Self-similar "scalar" collapse regimes

V. M. Malkin and V. N. Khudik

Institute of Nuclear Physics, Siberian Branch, Academy of Sciences of the USSR
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Under the customary assumption that the collapse of Langmuir waves is a self-similar process, the statistical properties of strong Langmuir turbulence are determined in the inertial range by means of a certain probability measure specified on a set of self-similar collapse regimes. This measure and set have not been studied previously. The only results available are isolated and furthermore extremely specialized examples of self-similar solutions of the Zakharov equations and their scalar model, found numerically. A simple method is proposed for reducing the nonlinear problem of seeking self-similar scalar collapse regimes, with an arbitrary cavity shape, to a well-known linear problem: solving a Poisson equation. This approach reveals the structure of the entire set of self-similar scalar collapse regimes and permits an analytic derivation of these regimes for the case of ellipsoidal cavities.

1. INTRODUCTION

Most attempts to describe strong Langmuir turbulence have made use of the hypothesis that the collapse of the Langmuir waves is of a self-similar supersonic nature (Refs. 1 and 2, for example). According to this hypothesis, the shape of a collapsing cavity does not change in the inertial range, and the size decreases in accordance with

$$a \propto (t_s - t)^{2/3}, \tag{1}$$

where t_s is the time at which the singularity develops. Relation (1) agrees satisfactorily with the results of numerical calculations.[3–5] Although the conclusion that self-similar regimes like (1) are established for arbitrary initial conditions requires a more thorough test,[6] the study of these regimes will apparently prove useful for deriving a systematic theory of strong Langmuir turbulence. Despite the interest in self-similar regimes, which has been manifested since the very first paper on collapse,[7] we still do not know much about them. The very fact of their existence has remained in dispute until just recently. It has been only in the past few years that computers have made it possible to construct examples of self-similar solutions of the Zakharov equations and of the scalar analog of these equations proposed by Budneva *et al.*[3] The first step was to construct centrally symmetric solutions of a scalar collapse model and to solve one-dimensional equations giving an approximate description of the electric field on the "short" axis of the highly oblate cavity.[8] Later, the Zakharov equations with a centrally symmetric cavity and a triplet of identically populated bound states were solved.[9] These examples of course prove the existence of self-similar regimes of supersonic collapse. However, the solutions which have been found, which are extremely few in number and highly symmetric, do not tell us about typical representatives of the family of self-similar solutions or (especially) about the family as a whole. A systematic description of strong Langmuir turbulence will require not only studying this family but also finding the probabilities for the onset of all possible representatives of the family. In the present paper we attempt to solve the first part of this problem in a scalar model of collapse.

2. STRUCTURE OF THE FAMILY OF SELF-SIMILAR SOLUTIONS

Self-similar regimes like (1) of supersonic "scalar" collapse are described by the equations

$$(-\Delta + n + 1)E = 0, \tag{2}$$

$$\left(\frac{4}{3} + \frac{2}{3} r \frac{\partial}{\partial r}\right)\left(\frac{7}{3} + \frac{2}{3} r \frac{\partial}{\partial r}\right) n = \Delta |E|^2. \tag{3}$$

With each solution of this system of equations we can associate a region V,

$$n(\mathbf{r})|_{\mathbf{r}\in V} \leqslant -1, \quad n(\mathbf{r})|_{\mathbf{r}\notin V} > -1, \tag{4}$$

which is accessible to a particle with an energy of -1 during classical motion in the potential n. It turns out that the opposite is also true in a sense: If a region V has typical dimensions which are not too small, and if each point of its boundary S can be connected to the origin of coordinates by a straight line segment which lies entirely within V, then there exists a self-similar solution which satisfies conditions (4). In this section of the paper we are interested in regions V with typical dimensions which are large in comparison with unity. This restriction is sufficient for the existence of a solution of Eqs. (2) and (3) which corresponds to V, and it allows us to find this solution in a relatively simple way. The possibility of a simplification arises because the term ΔE in Eq. (2) is estimated to be small in comparison with E. If we omitted the term ΔE, we could easily find a solution E_*, n_* of the simplified version of system (2), (3). Specifically, in this case we would find from (2)

