Investigation of the period-doubling bifurcation cascade occurring during the development of magnetohydrodynamic instability in bismuth

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The paper reports the experimental investigation of the development of the instability due to the interaction between a current flowing through a sample and its magnetic field under conditions when the instability develops via a chain of period-doubling bifurcations. It was possible to record experimentally six doubling bifurcations (which corresponds to a 64-fold increase in the initial period). From the experimental results the values of the universal constants describing the process (the Feigenbaum numbers) could be found: $3.60 \lesssim \delta \lesssim 4.83$, and $\alpha \approx 2.65$.

INTRODUCTION

Significant successes have been achieved in the last few years in the theoretical study of the laws governing the transition of dynamical systems to chaos. This has aroused interest in the experimental study of this transition in different physical systems. The theoretical and experimental investigations have shown that the transition to chaos can occur in qualitatively different ways (scenarios).[1–3] Occupying a special place among them is the Feigenbaum scenario, according to which the randomization of the behavior of a number of nonlinear systems when some parameter varies monotonically can occur through successive period-doubling bifurcations. This doubling continues to infinity, and the sequence of bifurcation values of the parameter converges to some limit, beyond which chaotic behavior arises. In the vicinity of the critical point of the transition to chaos, the rate of convergence of the sequence of bifurcation values of the parameter and the shape of the spectrum are found to be universal, i.e., do not depend on the specific characteristics of the nonlinear systems.

The purpose of our experiment was to carry out a detailed quantitative investigation of the cascade of period-doubling bifurcations that occurs in the development of magnetohydrodynamic instability in bismuth. Earlier we reported the observation of an instability that develops during the passage of a constant electric current through a bismuth sample cooled down to helium temperatures.[4] The system of equations describing the instability consists of the constitutive equation

$$\mathbf{E}=\rho_0\mathbf{j}+\rho_1[\mathbf{H}\mathbf{j}]+\rho_2[\mathbf{H}[\mathbf{H}\mathbf{j}]]$$

(the first term describes the resistance; the second, the Hall effect; and the third, the magnetoresistance) and the Maxwell equations. Even before the onset of the instability, the magnetic self-field of the current and the presence of the magnetoresistance cause the current to be distributed nonuniformly over the cross section of the sample (the current density at the center is higher than at the periphery). Just before the onset of the instability in our experiments the parameter $\omega_c\tau$ (where ω_c is the cyclotron frequency and τ is the carrier relaxation time) attained values ranging from 20 to 30, and, thus, the current (and the associated magnetic field) were very nonuniform. The instability was due to the effect of the magnetic-field distribution on the current distribution over the cross section of the sample, and the reciprocal effect of the current on the magnetic field profiles.

The parameters on which the nature of the instability depended were the current through the sample, the external constant magnetic field, the temperature, and the temperature gradient. When one of the parameters was varied with the others fixed, the system behaved in a manner that depended qualitatively on the values of the fixed parameters, and underwent various transitions from laminar behavior to turbulence, including a transition to chaos via a series of period doubling bifurcations.

In our first experiments[5] we observed a series of signal-period doublings as the current through the sample was increased; in this case a suitable sample temperature was chosen, but the geomagnetic field (0.4 Oe) was not cancelled out. As subsequent experiments showed, small changes in the external magnetic field lead to a qualitative change in the behavior of the system. Because of the dependence of the thermal regime on the current strength, it turned out to be experimentally convenient to fix the current through the sample, and observe the development of the instability as the external longitudinal magnetic field was varied, and this is what was done in the present investigation.

