1+1-dimensional sigma model at finite temperatures

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The thermodynamic equations for the sigma model are derived, in the limit of infinite anisotropy, from the exact equations describing the thermodynamics of an anisotropic chiral field. The equations are then used to demonstrate factorization in the $U(1)$ gauge group. The low-temperature expansion of the free energy is then used to obtain a formula for the statistics of excitations. A representation of the excited-state wave function corresponding to this solution is discussed.

1. INTRODUCTION

The $O(3)$ symmetric sigma-model (n-field) is the $1+1$-dimensional theory of a Goldstone non-Abelian field:

$$S=\frac{1}{2\lambda}\int dt\,dx[(\partial_t n)^2-(\partial_x n)^2], \tag{1}$$

where $n(x,t)$ is a three-component unit vector.

The sigma-model (1) is a simple example of a theory in which a purely geometric limitation leads to strong interactions between particles: because of the non-Abelian character of the S^2 manifold, the effective coupling constant $\lambda(\varepsilon,p)$ increases as the energy scale ε (momentum p) is reduced.[1] The low-energy domain of the theory in which the perturbation theory is invalid can be described in terms of an exact solution; the model defined by (1) allows this because it has an infinite series of quantum conservation laws.[2,3] However, for well-known reasons, it is difficult to obtain directly the exact solution for (1). An exact solution has, in fact, now been obtained for a model of the anistropic chiral field that includes the sigma model as a limiting case.[4] The corresponding action takes the form

$$S=\int dt\,dx\left[\frac{1}{2\lambda_\perp}(\Omega_\mu{}^x\Omega^{\mu x}+\Omega_\mu{}^y\Omega^{\mu y})+\frac{1}{2\lambda_\parallel}\Omega_\mu{}^z\Omega^{\mu z}\right], \tag{2}$$

$$\Omega_\mu{}^a=\mathrm{Sp}\,(\sigma^a g^{-1}\partial_\mu g),\quad g\in SU(2),$$

where σ_a are the Pauli matrices. In the limit as $\lambda_\parallel\to\infty$, the action given by (2) becomes identical with (1) (Ref. 5). This can be verified by using the Hopf parametrization

$$\mathbf{n}\cdot\boldsymbol{\sigma}=g^{-1}\sigma^z g.$$

An exact solution of (1), regarded as a limiting case of (2), has been put forward by Wiegmann.[4] Excitations of the sigma-model have been found to be massive particles that transform in accordance with a representation of the group $O(3)$ with spin $S=1$. The two-particle S-matrix for these particles, obtained from the exact solution, is identical with the S-matrix proposed earlier[6] on the basis of phenomenological considerations.

The present paper is devoted to the derivation of thermodynamic equations describing the behavior of the sigma-model at finite temperatures, and to the examination of this behavior. At low temperatures $T\ll m$ (m is the mass of an excitation), the behavior of the system can be described in the language of single-particle excitations, and a perturbation theory in the density of these excitations can be developed. The low-temperature expansions for the free energy are used to determine the particle statistics and the interaction parameters. The high-temperature expansion for $\ln(T/m)\gg 1$, on the other hand, is used to determine the dimensions of the manifold to which the fields belong.

The plan of our paper is as follows. In Sec. 2, we discuss the qualitative picture of the limiting transition from model (2) to model (1) in the language of the particle spectrum. Section 3 is devoted to the derivation of the thermodynamic equations for model (2), and Sec. 4 gives a formal procedure for passing to the limit as $\lambda_\parallel\to\infty$ in these equations. Section 5 analyzes the resulting equations. The Conclusion discusses the final results.

2. THE PARTICLE SPECTRUM OF AN ANISOTROPIC CHIRAL FIELD IN THE LIMIT AS $\lambda_\parallel\to\infty$

It will be helpful in understanding the presentation given below to recall how the limit as $\lambda\to\infty$ is accomplished in the spectrum of physical particles of model (2).

Since the action given by (2) is invariant under global transformations of the form

$$g\to Ug\quad (U\in SU(2)),$$
$$g\to gV\quad (V\in U(1)),$$

it seems natural to suppose that the two-particle S-matrix of fundamental particles belonging to model (2) is the tensor product[4]

$$S(\theta)=S_{SU(2)}(\theta)\otimes S_{U(1)}(\theta;\gamma).$$

The co-factor matrices in this tensor product satisfy the Yang-Baxter equation as well as being unitary and crossing-symmetric. The matrix $S_{SU(2)}$ is invariant under the fundamental representation of the group $SU(2)$. The matrix $S_{U(1)}$ has the same dimension, but is invariant only under the group $U(1)$. It is, in fact, the soliton scattering matrix in the sine-Gordon model The parameter γ is expressed in terms of the constant $\lambda_\perp,\lambda_\parallel$ and $\gamma\to 0$ as $\lambda_\parallel\to\infty$.

