{21}$$

The central point is that the operators $\psi_i\bar{\varphi}$ are in fact not defined, since their action on $|\mathrm{vac}_{1,2}\rangle$ gives a non-normalizable wavefunction, i.e., goes outside the physical sector of the Hilbert space. We consider the matrix element

$$\langle\mathrm{vac}_2|\bar{Q}^1\psi_2\bar{\varphi}+\psi_2\bar{\varphi}\bar{Q}^1|\mathrm{vac}_1\rangle. \tag{22}$$

The second term in (22) is equal to zero, since the supercharge $\bar{Q}^\alpha$ annihilates the state $|\mathrm{vac}_1\rangle$. The first term in (22) can be rewritten as

$$\langle\mathrm{vac}_2|\bar{Q}^1\psi_2\bar{\varphi}|\mathrm{vac}_1\rangle\sim\int d\varphi\,d\bar{\varphi}\prod_{i=1,2}d\psi_\alpha\,d\bar{\psi}^\alpha$$
$$\times\left[\frac{\bar{\psi}^2}{1+\varphi\bar{\varphi}}\bar{\psi}^1(1+\varphi\bar{\varphi})\left(-i\frac{\partial}{\partial\bar{\varphi}}\right)\psi_2\bar{\varphi}\frac{\psi_1}{1+\varphi\bar{\varphi}}\right]. \tag{23}$$

It would also be equal to zero if it were possible to carry the action of $\bar{Q}^1$ over to the left (to "turn over" the derivative $\partial/\partial\bar{\varphi}$), in order to use $Q_1|\mathrm{vac}_2\rangle=0$. But the essential point is that in our case this is impossible, since the state $|X\rangle=\psi_2\bar{\varphi}|\mathrm{vac}_1\rangle$ is not normalizable and the matrix elements $\langle X|Q_1|\mathrm{vac}_2\rangle^*$ and $\langle \mathrm{vac}_2|\bar{Q}^1|X\rangle$ differ from each other. In other words, the operators Q_α and $\bar{Q}^\alpha$ cease to be Hermitian conjugates when they act on non-normalizable states. Specifically,

$$\langle \mathrm{vac}_2|\bar{Q}^1|X\rangle = \langle X|Q_1|\mathrm{vac}_2\rangle^* + i\int\bar{\partial}\left(\frac{\bar{\varphi}}{1+\varphi\bar{\varphi}}\right)d\varphi\, d\bar{\varphi}\left\{\int d\varphi\, d\bar{\varphi}\left(\frac{1}{1+\varphi\bar{\varphi}}\right)^2\right\}^{-1} = 0+i=i. \tag{24}$$

The contribution of the second term in (24) gives the matrix element of (19) after multiplication by i. The integral of the total derivative $\bar{\partial}(\bar{\varphi}/(1+\varphi\bar{\varphi}))$ coincides, to within a numerical factor, with the integral

$$\int Rh\, d\varphi\, d\bar{\varphi} \sim \int \bar{\partial}(\partial h/h)\, d\varphi\, d\bar{\varphi},$$

which determines the Euler characteristic of the manifold.

We note that, in contrast to the model analyzed in Sec. 2, in this case we need not impose any additional conditions on the wavefunctions. The usual normalizability condition for states of a discrete spectrum is sufficient.

4. SUPERSYMMETRIC YANG-MILLS FIELDS

We turn to the analysis of the question of the fermion condensate in SYM theory without matter. We note first of all that the operator $\lambda^{\alpha\beta}(\sigma_\mu)_\beta{}^\alpha A_\mu{}^a$ from the relation (1) is gauge-noninvariant, and, when it acts on the vacuum, gives a gauge-noninvariant nonphysical state. In a gauge theory, however the standard supersymmetry algebra acts only in the sector of physical states. Indeed, in the SYM theory, the anticommutator $\{Q_\alpha \bar{Q}^\beta\}$ contains not only the standard term $P_\mu(\sigma^\mu)_\alpha{}^\beta$ but also the extra term

$$\sim \int A_k{}^a(\mathbf{x})\, \mathcal{G}^a(\mathbf{x})\,(\sigma_k)_\alpha{}^{\dot\beta}\, d^3x, \tag{25}$$

where $\mathcal{G}^a(\mathbf{x})$ is the operator Gauss law $\mathcal{G}^a(\mathbf{x}) \propto (\operatorname{div}\mathbf{E}^a - j_0{}^a)$ coinciding with the generator of local gauge transformations. The extra term (25), proportional to the equation of motion, arises because we are considering a theory without the introduction of auxiliary fields, i.e., a theory in the Wess-Zumino gauge. In this case the composition of two supersymmetry transformations gives not only a translation but also a certain gauge rotation. In the sector of physical states $\mathcal{G}^a(\mathbf{x})|\Psi\rangle = 0$, the extra term (25) vanishes and the standard supersymmetry algebra applies.