EXPERIMENTAL PROCEDURE

In the experiments we used a long ($l = 10$ cm) nearly-cylindrical ($\varnothing = 1.5$ cm) bismuth single crystal with a resistance ratio $\rho_{300}/\rho_{4.2} \approx 600$, grown by the Czochralski method along the C_1 axis. Terminals were soldered to the ends of the sample with nonsuperconducting solder made from very pure copper. To decrease the effect of the magnetic field produced by the current in the leads, we used the symmetric current-supply scheme.[6] A storage battery with a transistorized control circuit served as the direct current source. The sample was vertically positioned directly in liquid helium. On the whole the experimental geometry was close to the one used earlier.[4–7]

The geomagnetic field compensation coils and the coils for the production of the longitudinal magnetic field were located outside the cryostat. The geomagnetic field was cancelled out to within 0.01 Oe. When the current strength exceeded some threshold value, a carrier-flow instability arose that manifested itself in the emission from the sample of electromagnetic radiation, which was detected with the aid of

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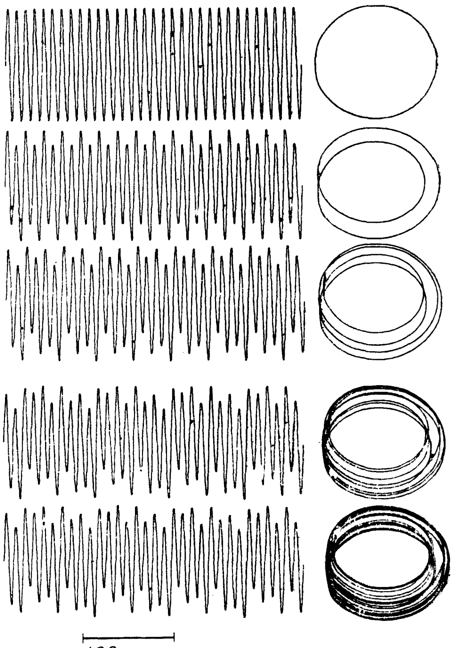

FIG. 1. Oscillograms of the signal and the phase portraits corresponding to 1, 2, 4, 8, and 16 cycles (from above down).

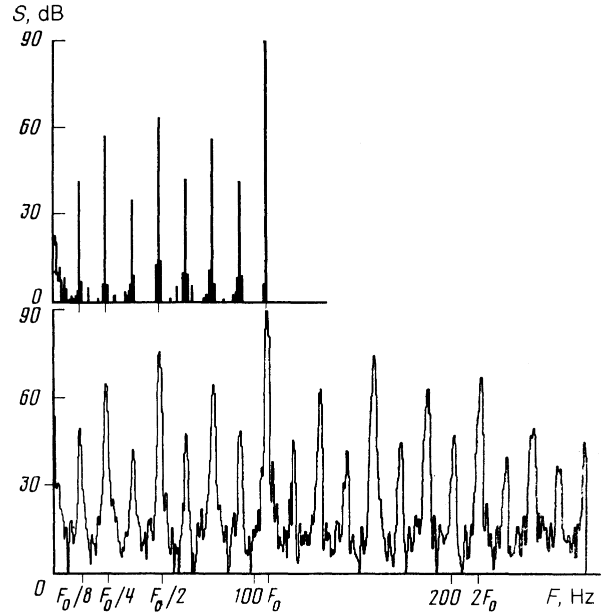

FIG. 2. Spectra obtained from a single realization of eight cycles: the top spectrum depicts the spectral power density S of a sequence of 256 signal extrema; there are 128 lines in the spectrum; and the duration of the sequence is 1.164 sec. The bottom spectrum is for a sequence of 1024 equidistant signal readouts; there are 512 lines in the spectrum; the sequence duration was 2.097 sec; and the readout rate was 488.28 Hz.

one of four coils wound in the middle part of the sample at distances ~ 1.5 cm from each other. Although the shapes of the signals obtained from the various coils differed from each other, the periods were the same. The signal from a coil, after being integrated, was amplified in the band 0.7–100 Hz. Thus, the quantity that was recorded in our experiments was the total variable magnetic flux $\Phi(t)$ through the sample cross section. The signal was transformed into a digital form by a 12-digit analog-to-digital converter coupled to a MERA-60 computer, and its oscillogram and phase portrait in the coordinates $\Phi(t)$ and $\int\Phi(t)dt$ were monitored simultaneously.