When $\gamma<8\pi$, the matrix $S_{U(1)}(\theta)$ has bands on the physical sheet that correspond to bound states of physical particles. The masses of the bound states ("breathers") are

$$m_j=2M_0\sin j\gamma/16,\quad j=1,\dots[8\pi/\gamma]-1.$$

The breathers do not carry the $U(1)$ charge, but constitute the tensorial square of the fundamental representation of the group $SU(2)$, i.e., they are a mixed state containing the singlet and triplet states of $SU(2)$.

For $\gamma\ll 1$, the mass of the first breather is $m_1\ll M_0$ (M_0 is the mass of the fundamental particle, i.e., the kink). If we pass to the limit as $\gamma\to 0$, we can consider the mass of the breather to be finite, in which case $M_0 \to \infty$ as $\gamma \to 0$ and kinks cease to be excited. This means that, as $\lambda_{\parallel} \to \infty$, model (2) exhibits the phenomenon of kink confinement (non-Hermitian), first discussed in this context in Refs. 2 and 7). Next, it turns out that, as $\gamma \to 0$, the triplet-triplet scattering splits from the singlet-singlet scattering. The matrix elements of the two-particle S-matrix corresponding to the singlet-triplet scattering are of order of $\gamma^{1/2}$ (Ref. 7). The S-matrix of triplet excitations becomes identical with the S-matrix proposed in Ref. 6 when $\gamma \to 0$.

3. DERIVATION OF THERMODYNAMIC EQUATIONS FOR A MODEL OF THE ANISOTROPIC CHIRAL FIELD

There are two ways of reducing model (2) to the model that can be solved using the Bethe ansatz.[4,8] The final result is the same in both cases. In particular, Wiegmann[4] used the model of relativistic fermions

$$\mathscr{L} = i\bar{\psi}_{\alpha k}\gamma_\mu \partial_\mu \psi_{\alpha k} - \lambda_\perp (J_\mu^x J^{x\mu} + J_\mu^y J^{y\mu}) - \lambda_\parallel J_\mu^z J^{z\mu},$$
$$J_\mu^a = \bar{\psi}_{\alpha k}\sigma_{\alpha\beta}^a \gamma_\mu \psi_{\beta k}, \quad k = 1, \ldots, N_f, \quad \alpha, \beta = 1, 2 \tag{3}$$

which is equivalent to model (2) in the $N_f \to \infty$ limit.

To find the eigenvalues of the Hamiltonian for model (3), we must place the system in a box of finite length L, and apply periodic boundary conditions to the N-particle wave function. The relativistic invariance that is violated by this can be restored by passing to the limit $L \to \infty$, $N \to \infty$, $M_0 L / N \to 0$, $M_0 = \text{const}$ in the expressions for the corresponding physical variables. The $N \to \infty$ and $N_f \to \infty$ limits do not commute and must be taken in the indicated order.

The energy eigenvalues of model (3) can be expressed in terms of the solutions of the following set of equations:[4,8]

$$E/L = \frac{N}{2iL} \sum_{\tau = \pm 1} \tau \sum_{\alpha = 1}^{M} \ln e_{N_f}(\lambda_\alpha + \tau/g), \tag{4a}$$

$$\{e_{N_f}(\lambda_\alpha + 1/g)\, e_{N_f}(\lambda_\alpha - 1/g)\}^{N/2} = \prod_{\alpha=1}^{M} e_2(\lambda_\alpha - \lambda_\beta),$$
$$e_n(\lambda) = \left\{\sinh \frac{\pi}{2p_0}(\lambda + in)\right\}^{-1} \sinh \frac{\pi}{2p_0}(\lambda - in). \tag{4b}$$

The number M can be expressed in terms of the number N of particles and the component S^z of the total spin angular momentum:

$$M = NN_f/2 - S^z.$$

The constants g and p_0 can be expressed in terms of $\lambda_\parallel$, $\lambda_\perp$.

Wiegmann[4] and Kirillov and Reshetikhin[9] have shown that, in the neighborhood of the limit in which we are interested, $p_0 = N_f + 1/\nu\ (\nu > 2)$, where $\lambda_\parallel \to \infty$ corresponds to $\nu \to \infty$. As before, $\gamma = 8\pi/\nu$. To avoid too many complexities, we shall assume that ν is an integer. The $\nu \to \infty$ and $N_f \to \infty$ limits commute.

The thermodynamic equations for model (3) can be written using the results reported in Ref. 9, which gives the equations for arbitrary N_f and p_0. The general equations given there are exceedingly unwieldy and inconvenient for analysis. We have rewritten them for our own case, $p_0 = N_f + 1/\nu$, after some rearrangement.

