Gluino condensation in supersymmetric theories with $SU(N)$ and $O(N)$ gauge groups

A. I. Vainshtein and M. A. Shifman

Institue of Theoretical and Experimental Physics
(Submitted 21 April 1987)
Zh. Eksp. Teor. Fiz. **93**, 1930–1942 (December 1987)

Supersymmetric $SU(N)$ and $O(N)$ gauge theories without matter are considered. A construction is proposed that can be used to evaluate the condensate $\langle \mathrm{Tr}\lambda\lambda \rangle$ in such theories. It is found that, in both these cases, $\langle \mathrm{Tr}\lambda\lambda \rangle_G = \tilde{\Lambda}_G{}^3 \exp(2\pi i k/T(G)) = 0$, where $\tilde{\Lambda}_G$ is a scale parameter, $G = SU(N)$ or $O(N)$, k is an integer, and $T(G)$ is the Dynkin index divided by two. In the case of unitary groups $\langle \mathrm{Tr}\lambda\lambda \rangle_{SU(N)}$ assumes N different values, whereas in the case of orthogonal groups, $\langle \mathrm{Tr}\lambda\lambda \rangle_{O(N)}$ assumes $N-2$ values. The degeneracy of the vacuum is then equal to N and $N-2$ (for $\theta = 0$), respectively.

1. INTRODUCTION

We shall examine supersymmetric Yang-Mills theories and will show that, in such theories with $SU(N)$ and $O(N)$ gauge groups (without matter), a nonzero vacuum expectation value of $\lambda\lambda$ arises dynamically, i.e.,

$$\langle 0|\mathrm{Tr}\,\lambda_\alpha\lambda^\alpha|0\rangle \neq 0, \qquad (1)$$

where λ is the gluino field.

There are now several well-known arguments in favor of gluino condensation, but they refer mostly to unitary groups (see below). The question of gluino condensations became particularly acute after it was suggested[1] that a condensate such as (1) could be used within the framework of the superstring approach for supersymmetry (*SUSY*) breaking. We shall show below that, in theories with unitary and orthogonal gauge groups, the condensate (1) appears in the strong coupling regime (symplectic and exclusive groups will be examined in a separate publication).

Our proof will be based on a combination of two ideas, namely, (a) the observation made in Ref. 2 that the introduction of matter leads to spontaneous (partial or complete) breaking of the gauge group and (b) the Ward identities of Ref. 3, which can be used to determine exactly the dependence of the condensates on the parameters of the Lagrangian.

Before we explain the essence of the proposed method in greater detail, we must say something about the literature on this topic. At the same time, we shall establish the connection with another interesting problem, namely, the evaluation of the Witten index[4] in supersymmetric gauge theories.

By definition, the index I_W is equal to the number of boson states with zero energy minus the number of fermion states with zero energy (we are assuming that infrared regularization has been introduced into the theory, so that the spectrum of the Hamiltonian is discrete). The analysis given in Ref. 4 shows that I_W is an invariant of the volume in which the supersymmetric system is considered, etc. (I_W does not change under a continuous variation of the parameters of the Hamiltonian). In other words, I_W can be evaluated in the weak coupling regime, which enables us to determine I_W for complex theories such as four-dimensional gauge models. In this situation, I_W is equal to the number of different vacuum states.

Without going into further details (see Ref. 4), we recall that, according to Ref. 4, supersymmetric gluodynamics (i.e., theories without matter) shows that

$$I_W = r(G) + 1, \qquad (2)$$

where $r(G)$ is the rank of the gauge group G. In the case of unitary groups, $G = SU(N)$, the rank is equal to $N-1$, and the result given by (1) reduces to $I_W = N$. This was obtained in Ref. 4 in two different ways, one of which ("twisted" boundary conditions) appears to be completely reliable. If this is so, then we have to ask: what is the significance of the existence of N-degenerate vacuums in the $SU(N)$ model?

The answer to this question will be given below. The point is that this model exhibits discrete Z_{2N} symmetry that is the "residue" of classical $U(1)$ invariance of the Lagrangian, broken by the axial quantum-mechanical anomaly. The nonzero expectation value (1) spontaneously breaks Z_{2N} to Z_2 and, in accordance with general rules, this results in N-degenerate vacuums.

Gluino condensation in the case of $SU(N)$ groups in thus seen to be at least in accord with (2). Unfortunately, in the case of $O(N)$ groups, the same hypothesis (1) yields $N-2$ vacuums, whereas (2) leads to $I_W = [N/2] + 1$ where the square brackets represent the integer part.

The results given below establish unambiguously the existence of the condensate (1) for $SU(N)$ and $O(N)$ groups. The consequence of this is that the correct value of I_W for *SUSY* gluodynamics with the $O(N)$ gauge group is

$$I_W = N-2, \quad G = O(N). \qquad (3)$$

The relation given by (2) is then incorrect.

