# Topological characteristics of singular points of the electric field accompanying sound propagation in piezoelectrics

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Singularities in the characteristics (the potential $\varphi$, the field strength $\mathbf{E}$, and the electric displacement $\mathbf{D}$) of the quasistatic electric field associated with elastic waves propagating near "acoustic axes" are analyzed. The "acoustic axes" are the directions in which the phase velocities of isonormal waves are degenerate. When the wave normal $\mathbf{m} = \mathbf{m}_0 + \Delta\mathbf{m}$ ($|\Delta\mathbf{m}| \ll 1$) is scanned around the direction of the acoustic axis $\mathbf{m}_0$, the elastic displacement vectors of the degenerate waves, $\mathbf{u}_i(\mathbf{m})$ ($i = 1,2$), rotate in the plane orthogonal to the vector $\mathbf{u}_3(\mathbf{m}_0)$. Correspondingly, the quantities $|\mathbf{E}_i(\mathbf{m})|$ and $\varphi_i(\mathbf{m})$ vary from zero to certain nonzero values (by no means small, in general) when $\mathbf{m}$ is scanned in this way over an infinitely small contour around the degeneracy point $\mathbf{m}_0$. Along the direction of the acoustic axis, $\mathbf{m}_0$, the same thing happens when an arbitrarily oriented vector $\mathbf{u}_i(\mathbf{m}_0)$ is rotated in the degeneracy plane. In contrast with the field $\mathbf{E}_i(\mathbf{m}) \| \mathbf{m}$, which has an amplitude singularity but not an orientational singularity near $\mathbf{m}_0$, the electric displacement $\mathbf{D}_i(\mathbf{m}) \perp \mathbf{m}$ forms a plane vector field with a singular point $\mathbf{m}_0$ in the degeneracy region. An expression is derived for calculating the rotation of this field (the index of the singular point). It is shown that in the case $\mathbf{D}_i(\mathbf{m}_0) \neq 0$ the indices of the vector fields $\mathbf{D}_i(\mathbf{m})$ and $\mathbf{u}_i(\mathbf{m})$ at the point $\mathbf{m}_0$ are equal in magnitude but may differ in sign. At the same time, in the case $\mathbf{m}_0 \| \bar{6}$, for example, with $\mathbf{D}_i(\mathbf{m}_0) = 0$, the rotation of the field $\mathbf{D}_i(\mathbf{m})$ around $\mathbf{m}_0$ turns out to be twice the rotation of the field $\mathbf{u}_i(\mathbf{m})$ and opposite in sign. General expressions are derived for determining the limits of the absolute value $|\mathbf{D}_i(\mathbf{m}_0)|$ on a contour $|\Delta\mathbf{m}| = \text{const} \ll 1$ [or for $\mathbf{m} = \mathbf{m}_0$ when the vector $\mathbf{u}_i(\mathbf{m}_0)$ is rotated in the degeneracy plane].

## INTRODUCTION

As an elastic wave propagates through a piezoelectric, it is known to generate a quasistatic electric wave field, characterized by a potential $\varphi$, a field strength $\mathbf{E}$, and an electric displacement $\mathbf{D}$. These characteristics depend on the orientation of the elastic displacement vector $\mathbf{u}$, which in turn exhibits singular behavior for wave normals $\mathbf{m}$ which lie near acoustic axes. A detailed study of the topology of the vector field $\mathbf{u}(\mathbf{m})$ near the directions of acoustic axes was carried out in Refs. 1 and 2. In the present paper we extend the results of Refs. 1 and 2 to discuss the singularities in the characteristics $\varphi(\mathbf{m})$, $\mathbf{E}(\mathbf{m})$, $\mathbf{D}(\mathbf{m})$ of the accompanying electrostatic wave near acoustic axes in piezoelectrics.

## INITIAL RELATIONS

A coupled acoustic-electric wave in a crystal lacking a center of symmetry is described in the quasistatic approximation by the system of equations (Ref. 3, for example)

$$\sigma_{ij,\,j}=\rho\ddot{u}_i, \quad D_{i,\,i}=0, \quad E_i=-\varphi_{,i},$$
$$\sigma_{ij}=e_{kij}E_k+c_{ijkl}u_{k,\,l}, \quad D_i=\varepsilon_{ik}E_k+4\pi e_{ikl}u_{k,\,l}, \tag{1}$$

where the subscript specifies a differentiation ($i\equiv\partial/\partial x_i$); $\rho$ is the density; and $\hat{\sigma}$, $\hat{e}$, $\hat{c}$, and $\hat{\varepsilon}$ are respectively the mechanical stress tensor, the piezoelectric-constant tensor, the elastic-modulus tensor, and the dielectric tensor. According to (1), in any direction $\mathbf{m}$ ($\mathbf{m}^2 = 1$) three elastic waves can propagate:

$$\mathbf{u}_\alpha=C_\alpha\mathbf{A}_\alpha\cos\chi_\alpha, \quad \chi_\alpha=k_\alpha(\mathbf{mr}-v_\alpha t), \quad \alpha=1,\,2,\,3, \tag{2}$$

with amplitudes $C_\alpha$, mutually orthogonal unit polarization vectors $\mathbf{A}_\alpha$, and phase velocities $v_\alpha$ given by the equation[4–7]

$$\hat{\Lambda}\mathbf{A}_\alpha=v_\alpha^2\mathbf{A}_\alpha, \quad \hat{\Lambda}=\frac{1}{\rho}\left(\mathbf{m}\hat{c}\mathbf{m}+\frac{4\pi}{\varepsilon}\mathbf{e}\cdot\mathbf{e}\right). \tag{3}$$