It is interesting to trace how the supersyememtry is lost in the gauge-noninvariant sector for a simple model. Following the approach outlined in Sec. 3, we shall reduce the theory to quantum mechanics: We place the system in a finite three-dimensional box, impose periodic boundary conditions on the fields, and discard all nonzero modes. We thus obtain the quantum mechanics of the zero modes. As shown in Ref. 1, in the gauge $A_0 = 0$ the dynamical degrees of freedom will be the three bosonic coordinates φ_i; these can be chosen to be $\varphi_i = A_i{}^3$ $(i = 1, 2, 3)$, where the superscript 3 indicates the orientation in color space. The fermionic degrees of freedom are λ_α^3 and $(\bar{\lambda}^3)^\alpha$. (In the following the index 3 will be omitted.) The Hamiltonian describing the dynamics of the zero modes,

$$H = -{}^1/_2\partial^2/\partial\varphi_i\partial\varphi_i, \tag{26}$$

corresponds to free motion in the space of the φ_i. The supercharges are written in the form

$$Q_\alpha = -i(\sigma_j)_\alpha{}^\gamma\lambda_\gamma\frac{\partial}{\partial\varphi_j}, \quad \bar{Q}^\alpha = -i\bar{\lambda}^\delta(\sigma_j)_\delta{}^\alpha\frac{\partial}{\partial\varphi_j}, \quad \bar{\lambda}^\delta = \frac{\partial}{\partial\lambda_\delta}. \tag{27}$$

It is easily verified that $\{Q_\alpha \bar{Q}^\beta\} = 2\delta_\alpha{}^\beta H$.

Up to now we have overlooked one very important circumstance: The variables φ_i do not vary within infinite limits, but are defined on a torus. In other words the points

$$\varphi_i \text{ and } \varphi_i + 2\pi m_i \quad (m_i \text{ are integers}) \tag{28}$$

are identified with each other.[1] This identification is a "recollection" of the gauge invariance of the original SUSY gluodynamics.

We emphasize that the model to be discussed is not simply gluodynamics in a finite volume. For the latter, the presence of nonzero (massive) modes (e.g., $\lambda^\pm$ in the SU(2) case) is extremely important. Treating these modes in the Born-Oppenheimer approximation, we note that they become massless on the boundary of the range of variation of φ_i. The effective lowest-approximation Hamiltonian (26) then acquires large corrections near $\varphi_i = 2\pi m_i$ (in Ref. 13, these were calculated explicitly in the very simple case of supersymmetric QED), and this, evidently, makes the Born-Oppenheimer approximation inapplicable for the analysis of the SYM model in a finite volume (see the detailed discussion in Refs. 5 and 13). Without dwelling further on this aspect, we propose to view the Hamiltonian (26)–(28) simply as a model which, while being a distant relative of the SYM model, has a number of features with analogs in field theory.

The wavefunctions of the physical states should be gauge-invariant. In the language of our model the condition that distinguishes the physical sector from the rest of the Hilbert space is as follows: Of all the eigenfunctions of the Hamiltonian (26) we should choose only those functions that are periodic in φ_i with period 2π.

The gauge invariance of the wavefunctions implies not only periodicity of the wavefunction $\Psi(\varphi_i)$ in φ_i, but also its invariance under G-parity transformations[1]:

$$\varphi_i \to -\varphi_i, \quad \lambda_\alpha \to -\lambda_\alpha. \tag{29}$$

Thus, the Schrödinger equation with Hamiltonian (26) has only two admissible solutions:

$$\Psi_{vac\,1}(\varphi_i, \lambda_\alpha) = C, \quad \Psi_{vac\,2}(\varphi_i, \lambda_\alpha) = C\lambda_\alpha\lambda^\alpha/2. \tag{30}$$

The solutions $C\lambda_\alpha$ are odd under the transformations (29), and are excluded. (Here C are normalization constants.)