We now turn to another line of argument, developed in Refs. 5 and 6. The discussion given in Ref. 5 is based on an analysis of the correlation function

$$\Pi(x_1, \ldots, x_T) = \langle 0|T\{\mathrm{Tr}\,\lambda\lambda(x_1), \ldots, \mathrm{Tr}\,\lambda\lambda(x_T)\}|0\rangle, \qquad (4)$$

where $T \equiv T(G)$ is one-half of the Dynkin index for the associated representation of the group G:

$$\mathrm{Tr}(T^a T^b) = T(G)\delta^{ab}; \qquad (5)$$

where T^a are the generators of the group in the associated representation. When the Lagrangian is supersymmetric, the correlator (4) cannot depend on the arguments x_i (even for spontaneous *SUSY* breaking[7]). In other words,

 0038-5646/87/121100-07$04.00

$$\Pi(x_1,\dots,x_T)=\text{const.} \tag{6}$$

If it were possible to show that the constant on the right-hand side of (6) differs from zero, then, using the clusterization properties for $x_i \to \infty$, we would conclude that

$$\Pi(x_1,\dots,x_T)=\langle \mathrm{Tr}\,\lambda\lambda\rangle^T=\text{const.} \tag{7}$$

The existence of the condensation (1) and the fact that

$$I_W=T(G) \tag{8}$$

follow directly from this. (We note that $\langle \mathrm{Tr}\lambda\lambda\rangle$ is then an order parameter that equals $1^{1/T}$, a total of T values.) The question as to the value of the right-hand side of (6) is a dynamic one. Theoretical advances in this area have been quite dramatic. First, it was shown in Ref. 5 that, for small values of x_i,i.e.,$x_i \ll \Lambda^{-1}$(Λ is a scale on which the gauge constant formally becomes infinite), an instanton of dimension $\rho \sim |x_i - x_j|$ provides a nonzero contribution to $\Pi(x_1,...,x_T)$ and, in accordance with general conditions, the result does not depend on the arguments x_i. Next, the following hypothesis was formulated in Ref. 6: to the extent that one is dealing with a correlation function at short distances, the one-instanton contribution (due to instantons with $\rho \sim |\Delta x|$) should completely saturate $\Pi(x_1,...,x_T)$. If this were the case, our own research would be necessary. Unfortunately, it was demonstrated in a subsequent paper[8] that the contribution of small instantons is compensated (partially or completely, depending on the particular model) by the contribution acquired over long distances (i.e., in supersymmetric gluodynamics under the strong coupling regime, which cannot by controlled theoretically).

It follows that an analysis performed within the framework of Refs. 5 and 6 can at best be looked upon as indicating the existence of the gluino condensate. An analogous status must be assigned to toron calculations[9], which are valid only in the case of $SU(N)$ groups, and to the arguments given in Ref. 10 and based on simple effective Lagrangians that implement the anomalous Ward identities.

The solution of supersymmetric gluodynamics, as a strong coupling theory, would provide a direct proof of (1) and (8), but this is well outside existing possibilities. We shall therefore outline an indirect method that will be used below to prove these results.

We start with $SU(2)$ gluodynamics. The first element of the construction is the introduction of two auxiliary superfields of matter in the fundamental (doublet) representation, namely, S and T. These additional fields are equipped with the mass term $mST|_F$ + h.c. In the limit $m \to \infty$, the matter fields leave the spectrum and we return to the original model (supersymmetric gluodynamics) in which we wish to determine $\langle \mathrm{Tr}\lambda\lambda\rangle$.

The second step is to consider the *opposite* limit, namely, $m \to \infty$. For small m, such that $m \ll \Lambda$, the $SU(2)$ gauge invariance of the model is spontaneously broken, the symmetry breaking is complete,[2] and all three gauge bosons acquire the mass m_V:

$$m_V \sim g\Lambda^{5/4} m^{-1/4}. \tag{9}$$

The important point is that, as $m \to 0$, the mass of the vector becomes $m_V \gg \Lambda$ and we enter the weak-coupling regime in which all the quantities in which we are interested are reliably determined. In particular,

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle = C\Lambda^{5/2} m^{1/2} e^{\pi i k}, \quad k=0,\ 1, \tag{10}$$

where C is a nonzero constant to be calculated.

The third step is to establish the general dependence of the $\lambda\lambda$ condensate on m. It follows that the Ward identities based on $SUSY$[3] that[1)]

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle \propto m^{1/2} \tag{11}$$

whatever the dependence on the mass parameter m. Combining (11) with (10), we then find that as $m \to \infty$ (i.e., in pure gluodynamics), we have

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle = C\Lambda^{5/2} m^{1/2} e^{\pi i k} = C\tilde{\Lambda}^3 e^{i\pi k}, \tag{12}$$

where $\tilde{\Lambda}$ is the scale parameter of supersymmetric gluodynamics. Having determined the gluino condensate in $SU(2)$ [or $O(3)$] gluodynamics, we can generalize the construction to an arbitrary unitary (orthogonal) group. Actually, the changes involve only a single point, namely, the particular choice of the auxiliary multiplets of matter.

2. *SU(2)* Model

In superfield language, action is given by

$$S = \frac{1}{2g^2}\mathrm{Tr}\int d^2\theta\, d^4x\, W^2 + \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\, d^4x\, \overline{S^f} e^V S^f + \frac{m}{4}\left(\int d^2\theta\, d^4x\, S^{\rho f} S_{\rho f} + \text{h.c.}\right), \tag{13}$$

where W is the chiral superfield

$$W_\alpha = i\lambda_\alpha - \theta_\alpha D - i\theta^\beta G_{\alpha\beta} + \theta^2 D_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}, \tag{14}$$

$\rho = 1, 2$ is the spinor index, and $f = 1, 2$ is the flavor index (we have introduced two chiral superfield doublets).