The free energy of the system is determined from the following set of nonlinear integral equations for the functions $\varepsilon(\lambda)$ ($\sigma = \pm\frac{1}{2}$), i.e., the energies of the kink and antikink, the breather energy $\varkappa_j(\lambda)$ ($j = 1,\ldots,\nu - 1$), and the auxiliary functions $\varepsilon_p(\lambda)$ ($p = 1,\ldots,N_f - 1$):

$$\frac{F}{L} = -\frac{T}{4}\sum_\sigma M_\sigma \int d\lambda \cosh\frac{\pi\lambda}{2} \ln\left(1 + \exp\left\{-\frac{\varepsilon_\sigma(\lambda)}{T}\right\}\right)$$
$$-\frac{T}{4}\sum_{j=1}^{\nu-1} m_j \int d\lambda \cosh\frac{\pi\lambda}{2} \ln\left(1 + \exp\left\{-\frac{\varkappa_j(\lambda)}{T}\right\}\right), \tag{5}$$

$$\varepsilon_\sigma(\lambda) - \sum_{\sigma'} K * T \ln\left(1 + \exp\left\{-\frac{\varepsilon_{\sigma'}(\lambda)}{T}\right\}\right) = M_0 \cosh\frac{\pi\lambda}{2}$$
$$- s * T \ln\left(1 + \exp\frac{\varepsilon_1(\lambda)}{T}\right) + b_j * T \ln\left(1 + \exp\left\{-\frac{\varkappa_j(\lambda)}{T}\right\}\right), \tag{6}$$

$$T \ln\left(1 + \exp\frac{\varkappa_j(\lambda)}{T}\right) - B_{jk} * T \ln\left(1 + \exp\left\{-\frac{\varkappa_k(\lambda)}{T}\right\}\right)$$
$$= m_j \cosh\frac{\pi\lambda}{2} - \xi_j * T \ln\left(1 + \exp\frac{\varepsilon_1(\lambda)}{T}\right)$$
$$+ b_j * \sum_\sigma T \ln\left(1 + \exp\left\{-\frac{\varepsilon_\sigma(\lambda)}{T}\right\}\right),$$
$$j, k = 1, \ldots, \nu - 1, \tag{7}$$

$$T \ln\left(1 + \exp\left\{-\frac{\varepsilon_p}{T}\right\}\right) - C_{pq} * T \ln\left(1 + \exp\frac{\varepsilon_q}{T}\right)$$
$$= -\delta_{p,1}\left[s * \sum_\sigma T \ln\left(1 + \exp\left\{-\frac{\varepsilon_\sigma}{T}\right\}\right) + \xi_j * T \ln\left(1 + \exp\left\{-\frac{\varkappa_j}{T}\right\}\right)\right],$$
$$p, q = 1, \ldots, N_f - 1. \tag{8}$$

We use the following notation:

$$f * g(\lambda) = \int_{-\infty}^{+\infty} f(\lambda - \lambda') g(\lambda')\, d\lambda',$$

$$m_j = 2M_0 \sin(\pi j/2\nu), \quad M_0 = \Lambda \exp(-\pi/g).$$

The Fourier transforms of the kernels of the integral equations are:

$$s(\omega) = (2\cosh\omega)^{-1},$$
$$K(\omega) = \sinh(1 - 1/\nu)\omega / 2\cosh\omega \sinh(\omega/\nu),$$
$$b_j(\omega) = \coth(\omega/\nu)\sinh j(\omega/\nu)/\cosh\omega,$$
$$\xi_j(\omega) = \cosh(1 - j/\nu)\omega/\cosh\omega,$$
$$B_{jk} = B_{kj},$$
$$B_{jk}(\omega) = 2\coth(\omega/\nu)\cosh^{-1}\omega \cosh(1 - j/\nu) \times \omega \sinh(k\omega/\nu) \quad (j > k),$$
$$C_{pq} = \delta_{pq} - s(\delta_{p,q-1} + \delta_{p,q+1}).$$

4. THE LIMIT OF MAXIMUM ANISOTROPY ($\nu \to \infty$, i.e., $\gamma \to 0$)

We shall now derive the equations describing the thermodynamics of the sigma-model (1). As indicated above, this can be done by passing to the limit as $\nu \to \infty$ in the equations describing the spectrum of physical particles, i.e., in Eqs. (5)–(8). We recall that, as $\nu \to \infty$, the mass of the first breather $m_1 \equiv m$ must be assumed finite.