Figure 1 shows examples of oscillograms and phase portraits obtained at different values of the magnetic field intensity. It can be seen that the variation of the magnetic field leads to successive doublings of the period of the observed signal; in this case each level of the maxima (minima) of the signal splits up into two levels of the maxima (minima), and the magnitude of the splitting decreases with each successive doubling of the period.

Because of the fact that the oscillograms and phase portraits of the 16, 32, and 64 cycles are practically indistinguishable, their identification was carried out with the aid of Fourier analysis. The sequence initially used for analysis could be obtained either as an equal-time-interval sequence by analog-to-digital conversion (not synchronized with the process), or through analog-digital conversion of the extrema. Figure 2 shows the spectra of one and the same signal computed from the sequences obtained by these two methods. It can be seen that the spectra are close, but the Fourier analysis of the sequence of extrema gives much better frequency resolution with a smaller number of points and shorter experimental time, which is due to the syfnchronization of the analog-to-digital conversion with the process under investigation.

The separation from the continuous signal of a sequence of extrema corresponds to obtaining a sequence with the aid of the Poincaré section in the phase space of the system. This sequence is a set of points of intersection of the trajectory of the system's motion with some hypersurface in the phase space. In our case this surface is clearly the surface determined by the condition for the coordinate in question (the magnetic flux) in the phase space to be extremal (i.e., for $\partial\Phi/\partial t = 0$). The sequence of extrema separated from the continuous signal is quite informative, and allows us to compare the theoretical and experimental results in terms of the Poincaré mapping. The extremal values of the signal were determined by means of a quadratic interpolation over three points closest to the extrema, and transformed into digital form with 256-μsec intervals. The assembly-language subroutine written for the Fourier analysis procedure using the fast-Fourier-transformation (FFT) algorithm allowed us to dynamically follow the spectrum of the signal from the sample (64 cycles were identified in the Fourier analysis of a sequence of 256 extrema (128 lines in the spectrum): the spectrum was displayed on the screen of the videomonitor every 4 sec.

The regions of existence of the high-order cycles are narrow in the magnetic field, which made it necessary in this case to vary the magnetic field slowly. To do this, we began by monitoring the signal with respect to its spectrum, oscillogram, and phase portrait, in order to establish such current-strength, temperature, and magnetic-field-intensity values as made it possible for us to observe the high-order cycles and the transition to chaos; then the magnetic field was reversed electronically with increasing speed.

Since 65536 extrema could be measured and stored (during ~ 5.5 min of observations) and in the process the parameters of the experiment could be monitored with the aid of a digital voltmeter, the input section of which could be connected with the aid of a commutator to temperature, current, and magnetic-field gauges, we observed the entire series of signal-period doubling bifurcations in the course of the variation of the external longitudinal magnetic field, meanwhile keeping fixed in the computer memory the values of both the signal extrema and the parameters.

In order to prevent the drift of the temperature and current while the series of signal-period doubling bifurcations was being recorded, we waited ~ 30 min after the current through the sample had been switched on before beginning the measurements, this time being the period necessary to establish thermal equilibrium in the helium heat bath and to warm up the current source. The fixed parameters—the current and temperature—were measured directly before and after the sequence of extrema was recorded while the variable magnetic field was measured when the measurements were being recorded after each group of 128 extrema.

The measurement of the cascade of period-doubling bifurcations and the parameters of the experiment were recorded on a floppy disk, and were processed with the objects of determining the numerical values of the universal constants. The software was written in interpreted BASIC, expanded by the necessary assembly subroutines.

EXPERIMENTAL RESULTS

Figures 3 and 4 show the sequence of signal values at the extrema, and the dependence of the magnetic field on the time. It can be seen that, as we increase the magnetic field (the initial value of which corresponds to the critical point of the transition to chaos), the behavior of the instability becomes simpler; the number of different values of the extrema (the number of intersections of the graph with the vertical), and, consequently, the period are successively halved. For convenience of exposition, and taking account of the fact that, as experiment showed, no hysteresis phenomena occur in the parameter-value region in question, below we shall speak of the behavior of the system as the magnetic field decreases.