$$n_*(\mathbf{r})|_{\mathbf{r}\in V} = -1, \quad E_*(\mathbf{r})|_{\mathbf{r}\notin V} = 0, \tag{5}$$

and from (3) we would find

$$\Delta E_*^2|_{\mathbf{r}\in V} = -{}^{28}/_9, \tag{6}$$

$$n_*(\mathbf{r})|_{\mathbf{r}\notin V} = g_1 r^{-2} + g_2 r^{-7/2}, \tag{7}$$

where g_1 and g_2 are functions of the coordinates which are independent of r. Equation (6) with the boundary condition

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$$E_*^2|_{r\in S}=0 \quad (8)$$

has an single-valued solution, which is automatically positive. The functions g_1 and g_2 in (7) can be expressed in a simple way in terms of the jump in the gradient of E_*^2 at surface S:

$$g_1=\left(-\frac{7}{3}+\frac{3}{2}\frac{A}{R\cos\alpha}\right)R^2, \quad g_2=\left(\frac{4}{3}-\frac{3}{2}\frac{A}{R\cos\alpha}\right)R^{7/2}. \quad (9)$$

Here R is the point in which the radius vector r intersects the surface S,

$$\mathbf{A}=-\nabla E_*^2|_{r=R-0},$$

and α is the angle between $\mathbf{R}$ and $\mathbf{A}$ (under the limitations specified above, this is an acute angle).

The functions E_*, n_* constructed here constitue a good approximation to the solution of Eqs. (2), (3) far from S. The term ΔE in (2) need be retained only in a narrow neighborhood of this surface, where the length scale for variations in E_* is quite small. To find the functions E, n in this neighborhood, it is convenient to analytically continue the solution of Eq. (6) which satisfies condition (8) beyond region V and, denoting the continued solution by G, to rewrite Eq. (3) as

$$\left(\frac{4}{3}+\frac{2}{3}r\frac{\partial}{\partial r}\right)\left(\frac{7}{3}+\frac{2}{3}r\frac{\partial}{\partial r}\right)(n+1)=\Delta(E^2-G). \quad (10)$$

Since the length scale for the variation of the quantities $n+1$ and E^2-G along the normal x_n to S near S is far smaller than the typical dimensions of region V, a double integration of (10) over x_n gives us

$$n+1\approx(9/4R^2\cos^2\alpha)(E^2-G). \quad (11)$$

In the range of applicability of (11), the function G can be assumed to depend linearly on x_n:

$$G\approx-Ax_n=-A(r-R)\cos\alpha. \quad (12)$$

After we substitute (11) and (12) into (2) and introduce the new notation

$$E(r)=(\delta A)^{1/2}f\left(\frac{r-R}{\delta}\cos\alpha\right), \quad \delta=\left(\frac{4R^2\cos^2\alpha}{9A}\right)^{1/3} \quad (13)$$

we find a so-called Painlevé equation of the second kind for the function $f(\xi)$:

$$d^2f/d\xi^2=(f^2+\xi)f \quad (14)$$

(see, e.g., Ref. 10). It is not difficult to see that the length scale δ for the variations of the field E near surface S is proportional to the square root of the linear dimensions of region V and is indeed small in comparison with these dimensions. At distances from S greater than δ, the functions E, n which satisfy relations (11)–(14) must become the same, to within small corrections, as E_*, n_*. Consequently, we have $f^2+\xi\to0$ as $\xi\to-\infty$ and $f(\xi)\to0$ as $\xi\to+\infty$. These conditions unambiguously determine a regular solution of Eq. (14). The function $f(\xi)$ is plotted in Fig. 1.

The problem of finding self-similar solutions with a large cavity V thus reduces to that of solving linear equation

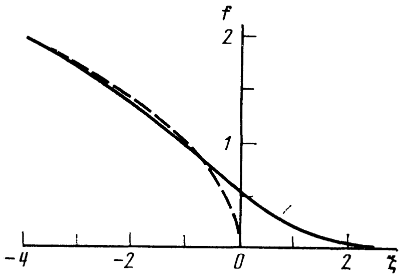

FIG. 1. Solution of Painlevé equation (14) which satisfies the boundary conditions $f(\xi)\to0$ as $\xi\to\infty$ and $f^2(\xi)+\xi\to0$ as $\xi\to-\infty$. The dashed line is the function $f=(-\xi)^{1/2}$.