Here we are using the notation $\mathbf{e} = \mathbf{m}\hat{e}\mathbf{m}$, $\varepsilon = \mathbf{m}\hat{\varepsilon}\mathbf{m}$. A dot between vectors means the scalar product. The characteristics $\varphi_\alpha$, $\mathbf{E}_\alpha$, and $\mathbf{D}_\alpha$ of the accompanying quasistatic electric-field wave are given by

$$\varphi_\alpha=\varphi_\alpha^0\cos\chi_\alpha, \quad \mathbf{E}_\alpha=\mathbf{E}_\alpha^0\sin\chi_\alpha, \quad \mathbf{D}_\alpha=\mathbf{D}_\alpha^0\sin\chi_\alpha, \tag{4}$$

where

$$\varphi_\alpha^0=4\pi C_\alpha(\mathbf{e}\mathbf{A}_\alpha)/\varepsilon, \quad \mathbf{E}_\alpha^0=k_\alpha\varphi_\alpha^0\mathbf{m}, \tag{5}$$
$$\mathbf{D}_\alpha^0=4\pi C_\alpha k_\alpha\hat{B}\mathbf{A}_\alpha, \quad \hat{B}(\mathbf{m})=\hat{e}\mathbf{m}-\hat{\varepsilon}\mathbf{m}\cdot\mathbf{e}/\varepsilon; \tag{6}$$

i.e., they are related in a linear way to the vector amplitude $C_\alpha\mathbf{A}_\alpha$ of the corresponding elastic wave.

Let us examine the orientational properties of the polarization vectors near the acoustic axis $\mathbf{m}_0$ [we assume, for example, $v_{01} = v_{02} \neq v_{03}$, $v_{0\alpha} \equiv v_\alpha(\mathbf{m}_0)$]. According to (3), at $\mathbf{m} = \mathbf{m}_0$ the polarization of the degenerate elastic wave $\mathbf{u}$ can be arbitrary in the plane orthogonal to the vector $\mathbf{A}_3(\mathbf{m}_0) \equiv \mathbf{A}_{03}$. Setting $\mathbf{m} = \mathbf{m}_0 + \Delta\mathbf{m}$, solving (3), and making use of the small quantity $|\Delta\mathbf{m}| \ll 1$, we find the approximate result[2]

$$\mathbf{A}_i=\mathbf{A}_{01}\cos\Phi_i+\mathbf{A}_{02}\sin\Phi_i+\mathbf{A}_{03}(\mathbf{t}_i\Delta\mathbf{m}), \quad i=1,\,2, \tag{7}$$

where $\mathbf{A}_{01}$, $\mathbf{A}_{02}$ is any pair of unit vectors which form a right-handed orthogonal triad with the vector $\mathbf{A}_{03}$:

$$\operatorname{tg}2\Phi_{1,\,2}=(2\mathbf{q}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{G}\Delta\mathbf{m})/(2\mathbf{p}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{F}\Delta\mathbf{m}), \tag{8}$$
$$\mathbf{t}_i=2v_{01}(\mathbf{q}^{(1)}\cos\Phi_i+\mathbf{q}^{(2)}\sin\Phi_i)/(v_{01}^2-v_{03}^2). \tag{9}$$

Here and below, the index $i = 1, 2$ is used for quantities which refer to the characteristics of the wave branches

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which are degenerate in terms of velocity along an acoustic axis. In (8) and (9) we have introduced the following notation:

$$p_j=\frac{1}{4v_{01}}\frac{\partial(\Lambda_{11}-\Lambda_{22})}{\partial m_{0j}},\quad q_j=\frac{1}{2v_{01}}\frac{\partial\Lambda_{12}}{\partial m_{0j}},$$

$$q_j^{(l)}=\frac{1}{2v_{01}}\frac{\partial\Lambda_{l3}}{\partial m_{0j}},\quad l=1,2,$$

$$F_{jk}=\frac{1}{4v_{01}}\frac{\partial^2(\Lambda_{11}-\Lambda_{22})}{\partial m_{0j}\partial m_{0k}}+\frac{2v_{01}}{v_{01}^2-v_{03}^2}(q_j^{(1)}q_k^{(1)}-q_j^{(2)}q_k^{(2)}),$$

$$G_{jk}=\frac{1}{2v_{01}}\frac{\partial^2\Lambda_{12}}{\partial m_{0j}\partial m_{0k}}+\frac{2v_{01}}{v_{01}^2-v_{03}^2}(q_j^{(1)}q_k^{(2)}+q_j^{(2)}q_k^{(1)}),$$

$$\Lambda_{\alpha\beta}=\mathbf{A}_{0\alpha}\hat{\Lambda}\mathbf{A}_{0\beta}\quad(\alpha,\beta=1,2),$$

$$\partial/\partial m_{0j}\equiv(\partial/\partial m_j)_{\mathbf{m}_0}\quad(j,k=1,2,3).\qquad(10)$$

In a derivative with respect to the components $m_j$, the latter are assumed to be independent; i.e.,the condition $\mathbf{m}^2=1$ is ignored.

It can be seen from (7) and (8) that near $\mathbf{m}_0$ the vectors $\mathbf{A}_i$ have a singular dependence on $\mathbf{m}$. When $\mathbf{m}=\mathbf{m}_0+\Delta\mathbf{m}$ is scanned around $\mathbf{m}_0$ in a cone with an infinitesimal vertex angle, the vectors $\mathbf{A}_i$ undergo large rotations. On the other hand, the components of the vectors $\mathbf{A}_i\perp\mathbf{A}_3$ along the direction $\mathbf{A}_{03}$ remain small (on the order of $|\Delta\mathbf{m}|$). Near $\mathbf{m}_0$, the vectors $A_i(\mathbf{m})$ can thus be replaced approximately by their projections $\mathbf{a}_{1,2}(\mathbf{m})$, onto the plane orthogonal to $\mathbf{A}_{03}$:

$$\mathbf{a}_i(\mathbf{m})=\mathbf{A}_{01}\cos\Phi_i+\mathbf{A}_{02}\sin\Phi_i.\qquad(11)$$