The decrease of the field gives rise to successive period doublings, doubling of the number (i.e., the splitting) of the levels of the extrema, and successive decreases in the distances between each pair of split levels. In the plots we can clearly see the bifurcations in which the cycles of order 2, 4, 8, and 16 are produced. As the order of the cycle increases, the corresponding magnetic field interval decreases.

In the region of magnetic fields corresponding to the high-order cycles, the bifurcation diagram (in the left-hand side of Fig. 4) reveals considerable noise, which manifests itself in smearing of individual levels exceeding the magnitude of the level splitting. At a fixed magnetic field in this region the spectra of the individual realizations do not coincide with each other, and there occur, besides the "pure" spectra (not containing a continuous component), spectra with a high level of noise; in this case the stability of the cycles (the rate at which the "pure" spectra arise) decreases as the magnetic field approaches the region of chaos.

Figure 5 shows the spectra of the signal-extremum sequences taken from the realization at some values of the magnetic field. It can be seen from Fig. 5 that, as the magnetic field value approaches the point of accumulation, the amplitudes of the subharmonics $f_0/2, f_0/4, f_0/8, \ldots$ (and their harmonics), which appear successively, decrease, but do not vary as the field intensity is decreased further.

DISCUSSION OF THE RESULTS

As the above results indicate, under our experimental conditions as the magnetic field is varied the instability develops via a chain of period doublings. Such behavior is characteristic of the scenario proposed by Feigenbaum.[3] The decrease of the level splittings and the values of the magnetic field for which the cycles of increasing order exist also point to this interpretation. Of obvious interest is the quantitative comparison of the experimental results with the theoretical predictions, according to which the sequence of the ratios of the maximum values of the level splittings at each subsequent bifurcation tends to a universal value $\alpha \approx 2.50$ (i.e., a value that does not depend on the specific system). For this comparison we took 256 terms of the sequence of extremum values a_i^{64} corresponding to 64 cycles (the spectrum 7 in Fig. 5), and from them we obtained by averaging the first and second halves of the realization a sequence corresponding to 32 cycles:

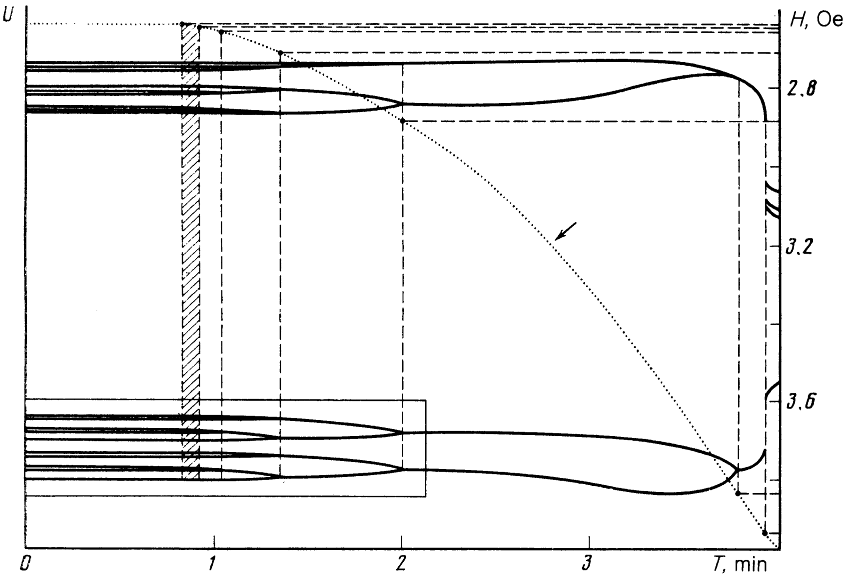

FIG. 3. Time dependence of the extremal values of a signal proportional to the magnetic flux through the sample cross section for different values of the external longitudinal magnetic field at $I = 50.18$ A and $T = 1.34$ K. The arrow indicates the time dependence of the magnetic field (right scale). The vertical dashed lines indicate the moments of the period doubling bifurcations; the horizontal dashed lines, the corresponding magnetic field values. The region of transition from 16 to 32 cycles is hatched. The part of the figure enclosed in the frame is shown in Fig. 4 in a magnified form.