As $\nu \to \infty$, the kernels of the integral equations degenerate:

$$B_{jk}(\lambda) = 2\min(j, k)\delta(\lambda), \quad \xi_j(\lambda) = \delta(\lambda).$$

Equations (7) thus become algebraic:

$$T\ln\left(1+\exp\frac{\varkappa_j(\lambda)}{T}\right)$$

$$-2T\sum_{k=1}^{\infty}\min(j,k)\ln\left(1+\exp\left\{-\frac{\varkappa_k(\lambda)}{T}\right\}\right)'$$

$$=jm\cosh\frac{\pi\lambda}{2}-T\ln\left(1+\exp\frac{\varepsilon_1(\lambda)}{T}\right),\quad j,k=1,2,\ldots. \tag{9}$$

Inverting the kernel, we obtain (9) in the more convenient form

$$C_{jk}(0)T\ln\left(1+\exp\frac{\varkappa_k}{T}\right)-T\ln\left(1+\exp\left\{-\frac{\varkappa_j}{T}\right\}\right)$$

$$=-\frac{1}{2}\delta_{j,1}T\ln\left(1+\exp\frac{\varepsilon_1}{T}\right),$$

$$\lim_{j\to\infty}\varkappa_j(\lambda)/j=m\cosh(\pi\lambda/2)\equiv\varepsilon_0(\lambda), \tag{10}$$

$$C_{jk}(0)=\delta_{jk}-{}^1/_2(\delta_{j,k-1}+\delta_{j,k+1}).$$

This set of equations was first obtained by Takahashi:[10]

$$1+\exp\frac{\varkappa_j(\lambda)}{T}=\left\{\sinh\left[\frac{\varepsilon_0(\lambda)}{2T}(j+1)-u(\lambda)\right]\Big/\sinh\frac{\varepsilon_0(\lambda)}{2T}\right\}^2. \tag{11}$$

We can now use (10) with $j=1$ to express $u(\lambda)$ in terms of ε_1:

$$\left(1+\exp\frac{\varepsilon_1}{T}\right)^{-1}=\left\{\sinh\left(\frac{\varepsilon_0}{2T}-u\right)\Big/\sinh\frac{\varepsilon_0}{2T}\right\}^2. \tag{12}$$

Next, the sum in (8) must be expressed in terms of u and ε_0:

$$\sum_j^{\infty}\ln(1+e^{-\varkappa/T})$$

$$=\sum_{j=1}^{\infty}\ln\left\{\sinh^2\left(\frac{\varepsilon_0}{2T}(j+1)-u\right)\Big/\sinh\left(\frac{\varepsilon_0}{2T}j-u\right)\right.$$

$$\left.\times\sinh\left(\frac{\varepsilon_0}{2T}(j+2)-u\right)\right\}$$

$$=\ln\left\{e^{-\varepsilon_0/2T}\sinh\left(\frac{\varepsilon_0}{T}-u\right)\Big/\sinh\left(\frac{\varepsilon_0}{2T}-u\right)\right\}. \tag{13}$$

An analogous sum appears in the expression for the free energy (5):

$$\sum_{j=1}^{\infty}j\ln(1+e^{-\varkappa_j/T})=-\ln(1-e^{2u-\varepsilon_0/T})$$

$$=\ln(1+e^{-\zeta/T})/(1-e^{-\varepsilon_0/T}). \tag{14}$$

In Eqs. (8), we must put $N_f=\infty$. Equations (6) are unnecessary because the mass of the kink satisfies $M_0\to\infty$ and the kink is not excited.

Finally, we obtain the following set of equations:

$$\zeta(\lambda)=m\cosh\frac{\pi\lambda}{2}-s*T\ln\left(1+\exp\frac{\varepsilon_2(\lambda)}{T}\right), \tag{15}$$

$$\varepsilon_j(\lambda)=Ts*\ln\left(1+\exp\frac{\varepsilon_{j-1}(\lambda)}{T}\right)\left(1+\exp\frac{\varepsilon_{j+1}(\lambda)}{T}\right)$$

$$+\delta_{j,2}Ts*\ln\left(1+\exp\left\{-\frac{\zeta(\lambda)}{T}\right\}\right),\quad j=1,2,\ldots, \tag{16}$$

$$\lim_{j\to\infty}\varepsilon_j/j=H, \tag{17}$$

$$F=F_1+F_2, \tag{18a}$$

$$\frac{F_1}{L}=-\frac{Tm}{4}\int d\lambda\cosh\frac{\pi\lambda}{2}\ln\left(1+\exp\left\{-\frac{\zeta(\lambda)}{T}\right\}\right), \tag{18b}$$

$$\frac{F_2}{L}=\frac{T}{2\pi}\int dp\ln\left(1-\exp\left(-\frac{(m^2+p^2)^{1/2}}{T}\right)\right). \tag{18c}$$

Thus, the free energy has split into the two independent parts F_1 and F_2. The latter is the free energy of a free massive Bose-field. This part of the energy corresponds to the degrees of freedom of the system that are singlets in $SU(2)$ color.