It is not difficult to show that, according to (8), we have $\mathrm{tg}\Phi_1\ \mathrm{tg}\ \Phi_2=-1$. Hence $\Phi_1=\Phi_2\pm\pi/2$ and $\mathbf{a}_1\perp\mathbf{a}_2$ hold. The change in the angle $\Phi_i(\mathbf{m})$ as the point $\mathbf{m}_0$ is circumvented on the sphere $\mathbf{m}^2=1$ along a small contour $\Gamma$ determines the rotation of the polarization fields (11) near an isolated degeneracy point $\mathbf{m}_0$. The complete change in the angle $\Phi_i$ when this contour is traced, expressed in units of $2\pi$, is called the "Poincaré index" $n$ of a singular point of a plane vector field: $n=\gamma(2\pi)$, where $\gamma(\psi)=[\Phi(\psi)-\Phi(0)]/2\pi$, and $\psi$ is the angle through which the vector $\Delta\mathbf{m}$ is rotated on contour $\Gamma$. According to Refs. 1 and 2, the only possible values of this index are $n=0$, $\pm\frac{1}{2}$, $\pm1$. The configuration of the polarization fields near an acoustic axis also depends on the geometry of the contact of the velocity regions $v_1(\mathbf{m})\leqslant v_2(\mathbf{m})$, which is determined by the properties of the vectors $\mathbf{p}$, $\mathbf{q}$: conical contact in the case $\mathbf{p}\|\mathbf{q}$, "wedge" contact in the case $\mathbf{p}\|\mathbf{q}$, and tangency in the case $\mathbf{p}=\mathbf{q}=0$ (Ref. 1). Figure 1 shows various types of singularities of the polarization fields near isolated degeneracy points. Incorporating the piezoelectric effect does not expand the class of degeneracies of elastic waves in crystals.[2] Except for renormalization of the parameters (10), incorporating the piezoelectric effect does not change the explicit form of the equations derived in Ref. 1 for calculating the indices $n$ corresponding to degeneracies of various types.

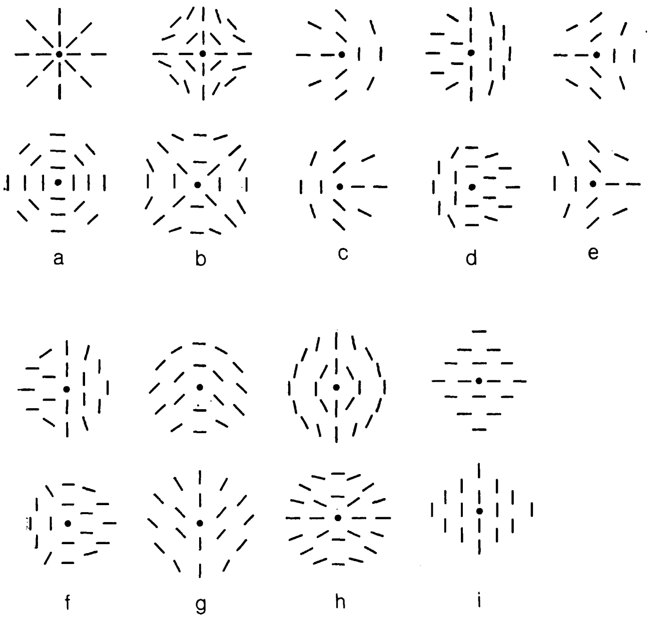

FIG. 1. The vector polarization field $\mathbf{a}_i(\mathbf{m})$, $i=1,2$, near isolated acoustic axes $\mathbf{m}_0$ of various types [top view of the plane orthogonal to $\mathbf{A}_{03}=\mathbf{A}_3(\mathbf{m}_0)$; the point corresponds to the direction of $\mathbf{m}_0$]. Cases a, b, h, and i are degeneracies of the tangency type; c, e—of the conical type; d, f, g—of the wedge type. The values of the index $n$ of the singular point $\mathbf{m}_0$ are as follows. a:$n=1$; b:$n=-1$; c, d:$n=1/2$; e, f:$n=-1/2$; g, h:$n=0$, $\gamma(\psi)\not\equiv0$; i:$n=0$, $\gamma(\psi)\equiv0$.

## POTENTIAL AND ELECTRIC-FIELD WAVES NEAR AN ACOUSTIC AXIS

We first consider the properties of a wave of the electric potential $\varphi$ which arises when a degenerate elastic wave $\mathbf{u}$ propagates strictly along the acoustic axis $\mathbf{m}_0$. We write the polarization vector $\mathbf{A}$, with an arbitrary orientation in the degeneracy plane, as

$$\mathbf{A}=\mathbf{A}_{01}\cos\Psi+\mathbf{A}_{02}\sin\Psi.\qquad(12)$$

We wish to emphasize that the quantity $\Psi$ in (12) itself depends on the choice of the orientation of the vector $\mathbf{A}$, in contrast with the angle $\Phi_i$ in (7), (11), which is function of $\mathbf{m}$ [see (8)] and which specifies the orientation of the vector $\mathbf{a}_i(\mathbf{m})$. According to (5), in the case $\mathbf{m}_0\hat{e}\mathbf{m}_0\equiv\mathbf{e}_0\|\mathbf{A}_{03}$ we have $\varphi=0$ and $\mathbf{E}=0$ for arbitrary $\mathbf{A}$ in (12). A situation of this sort arises, for example, for degeneracy directions which coincide with symmetry axes. In this case we have $\mathbf{e}_0\|\mathbf{A}_{03}\|\mathbf{m}_0$. We now consider the case of symmetrically oriented degeneracy directions, for which we have $\mathbf{e}_0\|A_{03}$. Making use of the arbitrariness in the choice of the vector $\mathbf{A}_{01}\perp\mathbf{A}_{03}$, we choose $\mathbf{A}_{01}\|[\mathbf{e}_0\mathbf{A}_{03}]$ for convenience. Substituting (12) into (5), we then find

$$\varphi^0(\mathbf{m}_0)=\frac{4\pi C}{\varepsilon_0}[\mathbf{e}_0^2-(\mathbf{e}_0\mathbf{A}_{03})^2]^{1/2}\sin\Psi.\qquad(13)$$

For a given form of the matrix $\hat{\Lambda}(\mathbf{m}_0)$ with degenerate eigenvalues, the orientation of the vector $\mathbf{A}_{03}$ is determined from the known formulas.[8]