A detailed description of the situation in the theory with the above Lagrangian can be found in the review given in Ref. 11 (which also provides a full explanation of the notation used below). Here, we confine our attention to the following remarks.

As $m \to 0$, the scalar-field self-action potential obtained from (13) has valleys, i.e., planar directions along which the D-terms vanish. Specifically, these directions can be parametrized as follows:

$$\varphi_1 = v\begin{pmatrix}1\\0\end{pmatrix}, \quad \varphi_2 = v\begin{pmatrix}0\\1\end{pmatrix}, \tag{15}$$

where ϕ_f is the lower component of the superfield $S_F(f = 1, 2)$. The energy of the configuration (15) is zero for arbitrary v. Moreover, when $m = 0$, the values of the F-terms are also zero (effects connected with the fact that m is not zero will be discussed later).

The infinite degeneracy of vacuum for $m = 0$, i.e., the existence of the valley (15), occurs not only at the classical level, but also when all orders of perturbation theory are taken into account, and is a consequence of the renormalization theorems.[12] The question now is: is degeneracy lifted by nonperturbative effects?

Before we answer this question, we must examine the structure of the model for $v \neq 0$ in greater detail. For $gv \gg \Lambda$, the theory splits into two sectors, namely, heavy particles, i.e., the triplet of massive gluons, $m_v = gv$, the triplet of Dirac spinors, and the triplet of real scalar fields of the same mass, and light particles, which involves only one superfield $\sim S_{af}(\equiv S^2)$ that is a singlet in both the color and the flavor $SU(2)$ groups. At energies much smaller than gv, only light particles belonging the S^2 supermultiplet survive in the spectrum of the theory. The structure of the vacuum of the theory and, in particular, the vacuum condensates, is uniquely determined by the Lagrangian describing the dynamics of the "sterile" light field S^2. We emphasize that, when $gv \gg \Lambda$, integration over the heavy degrees of freedom and the construction of the effective low-energy Lagrangian for S^2 constitute a procedure that has been justified parametrically.

We now return to the superpotential problem. This contains the classical term $mS^2|_F$ + h.c. which, strictly speaking, destroys the valley and raises its bottom, so that

$$\Delta V_{nom}=m^2v^2, \tag{16}$$

while tending to "bend" the theory toward the "unfavorable" point $v = 0$. Fortunately, there is also the reverse tendency because instantons generate the superpotential, tending to increase v. Specifically,[2,11]

$$\Delta S_{inst}=C\int d^2\theta\, d^4x \frac{\Lambda^5}{S^2(x_L,\theta)}+\text{h.c.}, \tag{17}$$

where ΔS_{inst} is the instanton contribution to action and C is a nonzero constant to be calculated. The important point is that, when $gv \gg \Lambda$, the theory is completely determined in the infrared region. In particular, the superpotential (17) is due to an instanton of dimension $\rho \sim v^{-1}$, and all the calculations are completely controlled theoretically.

When the superpotential contains two terms, namely, the mass term and the instanton term (17), the functional degeneracy of vacuum is found to disappear, and the potential energy for the configurations (15) vanishes only for certain values of v. These values make the F-term vanish:

$$\frac{m}{2}S-C\frac{\Lambda^5}{(S^2)^2}2S=0,$$

or, in other words,

$$v^2=\pm 2(C\Lambda^5/m)^{1/3}. \tag{18}$$

When m is small, the solution for the vacuum field does actually satisfy the condition $gv \gg \Lambda$, which justifies the single-instanton approximation in (17), and the procedure as a whole. We draw attention to the fact that v^2 assumes two values in vacuum (v^2 is a gauge invariant). These are the same two vacuums that are predicted by the Witten index for the $SU(2)$ theory.

Finally, to determine the gluino condensate, we use an operator relation,[13] frequently referred to as the Konishi anomaly in the literature:

$$\frac{1}{8}\bar{D}^2\overline{S^{af}}e^V S^{af}=\frac{m}{2}S_{af}S^{af}+\frac{1}{16\pi^2}\text{Tr}\,W^2. \tag{19}$$

When the average is taken over the supersymmetric vacuum state, the left-hand side vanishes and, consequently,

$$\frac{1}{16\pi^2}\langle\text{Tr}\,\lambda\lambda\rangle=\frac{m}{2}\langle S^2\rangle=\pm(C\Lambda^5 m)^{1/3}. \tag{20}$$

Let us summarize the situation so far. The value of the condensate (1) has been found in the theory with auxiliary matter multiplets. In the limit $m \ll \Lambda$, the right hand side of (20) is known reliably and has two values that are definitely nonzero. Our aim is to pass to the limit as $m \to \infty$. This can be done by exploiting the observation made in Ref. 3 whereby we know precisely how $\langle \text{Tr}\lambda\lambda \rangle$ depends on a parameter of the Lagrangian, such as m.

