Decay of metastable states in a situation with close-lying tunneling trajectories

B. I. Ivlev and Yu. N. Ovchinnikov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
(Submitted 23 December 1986)
Zh. Eksp. Teor. Fiz. **93**, 668–679 (August 1987)

The probability for the decay of a current-carrying state of two coupled Josephson junctions exhibits essential singularities as a function of the temperature and the magnitude of the current. The reason is that the two-dimensional system contains close-lying classical trajectories in the imaginary time. These trajectories may merge, smoothly or discontinuously, to form a single trajectory at certain values of the current and the temperature. The behavior of the probability for the decay of a metastable state near such singularities is analyzed.

1. INTRODUCTION

Research on the decay of current-carrying states of Josephson junctions has spurred new developments in the physics of long-lived metastable states.[1–13] A Josephson junction is a quantum-mechanical system with one degree of freedom, when the system overcomes a potential barrier. In this case the decay probability is known to be determined in the exponential approximation by the action on an extremal trajectory in imaginary time. The coefficient of the exponential function is found through functional integration along trajectories to the extremal trajectory.

In the physically interesting case of two coupled Josephson junctions (a two-junction interferometer) the corresponding two-dimensional quantum-mechanical system may differ from a one-dimensional system qualitative. Specifically, there may exist several spatially separated trajectories on which the action is extremal (instantons). As the parameters of the system are varied, two such instantons may approach each other, and at certain critical parameter values they may merge. Near this critical point the fluctuations are large, and the coefficient of the exponential function grows.

In this paper we analyze a system of two identical Josephson junctions. If the coupling between these junctions is sufficiently strong, the phases of the junctions vary in identical ways on the extremal trajectory. As the coupling becomes weaker, the symmetric solution always remains, but at the critical point two close solutions split off from it, and the decay ceases to be symmetric.

This picture prevails at essentially all temperatures $T<T^*$. Because of the numerical coefficients, the temperature T^* is very close to the temperature T_0 at which the quantum-mechanical regime gives way to a classical regime. For temperatures $T^*<T<T_0$ the situation is analogous to a first-order phase transition. In this region, the splitting off of a trajectory from the symmetric trajectory occurs discontinuously at a finite distance.

We will derive the probability for the decay of a metastable state in the two-dimensional case for $T<T^*$. We will study the behavior of the system near the point T^*.

2. EXPRESSION FOR THE ACTION; BEHAVIOR OF THE TRAJECTORIES

We restrict the discussion to the case in which the viscosity is negligible. In this approximation the Lagrangian of a system of two identical junctions is known to be

$$L=V\left[\frac{1}{2\omega^2}\left(\frac{\partial\varphi_1}{\partial t}\right)^2+\frac{1}{2\omega^2}\left(\frac{\partial\varphi_2}{\partial t}\right)^2+\cos\varphi_1+\cos\varphi_2+\frac{I}{2I_c}(\varphi_1+\varphi_2)-\frac{1}{2\beta}(\varphi_1-\varphi_2)^2\right], \quad (1)$$

where φ_1 and φ_2 are the phase differences at the junctions,

$$V=\hbar I_c/2e,\quad \omega^2=2eI_c/\hbar C,\quad \beta=2\pi L_0I_c/\Phi_0.$$

Here L_0 and C are the inductance and capacitance of each junction, and $\Phi_0=\pi\hbar c/e$ is the magnetic flux quantum.

The Lagrangian (1) corresponds to a zero external magnetic flux Φ_e through the interferometer. In the opposite case, the Lagrangian would acquire another term

$$\frac{V}{\beta}\frac{2\pi\Phi_e}{\Phi_0}(\varphi_1-\varphi_2).$$

If the external flux is equal to an integer number of quanta ϕ_0, the Lagrangian can still be reduced to the form in (1) through a simple change of variables.

In this paper we assume that the external magnetic flux through the interferometer is either zero or equal to an integral number of quanta. If this condition does not hold, the interferometer acquires an effective asymmetry, but the splitting of the instanton still occurs.

In general the action is proportional to the parameters $V/\hbar\omega$, which is generally large. Experimentally, therefore, a macroscopic quantum-mechanical tunneling is studied at currents close to the critical value, $2I_c-I\ll I_c$, where the potential barrier is substantially lowered. In this approximation the potential energy is approximately cubic.

The general expression for the probability for the decay of a metastable state at a nonzero temperature is[14–16]

$$\Gamma=2TZ^{-1}\,\mathrm{Im}\,Z, \quad (2)$$

where the partition function Z is defined by the functional integral

$$Z=\int D\varphi_1 D\varphi_2\exp(-S). \quad (3)$$

The functional integral (3) involved periodic functions $\varphi_{1,2}(t)$ with a period of $\hbar/T$.