Equations (15)–(17), (18b) describe the thermodynamics of the sigma-model. They include condition (17), which does not follow from the foregoing presentation and requires further elucidation.

The transition to the $N_f\to\infty$ limit in (8) makes this set of equations incomplete. The requirement that the solution of (15), (16) is unique can be satisfied only by imposing a condition on the behavior of ε_j for $j\to\infty$. Equation (17) is, in fact, this condition. The constant J has the following significance. The model (1) has the following conservation law:

$$\mathbf{I}_0=\int dx[\mathbf{n}\times\partial_t\mathbf{n}],$$

and the field H is the conjugate of it. This constant of motion did not appear in model (3) for finite N_f, so that the field H did not appear at the preceding stages of the solution.

It is important to note that the thermodynamic equations (15)–(17), (18b) that we have obtained are precisely the same as would have been obtained had we examined the set of particles with relativistic spectrum $\varepsilon(\theta)=m\cosh\theta$, for which the factorized scattering is described by the two-particle S–matrix of Ref. 6.

5. ANALYSIS OF THERMODYNAMIC EQUATIONS

The thermodynamic equations contain sufficient information on the spectrum of the particles and their interaction. In particular, the expansion of the free energy in terms of $\exp(-m/T)$ for $T\ll m$ can be used to obtain the formula for the statistics of single-particle excitations, and to help understand how the interaction renormalizes the density of states.

At high temperatures $\ln(T/m)\gg1$, the interaction between the particles is small and the behavior of the free energy is determined by the dimension of the manifold G of the sigma-model:

$$F=-\frac{\pi}{6}(\dim G)T^2+\ldots. \tag{19}$$

In our case, $G=S^2$, $\dim G=2$.

To achieve a better understanding of confinement physics, it is interesting to evaluate the correction to the free energy in (18) for small but finite γ. We shall now evaluate the first correction in γ.

(A) Thus, we shall solve (15)–(17), (18b) by iteration in $\exp(-m/T)$. In the zero-order approximation, the term containing $\ln[1+\exp(-\zeta/T)]$ can be neglected in (16). The set of equations given by (16) then becomes algebraic and its solution is

$$1+\exp\frac{\varepsilon_j}{T}=\Phi^2(j), \tag{20}$$

$$\Phi(j)=\sinh(H(j+1)/2T)/\sinh(H/2T).$$

We now use this solution to obtain from (15) the expression for the first iteration:

$$\zeta^{(1)}=\varepsilon_0-T\ln\Phi(2). \tag{21}$$

Let us now calculate the first correction in $\exp(-m/T)$ to the function

$$d_j=T\ln(1+\exp(\varepsilon_j/T))=2T\ln\Phi(j)+d_j^{(1)}+\dots$$

Linearizing (16), we obtain the following set of equations for $d_j^{(1)}$:

$$[\Phi^2(j)/\Phi(j-1)\Phi(j+1)]d_j^{(1)}-s*(d_{j-1}^{(1)}+d_{j+1}^{(1)})$$
$$=\delta_{j,2}s*T\ln(1+e^{-\zeta/T}). \tag{22}$$

The solution of this is[10]

$$d_j^{(1)}=\frac{\Phi(1)}{\Phi(2)\Phi(j)}(\Phi(j+1)\hat{a}_j-\Phi(j-1)\hat{a}_{j+2})s^{-1}*T$$
$$\times\ln\left(1+\exp\left(-\frac{\zeta}{T}\right)\right), \tag{23}$$

where the Fourier transform of the kernel is

$$a_j(\omega)=e^{-j|\omega|}.$$

The second interation for ζ is obtained from (23):

$$\zeta^{(2)}=-T\frac{\Phi(1)}{\Phi^2(2)}(\Phi(3)\hat{a}_2-\Phi(1)\hat{a}_4)*\ln\left(1+\exp\left(-\frac{\zeta}{T}\right)\right). \tag{24}$$

This formula contains integration with respect to rapidity, which gives a contribution of order $\sim(T/m)^{1/2}$. Correspondingly, the third iteration (the formula which is given in the Appendix) contains two integrations and, consequently, is of order T/m. We can identify in the free energy the contribution due to the single-particle excitations by replacing the function ζ in (18b) with its first iteration (21):

$$\frac{F^{(1)}}{L}=-\frac{T}{4}\int d\lambda\cosh\frac{\pi\lambda}{2}\ln\left(1+\Phi(2)\exp\left\{-\frac{\varepsilon_0(\lambda)}{T}\right\}\right). \tag{25}$$

When interaction is taken into account, this results in corrections of the order of $(T/m)^{1/2}$ in the coefficients of $\exp(-nm/T)$ in the expansion of (25).