We now assume that elastic wave $\mathbf{u}_i$ is propagating along the direction $\mathbf{m}=\mathbf{m}_0+\Delta\mathbf{m}$, which is close to an acoustic axis. If $\mathbf{m}_0$ does not coincide with a symmetry axis ($\mathbf{e}_0\|\mathbf{A}_{03}$), by choosing $\mathbf{A}_{01}\|[\mathbf{e}_0\mathbf{A}_{03}]$, as before, we find

$$\varphi_i^0(\mathbf{m})\approx\frac{4\pi C_i}{\varepsilon_0}\mathbf{e}_0\mathbf{a}_i=\frac{4\pi C_i}{\varepsilon_0}[\mathbf{e}_0^2-(\mathbf{e}_0\mathbf{A}_{03})^2]^{1/2}\sin[\Phi_i(\mathbf{m})].\qquad(14)$$

The behavior of the electric potential near such degeneracies is therefore determined by the function $\Phi_i(\mathbf{m})$. Let us consider the most typical example of an asymmetrically orient-

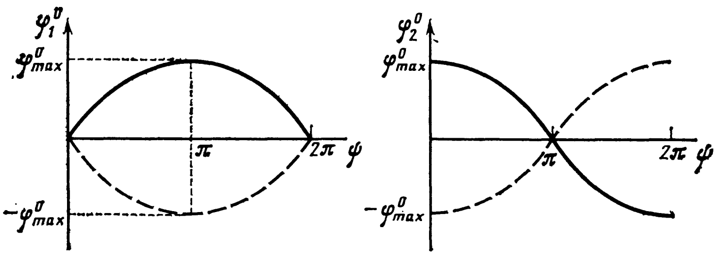

FIG. 2. Amplitude of the electric potential, $\varphi_i^0$ versus the propagation direction $\mathbf{m} = \mathbf{m}_0 + \Delta\mathbf{m}$ of isonormal elastic waves ($i = 1, 2$) near asymmetrically oriented acoustic axes $\mathbf{m}_0$ of the conical type. Here $\psi$ is the angle through which the vector $\Delta\mathbf{m}$ is rotated around $\mathbf{m}_0$, measured from the orientation of $\Delta\mathbf{m}$ corresponding to $\Phi_1 = 0$, i.e., $\mathbf{a}_i \| [\mathbf{e}_0\mathbf{A}_{03}]$. The solid and dashed lines refer to the cases $n = 1/2, -1/2$, respectively; $\varphi_{\max}^0 = (4\pi C/\varepsilon_0)[\mathbf{e}_0^2 - (\mathbf{A}_{03}\mathbf{e}_0)^2]^{1/2}$.

ed acoustic axis: a conical degeneracy ($n = \pm 1/2$), which occurs for a tensor $\hat{\Lambda}(\mathbf{m}_0)$ of "general position" and which is stable with respect to small perturbations $\Delta\hat{\Lambda}$ (Ref. 1). In this case the angle $\Phi_i$ takes on all values from zero to $\pi$ sign $n$ as $\Delta\mathbf{m}$ is rotated completely about $\mathbf{m}_0$ (Fig. 1, c and e); here we have $\Phi_i(-\Delta\mathbf{m}) = \Phi_i(\Delta\mathbf{m}) + (\pi/2)$ sign $n$. We thus find the plot of $\varphi_i^0$ versus the angle ($\psi$) through which the vector $\Delta\mathbf{m}$ is rotated around $\mathbf{m}_0$, shown in Fig. 2.

Near such a degeneracy the amplitude of the electric-potential wave $\varphi_i^0(\mathbf{m})$, like that of the field-strength wave $\mathbf{E}_i^0(\mathbf{m}) = k_i\varphi_i^0(\mathbf{m})\mathbf{m}$, has a singular behavior. As we let $\mathbf{m}\to\mathbf{m}_0$ along various paths (i.e., for various orientations $\Delta\mathbf{m}\to 0$, these quantities tend toward different limiting values, so that as the wave normal $\mathbf{m}$ sweeps around $\mathbf{m}_0$ in a cone of infinitely small vertex angles the amplitudes $\varphi_i^0(\mathbf{m})$ and $|\mathbf{E}_i^0(\mathbf{m})|$ vary from zero to nonzero—definitely not small—values. The corresponding "amplitude" singularity in the vector field $\mathbf{E}_i^0(\mathbf{m})$ near $\mathbf{m}_0$ is shown in Fig. 3a.

In the case $\mathbf{e}_0 \| \mathbf{A}_{03}$, in particular, for acoustic axes which coincide with symmetry axes of the crystal, this singularity disappears, since we have

$$\varphi_i^0(\mathbf{m}) \approx \frac{4\pi C_i}{\varepsilon_0}\left[(\mathbf{m}_0\hat{e}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{e}\mathbf{m}_0)\mathbf{a}_i+|\mathbf{e}_0|(\mathbf{t}_i\Delta\mathbf{m})\right], \qquad (15)$$

as follows from (5) and (7), and the quantities $\varphi_i^0(\mathbf{m})$, $\mathbf{E}_i^0(\mathbf{m})$ tend toward zero as $\Delta\mathbf{m}\to 0$, regardless of the orientation of $\Delta\mathbf{m}$, remaining continuous at the degeneracy point $\mathbf{m}_0$ (Fig. 3b). There is of course the possibility of a situation in which the linear term will also vanish when $\varphi_i^0(\mathbf{m})$ is expanded in powers of $\Delta\mathbf{m}$. For example, in the case $\mathbf{m}_0 \| \bar{6}$ we have

$$\varphi_1^0 \approx \frac{4\pi C_1}{\varepsilon_{33}}|\Delta\mathbf{m}|^2(e_{11}\sin 3\psi + e_{22}\cos 3\psi),$$

$$\varphi_2^0 \approx \frac{4\pi C_2}{\varepsilon_{33}}|\Delta\mathbf{m}|^2(e_{11}\cos 3\psi - e_{22}\sin 3\psi) \qquad (16)$$

(Fig. 3c). Furthermore, by virtue of the symmetry all the terms in this expansion may vanish. For example, we have $\varphi_i^0(\mathbf{m}) = 0$, $\mathbf{E}_i^0(\mathbf{m}) = 0$ for any propagation direction $\mathbf{m}$ of a purely transverse elastic wave in the classes $6mm$, $\infty m$ and for a quasitransverse wave in the classes 622, $\infty 2$.