Transforming to dimensionless variables, we write the action in terms of the imaginary time as

 0038-5646/87/080378-06$04.00

$$A=g\int_{-\tau_0}^{\tau_0} d\tau\left[\frac{1}{2}\left(\frac{\partial p}{\partial \tau}\right)^2+\frac{1}{2}\left(\frac{\partial q}{\partial \tau}\right)^2+p^2+q^2\right.$$
$$\left.-\frac{1}{2}(p^3+3pq^2)+\alpha q^2\right], \qquad (4)$$

where

$$\tau=\frac{it}{t_0}, \quad \tau_0=\frac{\hbar}{2Tt_0}, \quad t_0^2=\frac{2^{1/2}}{\omega^2}\left(1-\frac{I}{2I_c}\right)^{-1/2},$$
$$\varphi_{1,2}=\left[2\left(1-\frac{I}{2I_c}\right)\right]^{1/2}\left[\frac{3}{2}(p\pm q)-1\right]-\frac{3\pi}{2}. \qquad (5)$$

The parameter α is a measure of the interaction of two degrees of freedom, $p+q$ and $p-q$:

$$\alpha=\frac{2^{1/2}}{\beta}\left(1-\frac{I}{2I_c}\right)^{-1/2}. \qquad (6)$$

The decay probability is assumed to be small; accordingly, the semiclassical parameter

$$g=\frac{9}{2^{1/4}}\frac{V}{\hbar\omega}\left(1-\frac{I}{2I_c}\right)^{5/4} \qquad (7)$$

must be large: $g\gg 1$.

Figure 1 illustrates the behavior of an extremal trajectory as a function of the interaction parameter α. At $\alpha=0$, the variables $p+q$ and $p-q$ separate in Fig. 1a, and a transition occurs independently in terms of each of them. For $\alpha>1$ the potential has only a single saddle point. The solid lines are levels of a potential with a total energy such that the time of the motion between the separate lines would be τ_0. At a given temperature T, two close trajectories, 1 and 2, then split off from the trajectory $q=0$ as the parameter α decreases (Fig. 1b). This bifurcation of instantons occurs at $\alpha=\alpha_c(T)$. The critical value of the interaction parameter, $\alpha_c(T)$, will be derived below.

This picture is valid at temperatures $T<T^*$. At higher temperatures, the transition to the asymmetric trajectory occurs discontinuously as the coupling parameter α is reduced. The temperature T^* will be derived in Sec. 8.

3. DERIVATION OF THE INSTANTON

The classical trajectories of motion in imaginary time are found from the system of equations

$$\partial^2 p/\partial\tau^2=2p-{}^3/{}_2p^2-{}^3/{}_2q^2, \quad \partial^2 q/\partial\tau^2=2q-3pq+2\alpha q. \qquad (8)$$

For $\alpha>\alpha_c(T)$, the solution of this system which is periodic with a period of $2\tau_0$ is $p=p_0, q=q_0$, where $q_0=0$, while p_0 satisfies the equation

$$(\partial p_0/\partial\tau)^2=\text{const}+2p_0^2-p_0^3. \qquad (9)$$

Solving Eq. (9), we find

$$p_0(\tau)=s_2+(s_1-s_2)\,\text{cn}^2(z,k), \quad z={}^1/{}_2\tau(s_1-s_3)^{1/2}, \qquad (10)$$

where $\text{cn}(z,k)$ is the Jacobi elliptic function, and the quantities $s_{1,2,3}$ are the roots of the right side of Eq. (9). The quantity k in (10) is given by

$$k=[(s_1-s_2)/(s_1-s_3)]^{1/2}. \qquad (11)$$

We will be expressing all quantities in terms of this parameter. In particular, we can write

$$s_1={}^2/{}_3[1+(1+k^2)/(1-k^2+k^4)^{1/2}],$$
$$s_2={}^2/{}_3[1+(1-2k^2)/(1-k^2+k^4)^{1/2}], \qquad (12)$$
$$s_3={}^2/{}_3[1+(k^2-2)/(1-k^2+k^4)^{1/2}].$$

The temperature dependence of k is determined by the condition that the solution be periodic $p_0(\tau)=p_0(\tau+2\tau_0)$:

$$T_0/T=2\pi^{-1}(1-k^2+k^4)^{1/4}K(k). \qquad (13)$$

Here $K(k)$ is the elliptic integral of the first kind, and at the critical temperature T_0 which separates the quantum-mechanical regime from the classical regime, is given by

$$T_0=\frac{\hbar\omega}{2\pi}\left[1-\left(\frac{I}{2I_c}\right)^2\right]^{1/4}. \qquad (14)$$

The instanton is thus determined by Eqs. (10) with the implicit temperature dependence which follows from (12) and (13).