We now recall how this was done in Ref. 3. The theory defined by (13) exhibits anomaly-free $U(1)$ symmetry

$$\lambda\to\lambda e^{i\alpha},\quad \psi^f\to\psi^f e^{-2i\alpha},\quad S^f\to S^f e^{-i\alpha}, \tag{21}$$

which is broken only by the mass term in the Lagrangian. The fact that the chiral transformation (21) does not contain an anomaly is most readily verified by using an instanton (which, as always, displays all the anomalies). Actually, the chiral charge of the gluino is 1 and the chiral charge of matter fermions is -2. However, in the model that we are considering, the instanton configuration actually has four zero gluino modes and two matter modes: $4\times 1 + 2(-2) = 0$. The absence of an anomaly in the current

$$J_{\alpha\dot\alpha}=-(\lambda_\alpha\bar\lambda_{\dot\alpha}-2\psi_\alpha{}^f\bar\psi_{\dot\alpha}{}^f)+{}^1/_2 i\varphi^f\overleftrightarrow{\partial}_{\alpha\dot\alpha}\overline{\varphi^f} \tag{22}$$

can also be verified by standard methods [α and $\dot\alpha$ in (22) are the spinor indices, and summation over $f = 1, 2$ is implied; the color indices are not shown explicitly]. Next,

$$\partial_\mu J_\mu=i(2mS^2|_F-2\bar m\bar S^2|_F)$$

and, consequently,

$$\int d^4x\langle 0|T\{\lambda\lambda(0),\partial_\mu J_\mu(x)\}|0\rangle$$
$$=i\int d^4x\langle 0|T\{\lambda\lambda(0),2mS^2(x)|_F-2\bar m\bar S^2(x)|_{\bar F}\}|0\rangle_{\text{connected}}. \tag{23}$$

Since the spectrum does not contain massless particles, the left-hand side of (23) reduces to the commutator of $\lambda\lambda$ and the chiral charge, and is equal to $2\langle\lambda\lambda\rangle$.

The correlator $\langle T\{\lambda\lambda,\bar D^2\bar S^2\}\rangle_{\text{connected}}$, is obviously identically equal to zero by virtue of the above theorem on the correlation functions of the lower components of chiral superfields[7] (of the same chirality). Hence it follows that $\langle\lambda\lambda\rangle$ is independent of $\bar m$ and not merely of m, and the right-hand side of (23) is equal to $4m\partial\langle\lambda\lambda\rangle/\partial m$. Thus, finally, we may conclude that (23) is equivalent to the following relation:

$$m\frac{\partial}{\partial m}\langle\lambda\lambda\rangle=\frac{1}{2}\langle\lambda\lambda\rangle. \tag{24}$$

In other words, the square root dependence of $\langle\lambda\lambda\rangle$ on m, which is valid in (20) for small m, is in fact exact, which is clear from the general equations. It is now opportune to note that there are different parameters, and the question is which of them are considered constant under differentiation. To obtain the righthand side of (23) from $m\partial\langle\lambda\lambda\rangle/\partial m$, we must fix the gauge constant g_0 at the ultraviolet cutoff M_0. It is obvious that this determines the scale parameter Λ:

$$\Lambda\equiv M_0\exp(-8\pi^2/5g_0^2), \tag{25}$$

where the factor 5 in the denominator is the first coefficient

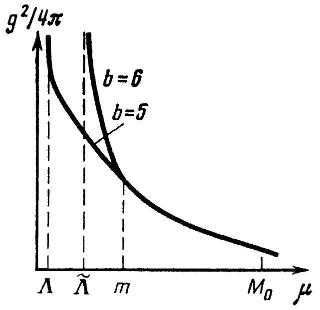

FIG. 1. Evolution of the effective gauge constant. Because of the difference in the first coefficient of the Gell-Mann function in the regions below and above m, the quantity $\tilde{\Lambda}$ is found to depend on m.

of the Gell-Mann–Low function of the theory (13) with $m\to 0$. For given M_0, g_0 (or, equivalently, given Λ) and variable but large m, the scale parameter $\tilde{\Lambda}$ characterizing the theory without matter, which survives below m, will vary (Fig. 1) in a way that can readily be determined by recalling that the first coefficient in the Gell-Mann–Low function is less than m, i.e., it is equal to 6 in the theory without matter. It is clear from Fig. 1 that

$$\Lambda = me^{-2\pi/5\alpha(m)}, \quad \tilde{\Lambda} = me^{-2\pi/6\alpha(m)} \tag{26}$$

and, consequently,

$$\Lambda^5 m = \tilde{\Lambda}^6. \tag{27}$$

In other words, in the limit as $m\to\infty$, the gluino condensate

$$\frac{1}{16\pi^2}\langle \mathrm{Tr}\,\lambda\lambda\rangle = \pm(\tilde{C}\Lambda^6)^{1/2}, \tag{28}$$

is a quantity that is absolutely natural for the theory in which there is hypothetical confinement at distances $\sim\tilde{\Lambda}^{-1}$.

3. GENERALIZATION OF ARBITRARY N

We begin by considering the chain $SU(2)\to SU(3)\to\ldots\to SU(N)$. The existence of the gluino condensate in the group $SU(N-1)$ unavoidably leads to the same effect in the group $SU(N)$. To prove this, we again introduce two auxiliary chiral superfields of matter in the fundamental representation, one N-tuplet S_α, and one anti-N-tuplet $\bar{T}^\alpha$. The mass term then has the form

$$\Delta S_m = \int d^4x\, d^2\theta \frac{m}{2} S_\alpha \bar{T}^\alpha + \text{h.c.} \tag{29}$$

For $m = 0$, there is a planar direction that is trivially parametrized:

$$S = v\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad \bar{T} = v\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}.$$