We now introduce the chemical potential into (15) by adding $-\mu$ on the right-hand side. Differentiating (18b) with respect to μ, we obtain the formula for the mean number of particles:

$$N=-\frac{1}{2\pi}\int dpn(\zeta(p))\left.\frac{\partial\zeta(p,\mu)}{\partial\mu}\right|_{\mu=0}=\frac{1}{2\pi}\int dpn_0(p)$$
$$+O\left(\left(\frac{T}{m}\right)^{1/2}\exp\left(-\frac{3m}{T}\right)\right),\quad\left(p=m\sinh\frac{\pi\lambda}{2}\right), \tag{26}$$

where

$$n_0(p)=\left[1+\frac{\sinh(H/2T)}{\sinh(3H/2T)}\exp\frac{\varepsilon_0(p)}{T}\right]^{-1} \tag{27}$$

specifies the formula for the statistics of the single-particle excitations.

We note that corrections of the order of $(T/m)^{1/2}\exp(-2m/T)$ in (26), due to particle interaction, cancel out.

(B) We now turn to the description of the high-temperature limit $\ln(T/m)\gg1$. The main contribution to the free energy of the system is then provided by the region $|\lambda|\gg1$, where $me^{\pi\lambda/2}\sim T$. In this region, the distribution densities of the particles and holes, $\rho_n,\tilde{\rho}_n$, are related to the energies ε_n by the following expressions that are valid for any system with a linear spectrum:

$$\rho_n(\lambda)=\frac{1}{2\pi}\operatorname{sign}\lambda\frac{\partial\varepsilon_n}{\partial\lambda}n(\varepsilon_n),$$
$$\tilde{\rho}_n(\lambda)=\frac{1}{2\pi}\operatorname{sign}\lambda\frac{\partial\varepsilon_n}{\partial\lambda}(1-n(\varepsilon_n)),\quad n(\varepsilon)=\left(1+\exp\frac{\varepsilon}{T}\right)^{-1}$$

We can now use these expressions to obtain the following formula for the entropy:[11]

$$S/TL=\frac{2}{\pi}\sum_n L\{\max(n(\varepsilon_n)),\min(n(\varepsilon_n))\}, \tag{28}$$

$$L(a,b)=-\frac{1}{2}\int_b^a dt\left(\frac{\ln t}{1-t}+\frac{\ln(1-t)}{t}\right),$$

where $L(a,b)$ is the Rogers dilogarithm.

According to (28), the entropy is determined only by the asymptotic values of the functions $\varepsilon_n(\lambda)$, which are related to the degree of degeneracy of the corresponding states. When $|\lambda|\to\infty$, the solution of (16) is given by (20), i.e., $n(\zeta)=0$ for $H=0$, and $n(\varepsilon_j)=(j+1)^{-2}$. When $|\lambda|\sim1$, the free term in (15) can be discarded to within $O[1/\ln(T/m)]$. In this case, (15) and (16) again become algebraic equations and their solution is

$$\zeta=-\infty,\ \varepsilon_j=+\infty,\quad \text{т. е. } n(\zeta)=1,\quad n(\varepsilon_j)=0. \tag{29}$$

Substituting these values in (28), we obtain

$$\frac{S}{TL}=\frac{2}{\pi}\left(L(1,0)+\sum_{j=1}^{\infty}L\left(\frac{1}{(j+1)^2},0\right)\right)=\frac{2\pi}{3}. \tag{30}$$

The high-temperature limit is thus correctly reproduced [see (19)].

(C) We must now evaluate the correction to the free energy for small but finite γ. We shall retain terms of order $1/\nu$ in the expansions for the kernels of (7) and (8):

$$B_{jk}(\omega)=2\min(j,k)-2j\frac{k}{\nu}\omega\operatorname{th}\omega+O\left(\frac{\omega^2}{\nu^2}\right), \tag{31}$$
$$\xi_j(\omega)=1-\frac{j}{\nu}\omega\operatorname{th}\omega.$$

As before, Eq. (6) is unnecessary because the inclusion of ε_σ yields the corrections to the free energy of order $\exp[-(m/\gamma)/T]$, which we shall not consider here.

Having used (31) in (7) and (8), we obtain a modified form of (15)–(18) after some simple rearrangement. In particular, the corrections to the free energy are expressed in terms of the function

$$\Delta=\frac{2}{\nu}f*T\ln\left\{\left(1+\exp\left\{-\frac{\zeta}{T}\right\}\right)\Big/\left(1-\exp\left\{-\frac{\varepsilon_0}{T}\right\}\right)\right\},$$
$$f(\omega)=\omega\tanh\omega. \tag{32}$$

The quantity $-\frac{1}{2}\Delta(\lambda)$ must be added to the right-hand side of (15), Eqs. (16) and (17) remain unaltered, and ε_0 in (18c) must be replaced with $\varepsilon_0-\Delta(\lambda)$.