4. TEMPERATURE DEPENDENCE

To evaluate the functional integrals in (2), we expand the deviations of the functions $\tilde{p}=p-p_0$ and $\tilde{q}=q$ from the classical solutions in eigenfunctions of the operators $\hat{P}$, $\hat{Q}$:

$$\hat{P}=-\partial^2/\partial\tau^2+2-3p_0(\tau), \quad \hat{Q}(\alpha)=\hat{P}+2\alpha. \qquad (15)$$

According to the general representations,[15] the operator $\hat{P}$ has, in addition to a zero eigenvalue, one negative eigenvalue, which gives rise to an imaginary part in the partition function. The value of α_c is found from the condition that adding $2\alpha_c$ to the operator $\hat{P}$ displaces the negative level to zero. The eigenvalue of the operator $\hat{Q}$ which corresponds to this "dangerous" mode is $2(\alpha-\alpha_c)$. The corresponding eigenfunction is

$$\psi(\tau)=\text{dn}(z)[\text{cn}^2(z)+\mu], \qquad (16)$$

where

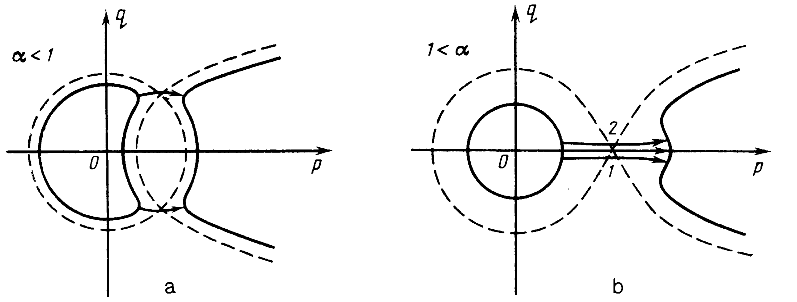

FIG. 1. a—Decay during a weak interaction; b—splitting off from the main trajectory, $q=0$, of two trajectories, 1 and 2, which lie close to the main trajectory. Solid lines) Curves at constant potential; lines with arrows) trajectories.

$$\mu=[(1-k^2+4k^4)^{1/2}+1-3k^2]/5k^2, \tag{17}$$

and $\mathrm{dn}(z)$ is the elliptic Jacobi function. For the critical value of the coupling parameter, α_c, we then find

$$\alpha_c={}^1/_4[2-k^2+2(1-k^2+4k^4)^{1/2}]/(1-k^2+k^4)^{1/2}. \tag{18}$$

From (13) and (18) we find a temperature dependence $\alpha_c(T)$. In the low-temperature limit $T \ll T_0$ we find

$$\alpha_c={}^5/_4-90\exp(-2\pi T_0/T). \tag{19}$$

5. DEPENDENCE OF THE DECAY PROBABILITY ON THE COUPLING CONSTANT NEAR THE CRITICAL REGIME

Since the eigenvalue of the dangerous mode is small near α_c, in evaluating the functional integral in (2) it is not sufficient to consider in the action only the terms quadratic in the deviations from the extremal trajectory. Singling out the integration over the dangerous mode, we find

$$\Gamma \propto \int_{-\infty}^{\infty} dx \exp[-g(\alpha-\alpha_c)x^2-g\delta x^4]e^{-A_0}, \tag{20}$$

where A_0 is action (4) calculated on the extremal trajectory (10) (corresponding to the point $\alpha = \alpha_c$):

$$A_0=g\frac{32\cdot 2^{1/2}}{15}\left\{\frac{E(k)}{(1-k^2+k^4)^{1/4}} + K(k)\left[\frac{(k^2-2)(13-13k^2+10k^4)}{36(1-k^2+k^4)^{5/4}} + \frac{5}{18}(1-k^2+k^4)^{1/4}\right]\right\}. \tag{21}$$

Along with (13), expression (21) describes the temperature dependence $A_0(T)$. To find the x^4 term in the argument we need to consider the cubic terms in the action:

$$\delta A=-\frac{3}{2}g\int d\tau\, \tilde{p}\tilde{q}^2.$$

The corresponding calculations are extremely laborious, since it is necessary to calculate sums over all matrix elements. However, an expression for the unknown quantity δ in (20) can be found in a far simpler way, by noting that under the condition $g^{1/2}(\alpha_c-\alpha) \gg 1$ this equation yields a renormalization of the action:

$$A=A_0+g(\alpha_c-\alpha)^2/4\delta. \tag{22}$$

The action (22) corresponds to the split-off trajectory 1 or 2 in Fig. 1b. The quantity $A - A_0$ can thus also be found from (4) with the help of (8):

$$A=A_0+\frac{g}{2}(\alpha-\alpha_c)\int_{-\tau_0}^{\tau_0} d\tau\, q^2. \tag{23}$$

Comparing expression (22) and (23), we can find the coefficient δ. An explicit expression has been derived for it.

At small values of $q, p-p_0$ and $\alpha_c-\alpha$, we find from (10)

$$\hat{P}(p-p_0)={}^3/_2q^2, \tag{24}$$

$$\hat{Q}(\alpha_c)q=3q(p-p_0)+2(\alpha_c-\alpha)q. \tag{25}$$

A solution of Eq. (25) is the function

$$q(\tau)=c\psi(\tau), \tag{26}$$

where the function $\psi(\tau)$ is given by (16).