The problem thus reformulated is extremely similar to the supersymmetric σ-model. It can be seen that the operator $\lambda_\alpha\lambda^\alpha$ has nonzero matrix elements

$$\langle \mathrm{vac}_2|\lambda_\alpha\lambda^\alpha|\mathrm{vac}_1\rangle = 2. \tag{31}$$

On the other hand,

$$\lambda_\alpha\lambda^\alpha = i\{Q_\alpha, \varphi_j\lambda^\beta(\sigma_j)_\beta{}^\alpha\}/3. \tag{32}$$

The right-hand side of (32) should not have matrix elements between the vacuum states that can be annihilated by the supercharges Q_α and $\bar{Q}^\alpha$, and this, obviously, contradicts (31).

The paradox is resolved by the observation that the pairs of operators Q_α, $\bar{Q}^\alpha$ are Hermitian conjugates only on the space of gauge-invariant wavefunctions periodic in φ_i. In fact, the Hermitian conjugacy of the supercharges (27) is connected with the hermiticity of the momentum operator. But the latter holds if, upon integration of the matrix element

$$\langle 1|P_i|2\rangle = -i\int \Psi_1^* \partial_i \Psi_2\, d^3\varphi$$

by parts, the surface term is equal to zero, i.e., $\Psi_1(\varphi_i)$ and $\Psi_2(\varphi_i)$ take the same values on the boundaries of the box. But the state

$$|X\rangle \equiv \varphi_j \lambda^\beta (\sigma_j)_\beta{}^\alpha |\mathrm{vac}_1\rangle \qquad (33)$$

is not gauge-invariant (the periodicity of the wavefunction is broken), and the matrix element $\langle \mathrm{vac}_2|Q_\alpha|X\rangle$ is given by the expression

$$\langle \mathrm{vac}_2|Q_\alpha|X\rangle = \langle X|\bar{Q}^\alpha|\mathrm{vac}_2\rangle^* - 2i\int d^3\varphi(\partial_j\varphi_j) \Big/ \int d^3\varphi = 0 - 6i = -6i \qquad (34)$$

(compare with (24)). When (34) is taken into account the right-hand side of (32) acquires a nonzero matrix element between the states $|\mathrm{vac}_1\rangle$ and $|\mathrm{vac}_2\rangle$, which coincides exactly with (31).

5. CONCLUSION

We have considered quantum-mechanical examples (two of these are related to important field-theory models) which show that a formal analysis of the system in the complete Hilbert space can give results that contradict the general consequences of SUSY. The principal lesson is that the supercharge operator, and also the other operators with which the manipulations are performed, should be well defined. This requirement in each concrete case leads to specific conditions on the choice of states from the Hilbert space. Besides normalizability, additional conditions on the physical states can arise. The action of the supercharge and of the "admissible" operators of the theory should not carry the states out of the physical sector. Supersymmetry holds only in part of the Hilbert space (the physical sector), and only those properties which stem from consideration of the admissible set of operators are fulfilled. The formal chains of conclusions using poorly defined operators lead to paradoxes. Although these results apply strictly to quantum mechanics, qualitatively the same situation also obtains in field theory.

[1)] In the case when there is one fermionic variable, the matrix representation $\psi = \sigma_- = (\sigma_1 - i\sigma_2)/2$, $\bar{\psi} = \sigma_+ = (\sigma_1 + i\sigma_2)/2$, where the σ_i are the Pauli matrices, is extremely convenient. The Hamiltonian then takes the form $H = (p^2 + W^2 + \sigma_3 W')/2$ and acts on a two-component column.

[2)] The relation (17) implies a so-called holomorphic representation. If ψ has the form $\Psi = a(\varphi, \bar{\varphi}) + b_1(\varphi, \bar{\varphi})\psi_1 + b_2(\varphi, \bar{\varphi})\psi_2 + \dots$, then Ψ^* must be taken to mean $\Psi^* = a^* + b_1^*\bar{\psi}^1 + b_2^*\bar{\psi}^2 + \dots$.

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