The correction to the free energy assumes the simplest form for $T\ll m$:

$$\frac{1}{L}\delta F=\frac{\pi m}{4\nu}(\Phi(2)+1)\left(1+\frac{\Phi(2)}{2}\right)\int d\lambda\,d\lambda'$$

$$\cdot\exp\left(-\frac{2m}{T}\cosh\frac{\pi\lambda}{4}\cosh\frac{\pi\lambda'}{4}\right)+O(e^{-4m/T}). \quad (33)$$

This is wholly due to the interaction, but the singularity of the kernel f [cf. (32)] leads to the appearance of the additional factor $\sim m/T$, which cancels the small quantity due to the integration with respect to rapidity.

It is clear that a perturbation theory in $1/\nu$ is a theory with singular kernels. All corrections of order $\sim\nu^{-n}$ are exclusively due to the interaction between the triplet and singlet degrees of freedom. The excitation of kinks leads to corrections that are exponential in $1/\nu$.

6. CONCLUSION

We have thus obtained Eqs. (15)–(17), (18b) that describe the thermodynamics of the sigma-model. Their form is typical of integrable theories. Hence, the formulas for the statistics of the excitations (27) and the low-temperature expansion are also typical of all integrable theories in which the spectum of excitations is separated by a gap from the ground state. These formulas can be used as a basis for reconstructing the wave function of the excitations.

It follows from (25) and (27) that (a) excitations transporting kinetic energy do not have internal degrees of freedom and (b) double occupation of a state with given rapidity is not possible. Hence, the N-particle wave function must be written as a product of functions describing the distribution of rapidities and the isotopic state of the system, respectively:

$$\Psi=\Psi_{ch}\Psi_{sp}, \quad (34a)$$

$$\Psi_{ch}(\theta_1,\ldots,\theta_N)=\prod_{\theta_l>\theta_{l+1}}R^+(\theta_i)|\Omega\rangle, \quad (34b)$$

$$\Psi_{sp}(a_1,\ldots a_N)=\{B(\lambda_1)\ldots B(\lambda_M)\}_{a_1,\ldots a_N}|0\rangle \quad (34c)$$
$$(a_i=-1,\ 0,\ +1).$$

The state $|\Omega\rangle$ is the physical vacuum and $|0\rangle$ is the isotopic vector corresponding to the maximum component of the total spin, $S^z=N, M=N-S^z$.

The energy of the system is

$$E=m\sum_{j=1}^{N}\cosh\theta_j-H(N-M). \quad (35)$$

When $H=0$, a state of given energy [given set of rapidities $\{\theta_j\}$] is degenerate and its isotopic structure is specified by the function Ψ_{sp}. This function is the eigenfunction of the N-particle scattering matrix:

$$T(\theta_1,\ldots\theta_N)=S_{12}(\theta_1-\theta_2)\ldots S_{1N}(\theta_1-\theta_N), \quad (36)$$

where $S(\theta)$ is the two-particle scattering matrix of the sigma-model.[6]

The matrix (36) can be diagonalized, and hence represention can be found for the operators $B(\lambda)$ by the method proposed in Ref. 12. It turns out that the S-matrix of the sigma-model satisfies the following relation:

$$S_{ab}{}^{nm}(\theta_1-\theta_2)\,\Xi_{n\sigma}{}^{\bar a\tau}(\theta_2-\theta_3)\,\Xi_{m\tau}{}^{\bar b\sigma}(\theta_1-\theta_3)=\Xi_{b\sigma}{}^{m\tau}(\theta_1-\theta_3)\cdot$$
$$\Xi_{a\tau}{}^{n\sigma}(\theta_2-\theta_3)S_{nm}{}^{\bar a\bar b}(\theta_1-\theta_2),$$
$$\Xi_{a\sigma}{}^{\bar a\bar\sigma}(\theta)=\theta\delta_{a\bar a}\delta_{\sigma\bar\sigma}+i\pi\boldsymbol{\sigma}_\sigma{}^{\bar\sigma}\mathbf{S}_a{}^{\bar a}, \quad (37)$$

where $\boldsymbol\sigma$, $\mathbf{S}$ are, respectively, the spin $\frac12$ and spin 1 operators. By virtue of (37), the scattering matrix (36) commutes with the trace of the monodromy matrix:

$$\mathscr{L}^{\beta\bar a_1,\ldots\bar a_M}_{\alpha,a_1\ldots a_M}(\lambda,\{\theta_j\})=\Xi^{\alpha_1\bar a_1}_{\alpha a_1}(\lambda-\theta_1)\ldots\Xi^{\beta\bar a_M}_{\alpha_M a_M}(\lambda-\theta_M) \quad (38)$$

so that the wave function Ψ_{sp} is also the eigenfunction of the operator

$$\mathrm{Sp}\,\mathscr{L}(\lambda)=\mathscr{L}_1{}^1(\lambda)+\mathscr{L}_2{}^2(\lambda). \quad (39)$$