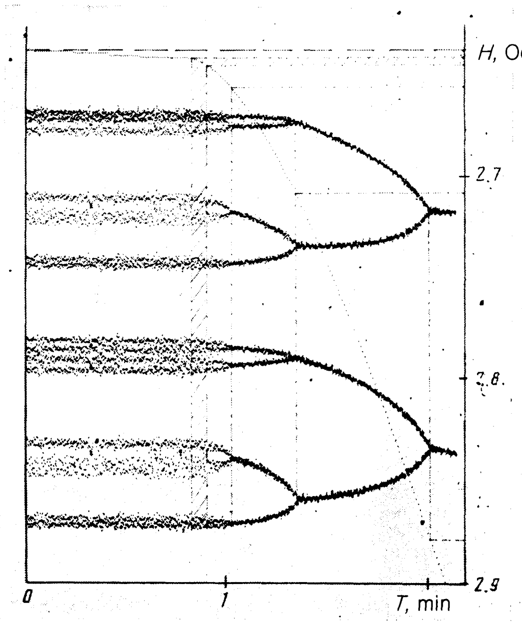

FIG. 4. Time dependence of the signal minima in magnetic fields corresponding to the high-order cycles. The designations are the same as in Fig. 3.

$$a^{32}_{i+128}=a_i^{32}=(a_i^{64}+a^{64}_{i+128})/2, \qquad i=0, 1, \ldots 127.$$

Applying a similar procedure six times, we obtained six sequences a_i^{2n} ($n = 0,1,\ldots,5$), and from them we found with the aid of the formula

$$\alpha_n=\max|a_i^{2^{n-1}}-a_{i+2^n}^{2^{n-1}}|/\max|a_i^{2^n}-a_{i+2^{n+1}}^{2^n}|$$

the sequence

$$\alpha_1=9.63,\ \alpha_2=1.81,\ \alpha_3=2.07,\ \alpha_4=2.74,\ \alpha_5=2.65.$$

As can be seen, the last two values deviate from the theoretical values by less than 10%, which, in our opinion, is quite good agreement. We verified the legitimacy of such a procedure for determining α from the experimental sequence by applying the same procedure to the sequence often used for the modeling of the Feigenbaum process, and given by the recursion formula $a_{i+1} = 1 - \mu a_i^2$, which for $\mu = 1.40115$ also gives 64 cycles. The same terms of the sequence obtained turned out to be equal to

$$\alpha_1=2.518,\ \alpha_2=2.510,\ \alpha_3=2.505,\ \alpha_4=2.504,\ \alpha_5=2.533.$$

Notice that the last term of this sequence deviates from the theoretical limit ($\alpha = 2.5029$) more than the preceding terms. This may be due to the roundoff in the computer calculations, which in our case used 32-bit words. This circumstance illustrates the fact that the determination of α from the experimental sequence obtained for high-order cycles can, with finite degree of precision, lead to a large error as a result of the fact that the level splittings are small (as compared to the measurement error). The determination of α from the low-order cycles does not make much sense, since the theory in this case does not lay claim to universality. Thus, for a given experimental accuracy there exists an optimal cycle order at which we realize the maximum accuracy of determination of the universal constant α, which determines the relation between the magnitudes of the extremum level splittings.

Another method of comparing the experimental results with the predictions of the theory uses the fact that the magnitudes of the level splittings determine the ratios of the subharmonics (and their harmonics) in the spectrum; in this case the amplitudes of the subharmonics (and their harmonics) of the high-order cycles decrease exponentially with the cycle order. Figure 6 shows the experimental spectrum of the 64 cycles and the theoretically computed spectrum of the sequence $a_{i+1} = 1 - \mu a_i^2$, with $\mu = 1.40115$. It can be seen that the amplitudes of the corresponding lines of the experimental and theoretical spectra are close. Common to the spectra shown are the "dips" in the spectra near the lines with large amplitude ($f_0, f_0/2, f_0/4, 3f_0/4$); in this case the dips near the lines with greater amplitude are deeper. The closeness of the spectra shown demonstrates good agreement between theory and experiment.