## ELECTRIC-DISPLACEMENT WAVE NEAR AN ACOUSTIC AXIS

By virtue of the condition div $\mathbf{D} = 0$, the displacement vector $\mathbf{D}$ always lies in the plane orthogonal to $\mathbf{m}$, and it rotates, according to (6), along with the polarization vector $\mathbf{A}$ as $\mathbf{m}$ is scanned around the acoustic axis $\mathbf{m}_0$ or in the case $\mathbf{m}\|\mathbf{m}_0$. Why are rotations of the vectors $\mathbf{D}$ and $\mathbf{A}$ coupled?

Let us consider the vector field $\mathbf{A}$ defined in the plane orthogonal to $\mathbf{A}_{03}$ and also the vector field $\mathbf{D}^0 = 4\pi Ck\hat{B}(\mathbf{m}_0)\mathbf{A}$, defined in the plane orthogonal to $\mathbf{m}_0$. We use the convention that the directions in which the vectors $\mathbf{A}$ and $\mathbf{D}^0$ are rotated are determined by looking at the rotation plane from the tips of the vectors $\mathbf{A}_{03}$ and $\mathbf{m}_0$, respectively. According to (6) we have $\mathbf{m}\hat{B}(\mathbf{m}) = 0$, so that the matrix $\hat{B}$ is degenerate: det $\hat{B} = 0$. We assume Rank $\hat{B} = 2$. It is also natural to assume that the vector $\mathbf{x}$ is the eigenvector of the matrix $\hat{B}$ which corresponds to zero eigenvalue: $\hat{B}\mathbf{x} = 0$ (not orthogonal to $\mathbf{A}_{03}$). Indeed, each of the

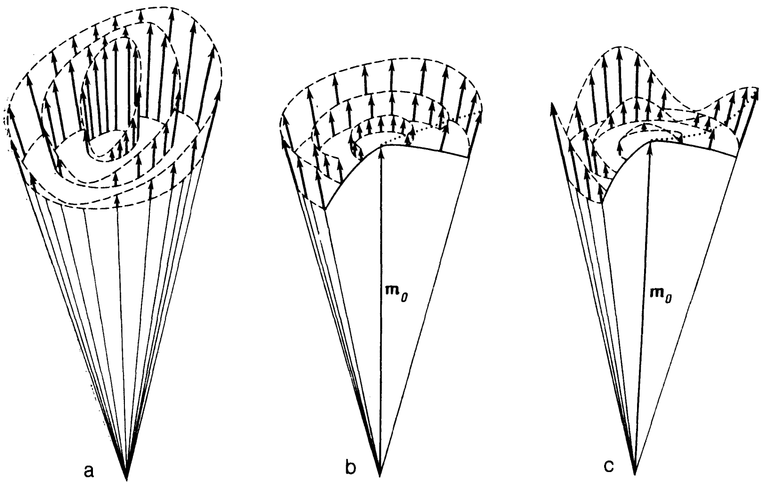

FIG. 3. The vector electric fields $\mathbf{E}_i^0$ near acoustic axes. A single wave branch, $i = 1$ or 2, is shown. a—asymmetrically oriented acoustic axis $\mathbf{m}_0$ of the conical type; b—$\mathbf{m}_0 \| \infty$, 6 (the branch corresponding to a quasitransverse elastic wave is shown in the case $\infty m$, $6mm$ and to the purely transverse wave in the case $\infty 2$, 622); c—$\mathbf{m}_0 \| \bar{6}$. For convenience, different scales have been used in drawing the lengths of the vectors $\mathbf{E}_i^0$ in parts a–c.

conditions Rank $\hat{B} < 2$, $\mathbf{x}\mathbf{A}_{03} = 0$ represents an additional equation for the components of the matrix $\hat{B}$. These conditions can be satisfied only if the material constants of the crystal satisfy certain special relations, even in the case of acoustic axes which coincide with high-symmetry directions of the medium [more on this below; the only exceptional case is the $\bar{6}$ axis, along which we have $\hat{B}(\mathbf{m}_0) = 0$]. If any of these conditions nevertheless is satisfied by chance, then the vector field $\mathbf{D}^0$ will degenerate into a straight line.

The rotation of the plane vector field produced by a nondegenerate mapping of the original field can be calculated quite easily on the basis of the appropriate theorems.[9] In this case the transformation $\hat{B}$ turns out to be degenerate. However, we can replace $\hat{B}(\mathbf{m}_0)$ by a matrix $\hat{B}'$ such that we have $\hat{B}'\mathbf{A} = \hat{B}\mathbf{A}$ for any $\mathbf{A} \perp \mathbf{A}_{03}$, but we also have $\det \hat{B}' \neq 0$. These conditions are satisfied by, for example, the matrix

$$\hat{B}' = \hat{B}(\mathbf{m}_0) + \mathbf{m}_0\mathbf{A}_{03}. \tag{17}$$

Indeed, we have $\det \hat{B}' = \mathbf{A}_{03}\hat{\bar{B}}\mathbf{m}_0$ (Ref. 8). The matrix $\hat{\bar{B}}$, the dual of $\hat{B}$, is proportional to $\mathbf{x}\mathbf{m}_0$, as can be shown, so we have $\det \hat{B}' \neq 0$ for $\mathbf{x}\mathbf{A}_{03} \neq 0$. Now using Ref. 9, we can easily find a result which we will formulate below separately for the cases of propagation directions strictly along the acoustic axis $\mathbf{m}_0$ and in its vicinity.

A) $\mathbf{m} \| \mathbf{m}_0$. When the polarization vector of a degenerate elastic wave, $\mathbf{A}(\Psi)$ [see (12)] is rotated through an angle of $2\pi$ in the plane orthogonal to $\mathbf{A}_{03}$, the vector $\mathbf{D}^0(\Psi) = 4\pi Ck\hat{B}(\mathbf{m}_0)A(\Psi)$ also undergoes a complete rotation in the plane orthogonal to $\mathbf{m}_0$, through an angle $2\pi$ sign det $\hat{B}'$, without vanishing in the process: $\mathbf{D}^0(\Psi) \neq 0$ holds for any $\Psi$.