It is well known that, in this case $B(\lambda)=\mathscr{L}_1^2(\lambda)$, and the parameters $\{\lambda_i\}$ satisfy the equations[3,12]

$$\prod_{j=1}^{N}\frac{\lambda_\alpha-\theta_j-i\pi}{\lambda_\alpha-\theta_j+i\pi}=\prod_{\beta=1}^{M}\frac{\lambda_\alpha-\lambda_\beta-i\pi}{\lambda_\alpha-\lambda_\beta+i\pi}. \quad (40)$$

We have thus constructed a representation of the operator $B(\lambda)$ in the form of an element of the monodromy matrix (38).

As far as the operator $R^+(\theta)$ and the conjugate operator $R(\theta)$ are concerned, it is reasonable to suppose that they satisfy the following commutation relations:

$$R^+(\theta_1)R^+(\theta_2)=S_0(\theta_1-\theta_2)R^+(\theta_2)R^+(\theta_1),$$
$$R(\theta_1)R(\theta_2)=S_0(\theta_1-\theta_2)R(\theta_2)R(\theta_1),$$
$$R^+(\theta_1)R(\theta_2)=S_0(\theta_2-\theta_1)R(\theta_2)R^+(\theta_1)+m\,\mathrm{ch}\,\theta_1\delta(\theta_1-\theta_2), \quad (41)$$

where

$$S_0(\theta)=(\theta-i\pi)/(\theta+i\pi)$$

is the unitarizing factor in the S-matrix of the sigma-model. Since $S_0(0)=-1$, it follows from (41) that $[R^+(\theta)]^2=0$ is a state with given rapidity that cannot be occupied twice.

A representation of the wave function analogous to that given above was constructed in Ref. 13 for the case of two excitations in the Heisenberg XXX-chain. Our representation of the wave function may be useful in the derivation of the Gel'fand-Levitan-Marchenko quantum equations for integrable models.

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APPENDIX

We shall now calculate the correction $\sim\exp(-2m/T)$ to the function $d_j=T\ln(1+e^{\varepsilon/T})$. It is determined by the following set of equations:

$$\frac{\Phi^2(j)}{\Phi(j-1)\Phi(j+1)}d_j^{(2)}-s*(d_{j-1}^{(2)}+d_{j+1}^{(2)})$$
$$=-\frac{1}{2}\left\{\frac{d_j^{(1)}\Phi(j)}{\Phi(j+1)\Phi(j-1)}\right\}^2. \quad (A1)$$

To evaluate this correction, we must know the Green's function of the operator on the left-hand side of (A1). The Fourier transform of the Green's function is

$$G_{jk}(\omega)=-\frac{1}{\Phi(j)}\frac{(\Phi(j+1)\sinh j\omega-\Phi(j-1)\sinh(j+2)\omega)(\Phi(k+1)-\Phi(k-1)e^{-2|\omega|})}{[\Phi(k)\sinh3\omega-\sinh\omega(\Phi(k+1)+\Phi(k-1)+\Phi(k-3))]}$$
$$\times s^{-1}(\omega)e^{-k|\omega|}(j\leqslant k),$$

$$G_{jk}(\omega) = -\frac{1}{\Phi(j)} \frac{(\Phi(j+1)e^{-j|\omega|} - \Phi(j-1)e^{-(j+2)|\omega|})}{[\Phi(k)\sinh 3\omega - \sinh\omega(\Phi(k+1) + \Phi(k-1) + \Phi(k-3))]}.$$
$$\times(\Phi(k+1)\sinh k\omega - \Phi(k-1)\sinh(k+2)\omega) \quad (j>k).$$

For the required function, we have

$$d_j^{(2)} = G_{jk} * (d_k^{(1)}\Phi(k)/\Phi(k+1)\Phi(k-1))^2.$$

The second iteration of the function ζ is

$$\zeta^{(2)} = -Ts * d_2^{(2)}.$$

When $T \ll m$,

$$\zeta^{(2)}/T \sim (T/m)^{3/2} e^{-2m/T},$$

i.e., when $\zeta^{(2)}$ is taken into account, the result is of order $(T/m)^{3/2}$, as compared with the leading term in the expansion for the free energy of order $\exp(-3m/T)$.

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