To find the normalization factor c we need to integrate Eq. (25) with a weight $\psi(\tau)$. Setting

$$a_k=\int_{-\tau_0}^{\tau_0} d\tau\, \psi^2(\tau), \quad b_k=-\frac{1}{c^2}\int_{-\tau_0}^{\tau_0} d\tau\, \psi^2(\tau)(p-p_0), \tag{27}$$

we find

$$c^2={}^2/_3(\alpha_c-\alpha)a_k/b_k. \tag{28}$$

Substituting (26) and (28) into (23), we find the following result for δ:

$$\delta={}^3/_4 b_k/a_k^2. \tag{29}$$

The function $p-p_0$ which appears in (27) is found from the inhomogeneous equation (24) with the function $q(\tau)$ as in the (26), on the basis of the parity and periodicity requirements on τ. The quantities a_k and b_k are found in the Appendix.

6. EXPRESSION FOR THE PROBABILITY FOR THE DECAY OF A STATE

Near the singularity α_c the probability for the decay of a metastable state can be writen

$$\Gamma=BFe^{-A_0}, \tag{30}$$

where A_0 is given by (21), and F by

$$F=\int_0^{\infty}\frac{dx}{\pi^{1/2}}\exp\{-g[(\alpha-\alpha_c)x^2+\delta x^4]\}. \tag{31}$$

In some limiting cases we have

$$F=\frac{1}{|2g(\alpha-\alpha_c)|^{1/2}}\begin{cases}2^{-1/2}, & (\alpha-\alpha_c)(g/\delta)^{1/2}\to\infty,\\ \exp[-g(\alpha_c-\alpha)^2/4\delta], & (\alpha-\alpha_c)(g/\delta)^{1/2}\to-\infty.\end{cases}$$

The coefficient B is expressed in terms of a product of eigenvalues of the operators $\hat{P}$ and $\hat{Q}$ (Ref. 15):

$$B=\frac{g}{t_0}\left|\int_{-\tau_0}^{\tau_0}\frac{d\tau}{\pi}\left(\frac{\partial p_0}{\partial\tau}\right)^2 D_1D_2\right|^{1/2}. \tag{32}$$

Here

$$D_1=\frac{\det(2-\partial^2/\partial\tau^2)}{\det'(2-3p_0-\partial^2/\partial\tau^2)},\qquad D_2=\frac{\det(2+2\alpha_c-\partial^2/\partial\tau^2)}{\det'(2+2\alpha_c-3p_0-\partial^2/\partial\tau^2)}. \tag{33}$$

The prime in (32) and (33) means that the zero eigenvalue is omitted.

We use the familiar method for calculating the ratio of the determinants in the problem with periodic boundary conditions and with an even potential

$$\frac{\det\left[V_1(\tau)-\dfrac{\partial^2}{\partial\tau^2}\right]}{\det\left[V_2(\tau)-\dfrac{\partial^2}{\partial\tau^2}\right]}=\frac{W_2}{W_1}\frac{\partial\psi_1}{\partial\tau}\varphi_1\left(\frac{\partial\psi_2}{\partial\tau}\varphi_2\right)^{-1}\Bigg|_{\tau=\tau_0}, \tag{34}$$

where $\psi_{1,2}$ are even solutions, and $\varphi_{1,2}$ odd solutions, of the equation $-\partial^2 f/\partial\tau^2 + V_{1,2}(\tau)f = 0$, and $W_{1,2}$ are the corresponding Wronskians.

Singling out the zero eigenvalue, using (34), we find

$$D_1 = -\frac{s_1-s_3}{6k^4}\,\frac{\mathrm{sh}^2(\pi T_0/T)}{J_1(k)\int_0^1 dx\, x^2[(1-x^2)(1-k^2x^2)]^{1/2}}, \tag{35}$$

$$D_2=(s_1-s_3)^{1/2}\,\mathrm{sh}^2[\pi T_0(1+\alpha_c)^{1/2}/T][a_k J_2(k)]^{-1},$$

where

$$J_1(k)=\int_1^\infty \frac{dx}{(x-1)^{1/2}}\,\frac{2x-k^2}{x^{5/2}(x-k^2)^{5/2}},$$

$$J_2(k)=\int_0^1 \frac{dx}{(1-x^2)^{1/2}(1-k^2x^2)^{3/2}(1+\mu-x^2)^2}. \tag{36}$$

Expressions for the functions $J_{1,2}(k)$ in terms of complete elliptic integrals are given in the Appendix.