From the results of the experiment we could determine another fundamental constant. According to the theory, the sequence of bifurcation values of the controlling constant (in our case the magnetic field) should converge to its limit geometrically, i.e., the difference between the terms of this sequence and the limiting value $\Delta H_n \equiv H_n - H_\infty$ should form a decreasing geometric progression (at large n) with the common ratio $1/\delta$, where $\delta \approx 4.669$. A plot for the determination of δ is shown in Fig. 7 in the form $\Delta H_n(n)$, where n is the serial number of the bifurcation. The magnetic-field values corresponding to the four left points were determined directly from the bifurcation diagram (Figs. 3 and 4). As a result of the subsequent decrease of the magnitude of the extremum splitting at each subsequent bifurcation, the tran-

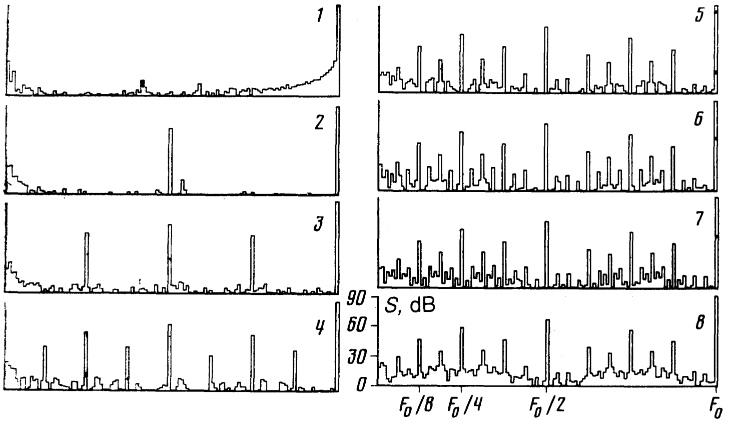

FIG. 5. Spectra of extrema sequences at different values of the magnetic field: 1) spectrum of 1 cycle, $H = 3.894$ Oe; 2) of 2 cycles, $H = 2.8956$ Oe; 3) of 4 cycles, $H = 2.7070$ Oe; 4) of 8 cycles, $H = 2.6784$ Oe; 5) of 16 cycles, $H = 2.6508$ Oe; 6) of 32 cycles, $H = 2.6427$ Oe; 7) of 64 cycles, $H = 2.6410$ Oe; 8) appearance of a continuous component in the spectrum, $H = 2.6405$ Oe.

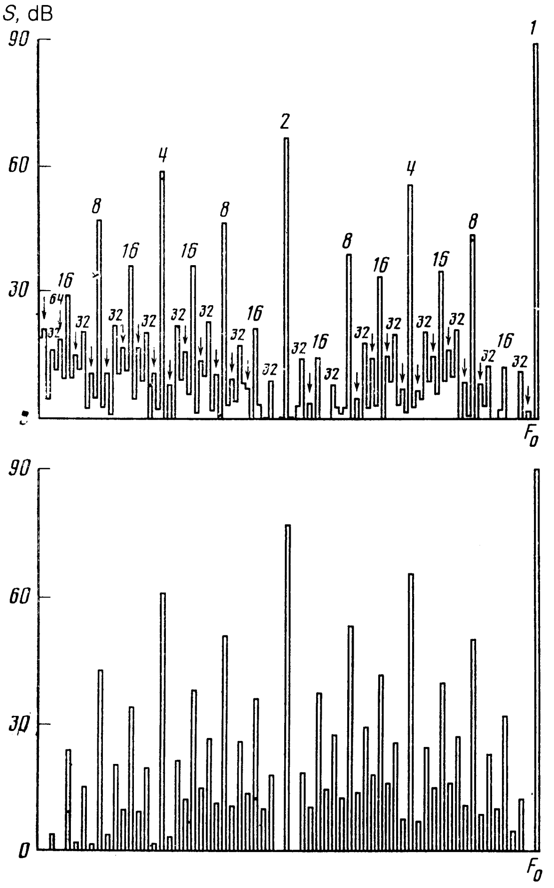

FIG. 6. Spectra of the 64th cycle. The top spectrum is the experimental spectrum for $H = 2.6410$ Oe; the bottom spectrum, the spectrum of the sequence $a_{i+1} = 1 - \mu a_i^2$ for $\mu = 1.40115$. The arrows indicate the frequency harmonic $f_0/64$.