When $gv\gg\Lambda$, the $SU(N)$ color group is spontaneously broken to $SU(N-1)$ and, at low energies, we have the $SU(N-1)$ gluodynamics plus the singlet sterile field $\overline{ST}$. The low-energy scale parameter $\tilde{\Lambda}$ is related to Λ by

$$\tilde{\Lambda}^3_{SU(N-1)} = \Lambda^{(3N-1)/(N-1)}(v^2)^{-1/(N-1)}. \tag{30}$$

At energies below $\tilde{\Lambda}$, only the singlet field $\overline{ST}$ survives in the

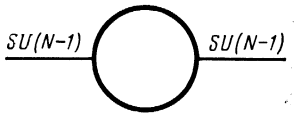

FIG. 2. Loop generating the term (31) in the effective Lagrangian. The thick line represents fields that have acquired mass under spontaneous $SU(N)\to SU(N-1)$ breaking.

spectrum and its effective Lagrangian determines the vacuum structure. The mass term (29) tends to reduce v to zero, but the gluino of the condensate in $SU(N-1)$ generates[2] a "repulsive" nonperturbative term in the superpotential. Actually, integrating over the heavy fields (with mass $m_v = gv$), we find from the loop in Fig. 2 that

$$\Delta S = \text{const}\int d^4x\, d^2\theta\,(\ln(M_0^2/S\bar{T}))\,\mathrm{Tr}\,W^2_{SU(N-1)} + \text{h.c.} \tag{31}$$

This interaction plus $\langle \mathrm{Tr}\,W^2\rangle_{SU(N-1)} = \tilde{\Lambda}^3 \neq 0$ guarantees that, at energies below $\tilde{\Lambda}$, the superpotential is[2]

$$\Delta S = \text{const}\int d^4x\, d^2\theta\, \exp(2\pi ik/(N-1)) \times \Lambda^{(3N-1)/(N-1)}(S\bar{T})^{-1/(N-1)}. \tag{32}$$

To prove this, we compare the F-terms in (31) with (32), subject to (30). The constant on the right-hand side of (32) can, in principle, be evaluated and is not equal to zero. The dependence on $(S\bar{T})$ on the right-hand side of (32) can be found from general considerations. The anomaly-free current that is the analog of (22) now assumes the form

$$J_{\alpha\dot{\alpha}} = -\{\lambda_\alpha\bar{\lambda}_{\dot{\alpha}} - N(\psi_T)_\alpha(\bar{\psi}_T)_{\dot{\alpha}} - N(\psi_S)_\alpha(\bar{\psi}_S)_{\dot{\alpha}}\} + \text{bosons} \tag{33}$$

where the coefficients can readily be established from instanton calculus: $2N$ gluino modes protrude from the instanton, namely, one ψ_t mode and one ψ_s mode. The combination given by (32) is the only invariant of the corresponding chiral rotations. Equating to zero the complete F-term [(32) + (29)], we obtain the vacuum values of $\overline{ST}$:

$$v^2 = \{m^{-1}\Lambda^{(3N-1)/(N-1)}e^{2\pi ik/(N-1)}\}^{(N-1)/N}. \tag{34}$$

The factor $N-1$ in the exponent annuls the $(N-1)$-valued factor $\exp[2\pi ik/(N-1)]$ in $\langle \mathrm{Tr}\lambda\lambda\rangle_{SU(N-1)}$, and the factor $1/N$ leads to the N-valued factor $\exp(2\pi ik/N)$ typical for $SU(N)$. For small m, we have $gv\gg\Lambda$, and all the approximations made above are justified.

So far, we have almost completely followed the line laid down in Ref. 2. We now use the anomalous relation given in Ref. 13 [which is essentially the same as (19)] and immediately arrive at the conclusion that as $m\to 0$, i.e., in the weak-coupling regime,

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle_{SU(N)} = m^{1/N}\Lambda^{(3N-1)/N}e^{2\pi ik/N}. \tag{35}$$

Finally, let us examine the Ward mass identities, which enable us to cover the entire range from $m\to 0$ to $m\to\infty$. The latter limit corresponds to the $SU(N)$ gluodynamics without matter. Starting with the current given by (33), and repeating the entire procedure described in Sec. 2, we find that

$$2\langle \mathrm{Tr}\,\lambda\lambda\rangle = 2Nm\frac{\partial}{\partial m}\langle \mathrm{Tr}\,\lambda\lambda\rangle|_{\Lambda\ \text{fixed}}, \tag{36}$$

which proves the validity of (35) for arbitrary m. We also note, that for fixed Λ, and variable but large m,

$$\Lambda^{(3N-1)/N} m^{1/N} = \tilde{\Lambda}^{3}_{SU(N)}, \tag{37}$$

where $\tilde{\Lambda}_{SU(N)}$ is the scale parameter of $SU(N)$ gluodynamics. As expected,

$$\langle \mathrm{Tr}\, \lambda\lambda \rangle_{SU(N)} = \tilde{\Lambda}^{3}_{SU(N)} e^{2\pi i k/N}.$$

The entire scheme was found to be completely self-consistent but, as the attentive reader may have noticed; it relies on the requirement that the various quantities made up of integers were equal. Before we proceed to the analysis of orthogonal groups, it will be useful to examine the corresponding arithmetic in general form.