The integral

$$\int_{-\tau_0}^{\tau_0} d\tau\left(\frac{\partial p_0}{\partial\tau}\right)^2 = \frac{16\cdot 2^{1/2}k^4}{(1-k^2+k^4)^{5/4}}\int_0^1 dx\, x^2[(1-x^2)(1-k^2x^2)]^{1/2}, \tag{37}$$

in (32) cancels the same integral in (35).

Substituting (35)–(37) into (32), we find an expression for the coefficient of the exponential function:

$$B=\frac{g}{t_0}\left[\frac{32}{3\pi a_k J_1(k)J_2(k)}\right]^{1/2}\frac{\mathrm{sh}(\pi T_0/T)\,\mathrm{sh}[\pi T_0(1+\alpha_c)^{1/2}/T]}{1-k^2+k^4}. \tag{38}$$

The decay probability is thus given by (30), where the argument of the exponential function, A_0, is given by expression (21), and the coefficient of the exponential function is given by (38). The function F in (30) describes a transition from one decay regime to another. The quantities α_c and δ which appear in it are given by (18) and (29), and the temperature dependence of k is found with the help of expression (13).

Here are the values of these quantities in the low-temperature limit $T \ll T_0$:

$$A_0=32\cdot 2^{1/2}g/15\approx 3.017g, \tag{39}$$

$$\delta=2^{1/2}\cdot 7^{-1}(^{15}/_{16})^2\approx 0.178, \tag{40}$$

$$B=96(50/\pi^2)^{1/4}g/t_0\approx 144.025g/t_0. \tag{41}$$

At some temperature T^* below T_0, the coefficient δ vanishes. The behavior of the decay probability near this point requires a special analysis, which is the subject of Sec. 8.

7. DECAY PROBABILITY AS A FUNCTION OF THE CURRENT AND THE TEMPERATURE

If the coupling coefficient β^{-1} is sufficiently small, there exists a threshold current I_t such that at currents $I < I_t$ there can be a change in the decay regime of the metastable state. This threshold current corresponds to the value $\alpha_c = 5/4$ and itself has the value

$$I_t=2I_c(1-32/25\beta^2). \tag{42}$$

It follows that at a large value of β the characteristic currents are close to the critical value $2I_c$.

The temperature $T_1(I)$, which separates the regime of symmetric decay, in which we have $\varphi_1 = \varphi_2$ on the extremal trajectory, from the regime of the asymmetric decay, is found from the condition $\alpha_c(T_1) = \alpha$. From (6) and (18) we find a universal functional dependence of the quantity $T_1\beta^{1/2}/\hbar\omega$ on $(1 - I/2I_c)\beta^2$, which is determined implicitly by

$$\left(1-\frac{I}{2I_c}\right)\beta^2=\frac{32(1-k^2+k^4)}{[2-k^2+2(1-k^2+4k^4)^{1/2}]^2},$$

$$\frac{T_1}{\hbar\omega}\beta^{1/2}=\{2^{1/2}K(k)[2-k^2+2(1-k^2+4k^4)^{1/2}]^{1/2}\}^{-1}. \tag{43}$$

This functional dependence is shown by curve 1 in Fig. 2; curve 2 there shows $T_0\beta^{1/2}/\hbar\omega$ as a function of $(1 - I/2I_c)\beta^2$, according to (14). Figure 2 may also be regarded as a phase diagram. Above curve 2 the decay is activated, while below this curve the decay is a tunneling-activated decay. To the left of curve 1 the decay is symmetric; to the right the symmetry is disrupted.

8. DECAY NEAR T^*

The behavior of the decay probability described in Sec. 7 prevails only up to the temperature T^* if we move along curve 1 in Fig. 2. The temperature T^* is found from the condition $\delta(T^*) = 0$. A numerical calculation yields for k the value $k^* = 0.454$. From (43) we can then find the coordinates of the point in Fig. 2 at which we have $T = T^*$:

$$T^*\beta^{1/2}/\hbar\omega\approx 0.2194, \quad (1-I^*/2I_c)\beta^2\approx 1.8958. \tag{44}$$

Because of the numerical parameter, the ratio $T^*/T_0(I^*) \approx 0.9879$ is very nearly equal to unity. By analogy with phase-transition theory,[17] the point R in Fig. 2 is a tricritical point, where the equilibrium curves for first- and second-order phase transitions converge. The classical action A serves as a thermodynamic potential here. Curve 1 below the point R corresponds to a second-order transition, according to this analogy, while above point R it corresponds to a first-order transition.