B) $\mathbf{m} = \mathbf{m}_0 + \Delta\mathbf{m}$, $|\Delta\mathbf{m}| \ll 1$. We denote by $n_A$ the Poincaré index which characterizes the rotation of plane vector fields $\mathbf{a}_i(\mathbf{m}) \perp \mathbf{A}_{03}$ [see (11)] near $\mathbf{m}_0$. Discarding terms $\sim |\Delta\mathbf{m}|$, we replace the vectors $\mathbf{D}_i^0(\mathbf{m})$ in (6) by the vector field

$$\mathbf{d}_i^0(\mathbf{m}) = 4\pi C_i k_i \hat{B}(\mathbf{m}_0)\mathbf{a}_i(\mathbf{m}), \tag{18}$$

which lies in the plane orthogonal to $\mathbf{m}_0$. The following assertion holds: Near an acoustic axis, the plane vector fields $\mathbf{d}_{1,2}(\mathbf{m}) \perp \mathbf{m}_0$ do not contain null vectors, and when $\mathbf{m}$ is rotated around $\mathbf{m}_0$ their rotation is characterized by the index

$$n_D = n_A \text{ sign det } \hat{B}'. \tag{19}$$

It should be noted that, in contrast with the mutually orthogonal vectors $\mathbf{A}_\alpha(\mathbf{m})$ ($\alpha = 1, 2, 3$), the three vectors $\mathbf{D}_\alpha(\mathbf{m}) \perp \mathbf{m}$ are coplanar and generally nonorthogonal in pairs. At the same time, it is easy to see that in the case $\mathbf{x}\mathbf{A}_3 \neq 0$ the vectors $\mathbf{D}_1(\mathbf{m})$, $\mathbf{D}_2(\mathbf{m})$ are not collinear with any $\mathbf{m}$. Consequently, the two vector fields $\mathbf{d}_1(\mathbf{m})$, $\mathbf{d}_2(\mathbf{m})$ are homotopic with each other; i.e., they correspond to the same value of the index $n_D$. We might also note that along an acoustic axis or in its vicinity, for orientations of the vectors $\mathbf{A}(\Psi)$ or $\mathbf{a}_i(\mathbf{m})$ parallel to $(1 - \mathbf{A}_{03} \cdot \mathbf{A}_{03})\mathbf{x}$, we have $\mathbf{D}^0(\Psi) \| \mathbf{D}_3^0(\mathbf{m}_0)$ or $\mathbf{d}_i^0(\mathbf{m}) \| \mathbf{D}_3^0(\mathbf{m}_0)$, respectively, where $\mathbf{D}_3^0(\mathbf{m}_0)$ is the amplitude of the displacement field which corresponds to a nondegenerate elastic wave propagating along $\mathbf{m}_0$.

As we have already mentioned, if the relation $\det \hat{B}' = 0$ holds by chance, then we have $\mathbf{x} \perp \mathbf{A}_{03}$, and the vector fields $\mathbf{D}^0(\Psi)$, $\mathbf{d}_i^0(\mathbf{m})$ degenerate into the straight line $\mathbf{L} \| \hat{B}\mathbf{x}_\perp$, where $\mathbf{x}_\perp \| [\mathbf{x}\mathbf{A}_{03}]$. In this case we have $\mathbf{D}^0(\Psi) = 0$ and $\mathbf{d}_i^0(\mathbf{m}) = 0$ for $\mathbf{A}(\Psi) \| \mathbf{x}$ and $\mathbf{a}_i(\mathbf{m}) \| \mathbf{x}$, respectively.

The method proposed above for constructing a nondegenerate matrix, through the use of (6) in place of $\hat{B}$, is not the only method possible. In particular, we could take the $2 \times 2$ matrix

$$B_{ij}'' = \mathbf{D}_{0i}\hat{B}\mathbf{A}_{0j}, \quad i, j = 1, 2, \tag{20}$$

which sends the column of coordinates of the vector $\mathbf{A}(\Psi)$ or $\mathbf{a}_i(\mathbf{m})$ in the orthonormal basis $\mathbf{A}_{01}$, $\mathbf{A}_{02} \perp \mathbf{A}_{03}$ into the column of coordinates of the vector $\mathbf{D}^0(\Psi)$ or $\mathbf{d}_i^0(\mathbf{m})$ in the orthonormal basis $\mathbf{D}_{01}$, $\mathbf{D}_{02} \perp \mathbf{m}_0$. The $2 \times 2$ matrix $\mathbf{B}_{ij}''$ is obviously not degenerate in general (for $\mathbf{x}\mathbf{A}_{03} \neq 0$); it is easy to verify that we have $\det \hat{B}'' = \det \hat{B}'$. Consequently, all the results obtained with the help of the matrix $\hat{B}'$ [including (19)] continue to hold when we replace $\hat{B}'$ by $\hat{B}''$.

Let us examine the question of determining extreme values of the absolute value of the displacement vector $\mathbf{D}^0$ near an acoustic axis. We now need to study the bilinear form $(\mathbf{D}^0)^2 \propto (\hat{B}\mathbf{A}, \hat{B}\mathbf{A}) = \mathbf{A}\hat{B}^T\hat{B}\mathbf{A}$ for extrema under the conditions $\mathbf{A}^2 = 1$ and $\mathbf{A}\mathbf{A}_{03} = 0$ ($\hat{B}^T$ is the transposed matrix). Constructing the Lagrange function $F = \mathbf{A}\hat{B}^T\hat{B}\mathbf{A} - \lambda(\mathbf{A}^2 - 1) - 2\mu(\mathbf{A}\mathbf{A}_{03})$, and equating the derivative $\partial F / \partial \mathbf{A}$ to zero, we find the following equation, making use of the symmetry of the matrix $\hat{B}^T\hat{B}$:

$$(\hat{B}^T\hat{B} - \lambda)\mathbf{A} = \mu\mathbf{A}_{03}. \tag{21}$$

Expanding $\mathbf{A}$ in the basis vectors $\mathbf{A}_{01}$ and $\mathbf{A}_{02}$, we multiply (21) from the left by $\mathbf{A}_{0i}$, $i = 1, 2$. Making use of the conditions $\mathbf{A}_{0i}\mathbf{A}_{03} = 0$, we find that the extreme (maximum and minimum) values of this bilinear form are realized for orientations of the vector $\mathbf{A}$ which correspond to the eigenvectors of the $2 \times 2$ matrix $\mathbf{T}_{ij} = \mathbf{A}_{0i}\hat{B}^T\hat{B}\mathbf{A}_{0j}$ ($i, j = 1, 2$) and which are equal to the eigenvalues $\lambda_{1,2} \geq 0$ of this matrix (in particular, $D_{\min}$ vanishes, even in the case of $\det \hat{T} = 0$; we evidently also have $\det \hat{B}' = 0$).

Consequently, near an acoustic axis of general position the vector fields $\mathbf{D}_i^0(\mathbf{m})$ have not only an orientation singularity but also an amplitude singularity, since the limiting value of $|\mathbf{D}_i^0(\mathbf{m})|$ in the limit $\Delta\mathbf{m} \to 0$ depends on the orientation of $\Delta\mathbf{m}$. This singularity, as in the case of the field $\mathbf{E}_i^0(\mathbf{m})$, disappears for directions of the acoustic axes which coincide

TABLE I. The amplitude of the electric displacement wave, $\mathbf{D}^0$, versus the orientation of the polarization vector $\mathbf{A}(\Psi)$ in (12) of a degenerate elastic wave which is propagating along a principal symmetry axis in a peizoelectric.

| Symmetry class | $\mathbf{D}^0/4\pi Ck$ |
|---|---|
| $\infty m$, $6mm$, $4mm$, $3m$ | $e_{15}\mathbf{A}(\Psi)$ |
| $\infty 2$, $622$, $422$, $32$ | $e_{14}[\mathbf{A}(\Psi)\mathbf{m}_0]$ |
| $\infty$, $6$, $4$, $3$ | $e_{15}\mathbf{A}(\Psi) + e_{14}[\mathbf{A}(\Psi)\mathbf{m}_0]$ |
| $23$ ($\mathbf{m}_0 \| 2$), $\bar{4}3m$, $\bar{4}2m$ ($\mathbf{m}_0 \| \bar{4}$) | $e_{14}[\mathbf{m}_0\mathbf{A}(-\Psi)]$ |
| $\bar{6}2m$, $\bar{6}$ | $0$ |
| $\bar{4}$ | $e_{15}\mathbf{A}(-\Psi) + e_{14}[\mathbf{m}_0\mathbf{A}(-\Psi)]$ |
| $23$, $\bar{4}3m$ ($\mathbf{m}_0 \| 3$) | $-e_{14}\mathbf{A}(\Psi)/\sqrt{3}$ |

Note. When $\mathbf{A}(\Psi)$ is replaced by $\mathbf{a}_i(\mathbf{m})$ as in (11), these expressions approximately (to within terms $\propto |\Delta\mathbf{m}|$) give the displacement vectors $\mathbf{D}_i^0(\mathbf{m})$ near the acoustic axes; see also (22) for the case $\mathbf{m}_0 \| \bar{6}$.

TABLE II. Rotations of the vector polarization fields of elastic waves (index $n_A$) and of the amplitudes of the accompanying electric displacement waves ($n_D$) near acoustic axes which coincide with symmetry axes $N$ of piezoelectric crystals.

| $N$ | ∞, 6 | $\bar{6}$ | 4 * | $\bar{4}$ and 2 ∈ 23 * | 3 |
|---|---|---|---|---|---|
| $n_A$ | 1 | 1 | ±1 | ±1 | −1/2 |
| $n_D$ | 1 | −2 | $n_A$ | $-n_A$ | −1/2 |

*The sign of $n_A$ in these cases is calculated from the equations given in Refs. 1 and 2.

with symmetry axes of the crystal. It is not difficult to show that for such directions the 2×2 matrix $T_{ij}$ is proportional to the unit matrix; i.e., its two eigenvalues are equal, so that the quantity $|\mathbf{D}_i^0(\mathbf{m})|$ under the condition $|\Delta\mathbf{m}| \ll 1$ does not depend on the orientation of $\Delta\mathbf{m}$, and $|\mathbf{D}_i^0(\Psi)|$ does not depend on the orientation of the polarization vector of the degenerate wave, **A**. Nevertheless, the orientational singularity of the vector field $\mathbf{D}_i^0(\mathbf{m})$ persists even near symmetry axes. Tables I and II show calculated values of the displacement vector for directions of the acoustic axes coinciding with symmetry axes of various piezoelectric classes.

An interesting configuration of the vector fields $\mathbf{D}_i^0$ arises near an acoustic axis $\mathbf{m}_0 \| \bar{6}$. In this case we have $\hat{B}(\mathbf{m}_0) = 0$ and

$$\mathbf{D}_1^0(\psi) \approx e_{11}\Delta\mathbf{m}(-2\psi) + e_{22}[\mathbf{m}_0\Delta\mathbf{m}(-2\psi)],$$
$$\mathbf{D}_2^0(\psi) \approx e_{22}\Delta\mathbf{m}(-2\psi) - e_{11}[\mathbf{m}_0\Delta\mathbf{m}(-2\psi)], \quad (22)$$

where $e_{22} = 0$ for the class $\bar{6}\,2m$. It is not difficult to verify that the vectors $\mathbf{D}_{1,2}^0$ in (22) lie in the plane perpendicular to $\mathbf{m}_0$ and are mutually orthogonal, while their absolute value is proportional to $|\Delta\mathbf{m}|$ and does not depend on the orientation $\psi$ of the vector $\Delta\mathbf{m}$. It follows directly from (22) that the singular point $\mathbf{m}_0$ which we have been discussing is characterized by an index $n_D = -2$. The corresponding configuration of the vector fields $\mathbf{D}_i^0$ is shown in Fig. 4.