The exponent x in $(S\overline{T})^{-x}$ in (32) was determined in two ways: (a) by joining the scale parameters of the theory with the original group G and light matter on the one hand, and the theory obtained after the $G \to G'$ breaking [see (30)], on the other, and (b) by considering the anomaly-free axial current. This leads to the requirement

$$x = T_M/(T - T_M) = (3T - 3T' - T_M)/2T', \tag{38}$$

where $T \equiv T(G), T' \equiv T(G')$, and T_m is the corresponding index for matter:

$$T_M = \sum_i T(R_i) \tag{39}$$

where the sum is evaluated over all the auxiliary multiplets of matter and $T(R_i)$ is defined as follows:

$$\mathrm{Tr}(T^a T^b) = T(R)\delta^{ab}, \tag{40}$$

where the generators T^a of the group G are taken in the given representation R. In the case examined above, $T(SU(N)) = N$, we have $T_M = 2 \times \frac{1}{2} = 1$. We note the fact that the indices T coincide with one-half of the number of zero fermion modes of a given type that "stick out" from the instanton. This appears to be practically the simplest way of evaluating T. Next, the dependence of $\langle \mathrm{Tr}\lambda\lambda \rangle_G$ on m is determined, on the one hand, dynamically [see (34) and (35)] and, on the other hand, by the Ward mass identities [see (36)]. For consistency, we must have

$$x/(x+1) = T_M/T.$$

Finally, for large m^2, the condensate $\langle \mathrm{Tr}\lambda\lambda \rangle_G$ must reduce to $\tilde{\Lambda}^3_G$, i.e., to the scale parameter of the group G without matter $(\tilde{\Lambda}^3_G = m^{T_M/T}\Lambda^{(3T-T_M)/T})$. Hence we find that

$$(x+1)T' = T. \tag{41}$$

In principle, this results in a set of restrictions that may contain a redundancy and may not have solutions. However, it is readily verified that all the relationships are simultaneously satisfied if

$$T' = T - T_M. \tag{42}$$

The proposed program of proof by induction was found to be successful for the groups $SU(N)$ because

$$G = SU(N), \quad G' = SU(N-1), \quad T = N, \quad T' = N-1, \quad T_M = 1.$$

We can now readily generalize our discussion to a chain

$$O(6)(\equiv SU(4)) \to \ldots \to O(N).$$

We now introduce a minor digression concerning the instanton in $O(N)$ groups, which will be useful when we count the indices. It is well known that an instanton can be imbedded in $O(N)$ in two different ways: (a) by selecting a subgroup $O(3)$ from $O(N)$ and identifying the $O(3)$ generators with the $SU(2)$ generators and (b) by selecting the subgroup $O(4) \equiv SU(2) \times SU(2)$ from $O(N)$ and placing the instanton in one of the $SU(2)$ "corners". The representation of $O(4)$ is the (2,2) representation of $SU(2) \times SU(2)$. The first method yields the configuration with topological charge greater by a factor of two than the minimum. We shall confine our attention to the second method and reserve the designation "instanton" for it.

For the group $O(6)$ that coincides with $SU(4)$, we already know that gluino condensation does occur, and $\langle \mathrm{Tr}\lambda\lambda \rangle$ assumes four possible values. We not introduce an auxiliary N-tuplet of matter into $O(N)$, i.e., a chiral superfield V_a with the mass term

$$\Delta S_m = \int d^4x\, d^2\theta \left(\frac{m}{2} V^a V^a + \text{h.c.}\right). \tag{43}$$

It is readily verified using the instanton, or directly, that $T(O(N)) = N - 2$, $T_M = 1$. We shall assume that $\langle \mathrm{Tr}\lambda\lambda \rangle \neq O$ in $O(N-1)$ gluodynamics. In that case, as $m \to 0$, the following superpotential is generated in the theory:[2]

$$\Delta S = \text{const} \int d^4x\, d^2\theta\, \Lambda^{(3N-7)/(N-3)} (V^2)^{-1/(N-3)} e^{2\pi i k/(N-3)} \tag{44}$$

and the gauge symmetry of $O(N)$ is spontaneously broken to $O(N-1)$. Since $T = N - 2$, $T' = N - 3$, and $T_M = 1$, condition (42) is satisfied. This means that we can repeat the entire analysis that was given for the $SU(N)$ chain: from (43) and (44) we find the vacuum value of V^2 which, as it turns out, is large for $m \to 0$. Next, we use the Konishi anomaly to determine $\langle \mathrm{Tr}\lambda\lambda \rangle$ in the weak-coupling regime $(m \to 0)$. Using the Ward mass identities, we find that, as $m \to \infty$.

$$\langle \mathrm{Tr}\, \lambda\lambda \rangle_{O(N)} = \tilde{\Lambda}^{3}_{O(N)} e^{2\pi i k/(N-2)} \neq 0 \tag{45}$$

which was to be proved.

4. MANY-LOOP EFFECTS

We shall now demonstrate that many-loop corrections do not in general arise if we use bare parameters. Although all the relationships given above were written in the single-loop approximation, the final results of the analysis are actually accurate for all loops. This is so because supersymmetry leads to exact formulas for the lower components of chiral superfields. In particular, this applies to the gluino condensate $\langle \lambda^2 \rangle$.