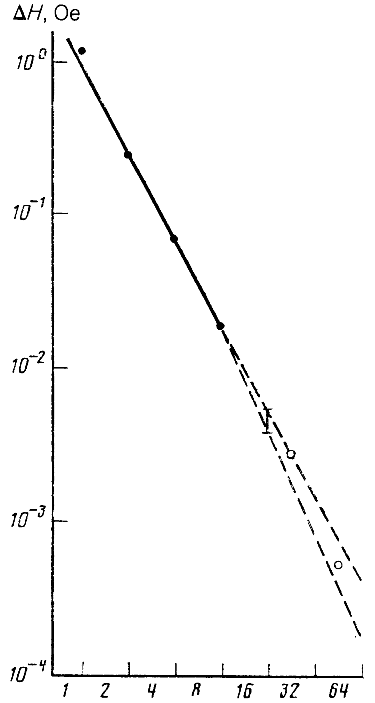

FIG. 7. Dependence of the bifurcation values of the magnetic field (measured from the accumulation point) on the cycle number (indicated by points). The vertical segment corresponds to the region of transition from the 16th to the 32nd cycle. The circles indicate the magnetic field values at which the spectra 6 and 7 in Fig. 5 were measured. The intervals along the abscissa, with the cycle numbers under them, correspond on the ordinate to magnetic field ranges within which the corresponding cycles are observed.

sition from 16 to 32 cycles cannot be seen from the realization (Figs. 3 and 4), but the extremum straggling is seen to be greater in this region. The boundaries of the region of the transition from 16 to 32 cycles were determined by means of Fourier analysis. In Figs. 3 and 4 this region is hatched; in Fig. 7 it is shown in the form of a vertical section. The angular coefficients of the dashed lines drawn through the ends of this section give the boundaries of the interval containing δ, and lie in the range from 3.60 to 4.83. Because of the noise (which can be seen in the left-hand side of the bifurcation diagram), in the region of the high-order cycles the magnetic-field point of transition from 32 to 64 cycles could not be determined; therefore, the circles in Fig. 7 correspond to the magnetic-field values at which the spectra of the 32 and 64 cycles were recorded (the spectra 6 and 7 in Fig. 5).

The fact that the plot shown in Fig. 7 is close to being a straight line in a broad range of magnetic field values (about three orders of magnitude) indicates both the right choice of the H_∞ value and the geometrical convergence of the sequence of bifurcation values of the parameter, which is in accord with the universality theory.

Notice that, as in the determination of α from the experimental data, the determination of δ from the first bifurcations does not have a special meaning because the theory does not lay claim to universality in this region, whereas the finite noise in real systems significantly lowers the accuracy with which the fundamental constants can be determined in the region of high-order cycles, where the extremum splittings and the parameter domains of existence of such cycles are small.

The high noise level in the region of parameter values corresponding to the high-order cycles may, in our opinion, be due to the fact that the individual sections of a cycle in phase space are close, and transitions to neighboring sections of the cycles are possible in the presence of noise in the system.

To conclude the discussion, let us note that another factor pointing to the fact that the development of the instability occurs according to the Feigenbaum scenario is the agreement between the experimentally observed migration of the extrema and the theoretically predicted course.

CONCLUSION

It has been shown that the development of the magnetohydrodynamic instability in bismuth can, under certain conditions, occur via a sequence of period doublings, and the possibility of observing high-order cycles (64 cycles) has been demonstrated. The values of the constants found experimentally correspond with satisfactory (in our opinion) accuracy to the theoretical values. The results presented are, apparently, the first observations of a chain of period doubling bifurcations in magnetohydrodynamic systems.

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