## CONCLUSION

In summary, this analysis of the characteristics of acoustic waves in piezoelectrics shows that near degeneracy directions the vector field $\mathbf{E}_i^0(\mathbf{m})$ has an amplitude singularity, the polarization field $\mathbf{A}_i(\mathbf{m})$ has a rotation in a plane, and, finally, the displacement field $\mathbf{D}_i^0(\mathbf{m})$ has singularities of both types. It has been shown that the amplitude singularities are retained only for acoustic axes which do not coincide with symmetry axes, while the orientational singularities of the vector fields $\mathbf{A}_i(\mathbf{m})$ and $\mathbf{D}_i(\mathbf{m})$ exist near any degeneracy directions. Singularities of the latter type have a close analog in the theory of liquid crystals, namely, the distribution of directors in the vicinity of disclinations in nematic liquid crystals. In both cases, vectors differing in sign are physically equivalent, so that the planar fields of undirected segments which we have been discussing here have a rotation which is a multiple of $\pi$ near singular points. This circumstance, combined with the classical definition of the index of a singular point, which measures the rotation of the vector field in units of $2\pi$ (the Poincaré index), allows us to use a definition in which the index is equal to the rotation in units of $\pi$ (the Frank index).

If we speak in terms of the polarization field, and when we note that the vectors $\mathbf{A}_\alpha$ ($\alpha = 1, 2, 3$) are nondirectional, we see that the topologically equivalent configurations of the triad $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ near a degeneracy are distinguished from each other by a rotation around the vector $\mathbf{A}_3$ through an angle which is a multiple of $\pi/2$. In other words, instead of the vector fields $\mathbf{A}_{1,2}$ we could examine a distribution of the orthogonal pair $\{\mathbf{A}_1, \mathbf{A}_2\}$, whose rotation (the index of the singularity, $n_+$) would naturally be measured in units of $\pi/2$. From this point of view, it would appear that the minimum possible index of the singular point of such a distribution would have to be $n_+ = \pm 1$ ($n = \pm 1/4$). However, this is not the case. According to Ref. 1 (see also Fig. 1), the minimum possible rotation around an isolated singular point corresponds to rotations of the eigenvectors $\mathbf{A}_i$ not through $\pm\pi/2$ but through $\pm\pi$ ($n_+ = \pm 2$, $n = \pm\frac{1}{2}$). There is nothing paradoxical at all here, since each of the vectors $\mathbf{A}_{1,2}$ belongs to its own neighboring branch ("fast" or "slow"), and no transition occurs between these branches on a contour enveloping the degeneracy point. In a sense we are dealing with an orthogonal pair which is formed in various ways by colored vectors. A rotation of the triad $\{\mathbf{A}_\alpha\}$ through an angle of $\pm\pi/2$ near the degeneracy point thus corresponds to a physically inequivalent configuration.

Near an acoustic axis, the tensor characteristics of elastic waves also have singularities. For example, let us examine the strain tensor associated with the elastic wave $\mathbf{u}_\alpha$ in (2):

$$U_{ln}^{(\alpha)} = \frac{1}{2}(u_{\alpha l,n} + u_{\alpha n,l}) = -\frac{k_\alpha}{2}\tau_{ln}^{(\alpha)}\sin\chi_\alpha, \quad (23)$$

where $\hat{\tau}^{(\alpha)} = \mathbf{m}\cdot\mathbf{A}_\alpha + \mathbf{A}_\alpha\cdot\mathbf{m}$. It is not difficult to show that the matrix $\hat{\tau}^{(\alpha)}$ has the eigenvalue $\gamma_{1,2}^{(\alpha)} = (\mathbf{A}_\alpha\mathbf{m}) \mp 1$, $\gamma_3^{(\alpha)} = 0$ and the eigenvectors[1)] $\mathbf{t}_{1,2}^{(\alpha)} \| \mathbf{A}_\alpha \mp \mathbf{m}$, $\mathbf{t}_3^{(\alpha)} \| [\mathbf{m}\mathbf{A}_\alpha]$. It follows in particular that near an acoustic axis the triads of eigenvectors $\{\mathbf{t}_1^{(i)}, \mathbf{t}_2^{(i)}, \mathbf{t}_3^{(i)}\}$ for each of the wave branches $i = 1, 2$ which are degenerate along $\mathbf{m}_0$ rotate along with $\mathbf{A}_i(\mathbf{m})$ when the vector $\Delta\mathbf{m}$ circumvents $\mathbf{m}_0$. In this case, however, we are dealing with something quite different from a two-dimensional rotation.

We have seen that the singular behavior of the elastic displacement field of a sound wave propagating along a degeneracy direction leads to several singularities in the electrical characteristics which are related to **u** in a linear way. These results also suggest that near an acoustic axis there

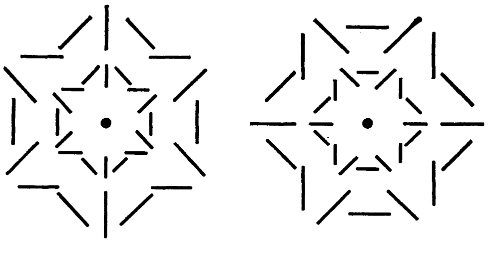

FIG. 4. Vector electric displacement fields $\mathbf{D}_i^0$, $i = 1, 2$, near the acoustic axis $\mathbf{m}_0 \| \bar{6}$ (top view of the plane orthogonal to $\mathbf{m}_0$; the point corresponds to the direction of $\mathbf{m}_0$).

may be singularities in other physical properties which are linear in **u**, e.g., the parameters of the acoustooptic interaction, which should result in a singular response to a rotation of the elastic displacement vector of a reference sound wave.

[1)]Here we are assuming $[\mathbf{A}_\alpha \mathbf{m}] \neq 0$. In the opposite case, we would have a degeneracy $\gamma_1^{(\alpha)} = \gamma_3^{(\alpha)} = 0$, and the corresponding eigenvectors would be oriented in an arbitrary way in the plane orthogonal to $\mathbf{t}_2^{(\alpha)} \| \mathbf{m}$.


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