The behavior of the decay probability near R can not be determined if we restrict the expansion of the action to terms

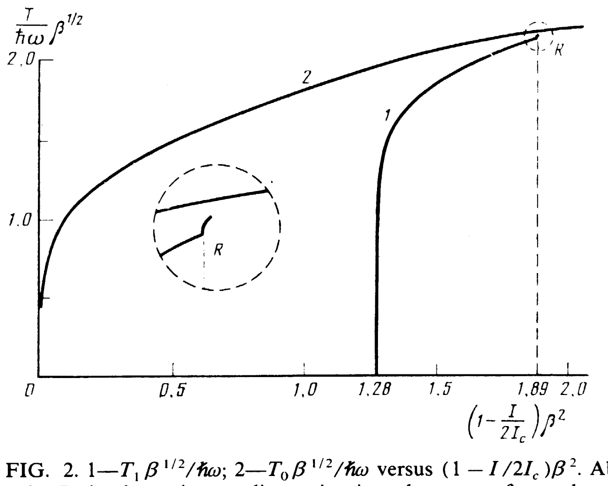

FIG. 2. 1—$T_1\beta^{1/2}/\hbar\omega$; 2—$T_0\beta^{1/2}/\hbar\omega$ versus $(1 - I/2I_c)\beta^2$. Above point R, the change in tunneling regime is analogous to a first-order phase transition. The point R corresponds to $T = T^*$.

of fourth order, as we did in (20). The incorporation of the sixth-order terms has the consequence that an additional term of order x^6 appears in the argument of the exponential function in (31), so that F is replaced by the expression

$$F \to \int_0^\infty \frac{dx}{\pi^{1/2}} \exp\{-g[(\alpha-\alpha_c)x^2+\delta x^4+\gamma x^6]\}. \quad (45)$$

We can determine γ by comparing the extremal action found for $\alpha<\alpha_c$ with the help of (45) with the extremal action calculated from (4) and (8). The value of γ at $T=T^*$ is

$$\gamma=-\frac{1}{2a_k^3}\left[\frac{1}{c^6}\int_{-\tau_0}^{\tau_0} d\tau(p-p_0)^3+\frac{3}{c^5}\int_{-\tau_0}^{\tau_0} d\tau(p-p_0)(q-c\psi)\psi\right], \quad (46)$$

where the deviation of the function q from $c\psi$, which is small by a factor on the order of c^2, satisfies the equation

$$(\partial^2/\partial\tau^2+3p_0-2-2\alpha_c)(q-c\psi)=-3c(p-p_0)\psi-2c(\alpha_c-\alpha)\psi. \quad (47)$$

The function $p-p_0$ in (46) and (47) satisfies Eq. (24) and is determined by Eq. (A8). As a result, the value of γ is given by

$$\gamma=-\frac{1}{2a_k^3}\left\{\frac{1}{c^6}\int_{-\tau_0}^{\tau_0} d\tau(p-p_0)^3 + \frac{9}{c^4}\int_{-\tau_0}^{\tau_0} d\tau\left[\frac{1}{\psi(\tau)}\int_0^\tau d\tau_1\,\psi^2(\tau_1)(p-p_0)\right]^2\right\}. \quad (48)$$

Evaluating the integrals in (48) numerically, we find $\gamma=1.694$. The quantity δ in (45) is given near the point T^* by the expression

$$\delta=1.924(1-T/T^*). \quad (49)$$

As a result, using the numerical values which we have found, we can write F as

$$F=\int_0^\infty \frac{dx}{\pi^{1/2}}\exp\left\{-g\left[(\alpha-\alpha_c)x^2+1.924\left(1-\frac{T}{T^*}\right)x^4+1{,}694x^6\right]\right\}. \quad (50)$$

Equation (30) with F from (50) thus determines the behavior of the decay probability near the tricritical point $T=T^*$.

The quantity x in (50) is proportional to the distance between the split-off trajectory and the symmetric position. The action with appears in the argument of the exponential function in (50) can have simultaneous minima at a nonzero x and at $x=0$ in the case $T>T^*$. The equality of the action values at these minima determines a phase equilibrium line, which is a continuation of curve 1 above point R:

$$T/T^*-1=1.353(\alpha-\alpha_c)^{1/2}. \quad (51)$$

It follows that the equilibrium curves intersect at right angles at the point R.

Our analysis describes the behavior of curve 1 in Fig. 2 only if the temperature is slightly above T^*. As the temperature is raised further, it becomes necessary to consider the pronounced nonlinearity of the action when the trajectory departs from the asymmetry position.

At the transition from one regime to the other at $T=T_1$, the argument of the exponential function in (30) is equal to $A_0\sim V\beta^{-5/2}/\hbar\omega$ in order of magnitude. This circumstance determines the choice of the most suitable coupling coefficients β^{-1} for observing these effects during the decay of a metastable state.

9. CONCLUSION

In the one-dimensional case, the decay of an activation nature gives way to a quantum-mechanical decay at a certain temperature T_0, because of the appearance of a nontrivial trajectory in imaginary time. In the multidimensional case the picture may become much more complicated. For certain values of the potential parameters, two extremal trajectories may come close together. Near such a point, the fluctuations of the mode corresponding to a transition between trajectories are large. For this reason, the quadratic approximation in the action is inadequate. As a result, the two-dimensional case which we have studied here contains in addition to T_0, another critical temperature, T_1, which corresponds to a change in tunneling regime.