When higher loops are included, the situation can be described in greater detail as follows. In Sec. 3, we used the single-loop approximation to obtain (35) which can be written in the following form for an arbitrary group:

$$\langle \mathrm{Tr}\, \lambda\lambda \rangle = \Lambda^3 (m/\Lambda)^{T_M/T(G)}. \tag{46}$$

In this approximation, the parameter Λ is related to the ultraviolet cutoff M_0 as follows:

$$\Lambda^{3T(G)-T_M}=M_0^{3T(G)-T_M}\left(\frac{8\pi^2}{g_0^2}\right)^{T(G)}\exp\left(-\frac{8\pi^2}{g_0^2}\right),$$

and the mass m of matter coincides with the bare mass m_0. If we express the right-hand side of (46) in terms of the bare parameters, we find that

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle = m_0^{T_M/T(G)}M_0^{3-(T_M/T(G))}\left(\frac{8\pi^2}{g_0^2}\right)\exp\left(-\frac{8\pi^2}{T(G)g_0^2}\right).$$

The central point is that the right-hand side, written in terms of the bare quantities, is exact. There are no higher-loop corrections. First, the exact dependence on m_0 was established above with the aid of the Ward identities (it is readily seen that it is precisely the bare parameter m_0 that appears in the proof). The proof that we have given forbids corrections that are logarithmic in M_0/m_0. Next, the absence of nonlogarithmic corrections follows from general theorems on the nonrenormalizability of the superpotential in perturbation theory. We must add, that the well-known exact β-function[14] is completely consistent with the law that we have obtained, and is equivalent to the following result: as M_0 is varied, the quantities M_0 and g_0 vary so that the given combination remains fixed. This fixed quantity is identical with the low-energy parameter $\widetilde{\Lambda}^3$ of gluodynamics with the group G, introduced earlier. As far as the parameter Λ is concerned (it appears in the theory with light matter), its determination with allowances for the higher-order loops is subject to arbitrariness because the renormalized mass m is a logarithmic function of Λ/m, and the logarithmic dependence on Λ/m may be distributed in different ways between Λ and m for different definitions of the renormalized mass. For example, the renormalized mass can be introduced as

$$m_R=m_0\exp\int_{10\Lambda}^{M_0}\gamma(\alpha(\mu))\,(d\mu/\mu),$$

where γ is the anomalous dimension of the matter field. The parameter Λ is then found from the equation

$$m_0^{T_M/T(G)}M_0^{3-(T_M/T(G))}(8\pi^2/g_0^2)\,e^{-8\pi^2/T(G)g_0^2} = m_R^{T_M/T(G)}\Lambda^{3-T_M/T(G)}$$

This procedure of introducing renormalized quantities involves a relatively complicated logarithmic dependence that does not appear in the language of bare quantities.

A few words now about the renormalization invariance of the left-hand side of (46). We not that, in the theory with matter that we are considering, the inclusion of higher-order loops ensures that $\langle \mathrm{Tr}\lambda\lambda\rangle$ ceases to be renormalization-invariant. The following combination is also independent of the normalization point:

$$\mathrm{Tr}\,W^2+\frac{\gamma}{3T(G)-T_M}\left[T_M\,\mathrm{Tr}\,W^2+16\pi^2 mS^2\right],$$

where γ is the anomalous dimension of the matter field S. If we look at the lower component of this operator, we see a mixing of λ^2 with $m\phi^2$. The fact that the above combination is renormalization-invariant follows, in particular, from the fact that it appears in the right-hand part of the anomaly in the supercurrent $\bar{D}^{\dot\alpha}J_{\alpha\dot\alpha}$ (see Ref. 15). We note that the addition to W^2 is identical to the right-hand side of the Konishi relation for $\bar{D}^2\bar{S}e^V S$, and is thus seen to reduce to the total derivative in superspace. Consequently, it disappears when the averages are evaluated over the supersymmetric vacuum. The addition also disappears as we pass to gluodynamics, i.e., in the limit as $m\to\infty$. We emphasize that the coefficient in front of W^2 is not proportional to the β-function (see Ref. 15).

5. CONCLUSION

It is thus clear that supersymmetric gauge theories exhibit a further suprising facet: without precise knowledge of the dynamics in the strong coupling regime, it is nevertheless possible to evaluate the gluino condensate absolutely reliably by using the inductive chain, i.e., by introducing the auxiliary matter field that reduces the gauge group $G\to G'$ as $m\to 0$. by generating the nonperturbative superpotential in the G theory with light matter, subject to the condition $\langle \mathrm{Tr}\lambda\lambda\rangle_{G'}\neq 0$: by valuating $\langle \mathrm{Tr}\lambda\lambda\rangle$ in the theory with the group G and light matter; and by extending the result with the aid of the Ward identities as $m\to\infty$.

We have thus found a gluino condensate in supersymmetric Yang-Mills theories with the $SU(N)$ and $O(N)$ gauge groups without matter. In all cases,

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle_G=e^{2\pi ik/T(G)}\widetilde{\Lambda}_G^3\neq 0,\quad G=SU(N),\ O(N). \qquad (47)$$

The condensate $\langle \mathrm{Tr}\lambda\lambda\rangle_G$ assumes $T(G)$ different values,

$$T(SU(N))=N,\quad T(O(N))=N-2,$$

which correspond to the $T(G)$-fold degeneracy of vacuum. The Witten index for $SU(N)$ is $I_w=N$. This is the same as the original formula[4] and we have $I_w=N-2$ for $O(N)$, which does not correspond to the result in Ref. 4, i.e., $I_w=r(G)+1$. The derivation given in Ref. 4 for orthogonal groups is erroneous because it makes use of the Born-Oppenheimer approximation in the form formulated there. This question will be examined in greater detail in a separate publication. We note that a strategy similar to our own was used in Ref. 16, where the aim was to establish the fact of spontaneous breaking of discrete Z_{2N-n} to Z_2 in $O(N)$ groups. The authors of Ref. 16 attempted to introduce auxiliary matter into $O(N)$ gluodynamics in an amount that would completely break the gauge group $O(N)$, and would reduce the theory to the situation in which weak coupling prevails. They thus find $N-2$ solutions for vacuum and, *en route*, evalute the condensate $\langle \mathrm{Tr}\lambda\lambda\rangle$. The only new aspect that we can introduce here is the use of the Ward mass identities to fix the magnitude of the condensate in the limit of infinite mass terms, so that one essentially returns to the theory without matter.