The decay probability given by (30) can be written as the product of the probability Γ_1, for a decay in the one-dimensional case, and some function proportional to $FD_2^{1/2}$. In the one-dimensional case, for a cubic potential, the probability Γ_1 is

$$\Gamma_1=B_1e^{-A_0}, \quad (52)$$

where the action on the extremal trajectory, A_0, is given by (21), and the coefficient of the exponential function is

$$B_1=\frac{1}{t_0}\left(\frac{2^{3/2}}{\pi}g\right)^{1/2}\operatorname{sh}\left(\frac{\pi T_0}{T}\right)\frac{k^2(1-k^2)^{1/4}}{(1-k^2+k^4)^{7/8}} \cdot\left[\frac{1-k^2+k^4}{1-k^2}E(k)-\left(1-\frac{k^2}{2}\right)K(k)\right]^{-1/2}. \quad (53)$$

In (53) we have $t_0=2^{1/2}/\omega_0$, $g=(27/2^{1/2}\cdot 8)U_0/\hbar\omega_0$, and $T_0=\hbar\omega_0/2\pi$, where ω_0 is the frequency of the small oscillations near the potential minimum, and U_0 is the height of the potential barrier. The experimental temperature dependence of the decay probability of the current-carrying state of an isolated Josephson junction, Γ_1, is in excellent agreement with the theoretical temperature dependence.[10] For the system of Josephson junctions which we have examined here, the ratio Γ/Γ_1 should increase substantially at temperatures below T_1.

Above the point R, the temperature or current dependence of the decay probability changes slope. However, this temperature interval would seem to be too narrow for a detailed experimental study.

APPENDIX

We can express the quantities which determine the lifetime Γ in (30) in terms of elliptic integrals:

$$J_1(k)=\frac{4}{3k^4(1-k^2)}\left[\frac{1-k^2+k^4}{1-k^2}E(k)-\left(1-\frac{k^2}{2}\right)K(k)\right], \quad (A1)$$

$$J_2(k)=\frac{1}{2\mu(1+\mu)(1-k^2-k^2\mu)^2}[E(k)-(1-k^2\mu)K(k)$$

$$-\frac{3k^2\mu^2+2(2k^2-1)\mu-1+k^2}{1+\mu}\Pi\left(\frac{\pi}{2},\frac{1}{1+\mu},k\right)\Bigg]$$

$$+\frac{k^4}{(1-k^2-k^2\mu)^2}\left[\frac{E(k)}{1-k^2}-\frac{1}{k^2(1+\mu)}\Pi\left(\frac{\pi}{2},\frac{1}{1+\mu},k\right)\right], \tag{A2}$$

$$a_k=2\sqrt{2}(1-k^2+k^4)^{1/4}\Bigg[\frac{E(k)-(1-k^2)K(k)}{k^2}\times\left(\frac{2}{3}\mu+\frac{2}{5}-\frac{2}{15k^2}\right)+\left(\mu^2+\frac{2}{3}\mu+\frac{1}{15k^2}+\frac{1}{5}\right)E(k)\Bigg]. \tag{A3}$$

We turn now to a calculation of the function b_k, determined by (27), where $p-p_0$ is an even solution of inhomogeneous equation (24) which satisfies the condition of periodicity with a period of $2\tau_0$. This solution can be written in the form

$$(p-p_0)(z)=\frac{6}{\varphi_1-\varphi_3}\frac{\partial\,\mathrm{cn}^2 z}{\partial z}(c_1+\chi c_2), \tag{A4}$$

where the odd function χ satisfies the equation

$$\partial\chi/\partial z=(\partial\,\mathrm{cn}^2 z/\partial z)^{-2},$$

and the functions c_1 and c_2 are found from the equations

$$\frac{\partial c_1}{\partial z}=\chi\frac{\partial\,\mathrm{cn}^2 z}{\partial z}q^2,\quad \frac{\partial c_2}{\partial z}=-\frac{\partial\,\mathrm{cn}^2 z}{\partial z}q^2. \tag{A5}$$

The solution of these equations should be determined from the condition on the parity and periodicity of the function $p-p_0$. The function $c_2(z)$ is

$$c_2(z)=-c^2\{{}^1/_4k^2\,\mathrm{cn}^8 z+{}^1/_3(1-k^2+2k^2\mu)\,\mathrm{cn}^6 z+[{}^1/_2k^2\mu^2+(1-k^2)\mu]\,\mathrm{cn}^4 z+(1-k^2)\mu^2\,\mathrm{cn}^2 z+\varkappa\}. \tag{A6}$$

The quantity χ is given by

$$\varkappa=-\frac{1-k^2}{6k^2}[(1-k^2)(3+2\mu)+k^2\mu^2]+\frac{1-k^2}{3k^2}[(1-k^2)^2(1+2\mu)-k^2(2-k^2)\mu^2]\cdot\frac{(1-2k^2)E(k)-(1-k^2)K(k)}{2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)}. \tag{A7}$$

The quantity $p-p_0$ can be written in the form

$$p-p_0=\frac{3}{s_1-s_3}\left\{\frac{(2y-1)c_2(y)}{1-k^2+k^2y}+[y(1-y)(1-k^2+k^2y)]^{1/2}\cdot\int_y^1\frac{2y_1-1}{[y_1(1-y_1)]^{1/2}}\frac{\partial}{\partial y_1}\frac{c_2(y_1)}{(1-k^2+k^2y_1)^{3/2}}dy_1\right\}, \tag{A8}$$

where $y=\mathrm{cn}^2 z$ and the function $c_2(y)$ is given by expression (A6), with $\mathrm{cn}^2 z$ replaced by y.

Using these relations along with (27), we find the following expression for b_k:

$$b_k=-\frac{3}{2^{1/2}}(1-k^2+k^4)^{3/4}\Bigg\{\frac{2\varkappa^2}{1-k^2}\left[\frac{1-k^2+k^4}{1-k^2}E(k)-\left(1-\frac{k^2}{2}\right)K(k)\right]$$
$$+\frac{k^4}{16}L_8+\frac{k^2}{6}(2k^2\mu+1-k^2)L_7+\left\{\frac{1}{9}(2k^2\mu+1-k^2)^2+\frac{k^2}{4}[2\mu(1-k^2)+k^2\mu^2]\right\}L_6+\left\{\frac{k^2}{2}(1-k^2)\mu^2+\frac{1}{3}(2k^2\mu+1-k^2)[k^2\mu^2+2\mu(1-k^2)]\right\}L_5 \tag{A9}$$
$$+\left\{\frac{1}{4}[k^2\mu^2+2\mu(1-k^2)]^2+\frac{2}{3}(1-k^2)\mu^2(2\mu k^2+1-k^2)\right\}L_4+(1-k^2)\mu^2[k^2\mu^2+2(1-k^2)\mu]L_3+(1-k^2)^2\mu^4L_2\Bigg\};$$

here

$$L_n=\int_0^1\frac{dy}{(1-y)^{1/2}}(2y-1)y^{n-3/2}(1-k^2+k^2y)^{-5/2}\times\left[\left(\frac{3}{2}-n\right)k^2y-n(1-k^2)\right]. \tag{A10}$$

These quantities satisfy the recurrence relation

$$L_n=\frac{2(n-3)}{2n-7}\frac{2k^2-1}{k^2}L_{n-1}+\frac{2n-5}{2n-7}\frac{1-k^2}{k^2}L_{n-2}. \tag{A11}$$

For L_2 and L_3 we have

$$L_2=K(k)-2E(k),\qquad L_3=[(1-2k^2)E(k)-(1-k^2)K(k)]/k^2. \tag{A12}$$

Expressions (A3) and (A9), along with (A7) and (A10), thus determine the value of δ according to expression (29).

[1] R. F. Yoss and R. A. Webb, Phys. Rev. Lett. **47**, 265 (1982).
[2] L. D. Jackel, J. P. Gordon, E. L. Ilu, *et al.*, Phys. Rev. Lett. **47**, 697 (1981).
[3] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1982).
[4] V. Ambegaokar, U. Eckern, and G. Schön, Phys. Rev. Lett. **18**, 1745 (1982).
[5] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **85**, 1510 (1983) [Sov. Phys. JETP **58**, 876 (1983)].
[6] A. Schmid, Ann. Phys. **170**, 333 (1986).
[7] V. I. Mel'nikov and S. V. Meshkov, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 111 (1983) [JETP Lett. **38**, 130 (1983)].
[8] M. Büttiker, E. P. Harris, and R. Landauer, Phys. Rev. B **28**, 1268 (1983).
[9] I. M. Dmitrenko, G. M. Tsoĭ, and V. I. Shnyrkov, Fiz. Nizk. Temp. **8**, 660 (1982) [Sov. J. Low Temp. Phys. **8**, 330 (1982)].
[10] N. Devoret, J. M. Martinis, and J. Clarke, Phys. Rev. Lett. **55**, 1908 (1985).
[11] K. K. Likharev, Physica B and C **108**, 1079 (1981).
[12] C. D. Tesche, J. Low Temp. Phys. **44**, 119 (1981).
[13] M. Klein and A. Mukherjee, Appl. Phys. Lett. **40**, 744 (1982).
[14] J. S. Langer, Ann. Phys. **41**, 108 (1967).
[15] C. Callan and S. Coleman, Phys. Rev. B **16**, 1762 (1977).
[16] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **86**, 719 (1984) [Sov. Phys. JETP **59**, 420 (1984)].
[17] A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, *Fluktuatsponnaya Teoriya Fazovykh Perekhodov* (*Fluctuation Theory of Phase Transitions*), Nauka, Moscow, 1982.

Translated by Dave Parsons