Let us now examine the degeneracy of the vacuum from a somewhat different standpoint. We know that gauge theories contain a hidden parameter, i.e., the vacuum angle θ, and the physics of the situation is periodic in θ (with a period of 2π). In supersymmetric theories (without matter), the massless gluinos that are present in the Lagrangian ensure that none of the physical observables such as the spectrum, vacuum energy, and so on depend on θ. When we speak of the degree of degeneracy, we actually mean the additional degeneracy of the vacuum for given θ.

In contrast to physically observable quantities, the condensate $\langle \mathrm{Tr}\lambda\lambda \rangle$ may also depend on θ in a particular way. Moreover, by investigating $\langle \mathrm{Tr}\lambda\lambda \rangle$ as a function of θ, we can independently determine the degree of degeneracy of the vacuum (starting, of course, with the fact that $\langle \mathrm{Tr}\lambda\lambda \rangle \neq 0$). To establish the law of variation of $\langle \mathrm{Tr}\lambda\lambda \rangle$, we take into account the fact that the θ-term in the Lagrangian has the form

$$\mathscr{L}_\theta = \frac{1}{32\pi^2}\theta G_{\alpha\beta}{}^a \tilde{G}_{\alpha\beta}{}^a. \tag{48}$$

We now redefine the field λ:

$$\lambda' = e^{i\alpha}\lambda. \tag{49}$$

By virtue of the triangular anomaly,

$$\partial_\mu a_\mu = T(G)\frac{1}{16\pi^2} G_{\alpha\beta}{}^a \tilde{G}_{\alpha\beta}{}^a, \qquad a_\mu = \bar{\lambda}\sigma_\mu\lambda \tag{50}$$

the θ-term in the Lagrangian for λ' vanishes if

$$\alpha = -\theta/2T(G),$$

so that

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle_\theta = \langle \mathrm{Tr}\,\lambda\lambda\rangle_{\theta=0} \exp\,(i\theta/T(G)). \tag{51}$$

The question is: how do we reconcile (51) with the periodicity in θ for $\theta = 2\pi$? The answer is relatively clear: there must be $T(G)$-degenerate states for which $\langle \mathrm{Tr}\lambda\lambda \rangle$ differs by the phase factor $\exp\,(2\pi ik/T(G))$. According to (51), evolution in θ from $\theta = 0$ to $\theta = 2\pi$ simply renumbers all the states within the cycle. A analogous situation was noted previously in quantum chromodynamics with N_j flavors.[17]

The authors are indebted to G. Veneziano for stimulating suggestions and to V. Novikov and V. Chernyak for discussions.

[1)]The idea of using these Ward identities in this context arose in discussions with G. Veneziano.

[1]M. Dine, R. Rohm, N. Seiberg, and E. Witten, Phys. Lett. **156**, 55 (1985).
[2]I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. B **241**, 493 (1984); **256**, 557 (1985).
[3]D. Amati, G. Rossi, and G. Veneziano, Nucl. Phys. B **249**, 1 (1985).
[4]E. Witten, Nucl. Phys. B **202**, 253 (1982).
[5]V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B **229**, 407 (1983).
[6]G. Rossi and G. Veneziano, Phys. Lett. B **138**, 195 (1984).
[7]A. Vaĭnshteĭn, V. Zakharov, and M. Shifman, Yad. Fiz. **42**, 554 (1985) [Sov. J. Nucl. Phys. **42**, 350 (1985)].
[8]V. Novikov, M. Shifman, A. Vaĭnshteĭn, and V. Zakharov, Nucl. Phys. B **260**, 157 (1985).
[9]E. Cohen and C. Gomes, Phys. Rev. Lett. **52**, 237 (1984).
[10]G. Veneziano and S. Yankielowicz, Phys. Lett. B **113**, 321 (1982).
[11]A. Vaĭnshteĭn, V. Zakharov, and M. Shifman, Usp. Fiz. Nauk **146**, 683 (1985) [Sov. Phys. Usp. **28**, 709 (1985)].
[12]M. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. B **159**, 429 (1979).
[13]T. E. Clark, O. Piguet, and K. Sibold, Nucl. Phys. B **159**, 1 (1979); K. Konishi, Phys. Lett. B **135**, 439 (1984).
[14]V. Novikov, M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Nucl. Phys. B **229**, 381 (1983); Phys. Lett. B **166**, 329 (1986).
[15]M. A. Shifman and A. I. Vaĭnshteĭn, Nucl. Phys. B **227**, 456 (1986).
[16]S. F. Cordes and M. Dine, Nucl. Phys. B **273**, 582 (1986).
[17]R. Crewther, Phys. Lett. B **70**, 349 (1977); **93**, 75 (1980).

Translated by S. Chomet