(6). That equation can be integrated numerically without difficulty in essentially any region V (see for example, Ref. 11). In the ellipsoid

$$F(\mathbf{r})\equiv1-x^2/a^2-y^2/b^2-z^2/c^2\geqslant0 \quad (15)$$

an analytic solution can be found:

$$E_*^2={}^{14}/_9(1/a^2+1/b^2+1/c^2)^{-1}F(\mathbf{r}). \quad (16)$$

Substituting (16) into the equations given above, we can easily calculate all the characteristics of self-similar solutions with ellipsoidal cavities.

If the dimensions of region V are small in comparison with unity, the system (2), (3) has no solutions. It is natural to suggest that each subfamily of regions V which are similar to each other contains a unique critical cavity dimension, separating regions in which solutions do and do not exist (for each shape there is a distinct critical cavity dimension). To test the validity of this suggestion and to see how well it agrees with the results of earlier studies, we turn now to centrally symmetric self-similar solutions.

3. CENTRALLY SYMMETRIC SOLUTIONS

Centrally symmetric, self-similar solutions with a large cavity radius ($R\gg1$) are[1)]

$$E(r)\approx[{}^{14}/_{27}(R^2-r^2)]^{1/2}, \quad n(r)\approx-1 \quad \text{for} \quad R-r\gg\delta;$$

$$E(r)\approx\left(\frac{28}{27}R\delta\right)^{1/2}f\left(\frac{r-R}{\delta}\right),$$

$$n(r)\approx-1+\frac{7}{3}\frac{\delta}{R}\left[f^2\left(\frac{r-R}{\delta}\right)+\frac{r-R}{\delta}\right] \quad \text{for} \quad |R-r|\ll R;$$

$$E(r)\approx0, \quad n(r)\approx-\frac{7}{9}\left(\frac{R}{r}\right)^2-\frac{2}{9}\left(\frac{R}{r}\right)^{7/2} \quad \text{for} \quad r-R\gg\delta. \quad (17)$$

Here $\delta=(3/7R)^{1/3}\ll R$, and f is the function plotted in Fig. 1. In accordance with the prediction in Ref. 6, a solution exists for a continuous set of values of the parameter R [or, equivalently, of the parameter $E_0\equiv E(0)$. The reason is a degeneracy, which causes the centrally general symmetric solutions of Eqs. (2) and (3) to be representable near the origin of coordinates ($r\to0$) not by series in even powers of r (as has been previously been assumed) but by double series in integer powers of r^2 and r^c, where

$$c={}^3/_2[(2E_0^2+{}^1/_4)^{1/2}-{}^1/_2-{}^4/_3] \quad (18)$$

is a positive and generally irrational number. As parameters determining the family of solutions which are bounded as $r \to 0$,

$$E(r) = \sum_{p,q} E_{p,q} r^{2p+cq}, \quad n(r) = \sum_{p,q} n_{p,q} r^{2p+cq}, \tag{19}$$

we might choose for example, the coefficients $E_{0,0} \equiv E_0$ and $n_{0,1} \equiv n_1$. The condition that there be no singularities of the type

$$E(r) = \pm \frac{2\sqrt{2}}{3} \frac{r_s}{r - r_s}$$

at finite r imposes a constraint on these coefficients:

$$r_s(E_0, n_1) = \infty. \tag{20}$$

As a result, we find a single-parameter family of centrally symmetric solutions which are bounded as $r \to 0$ and which vanish as $r \to \infty$. Taking the approach of Refs. 8 and 9, we can study this family in detail by numerical calculation. The line $n_1(E_0)$ determined by condition (20) is shown in Fig. 2. It lies entirely within the region $E_0^2 > 14/9$, since there are no self-similar solutions with $E_0^2 < 14/9$. The dependence of the solution on the parameter E_0 is smooth, despite the singularities in the function $n_1(E_0)$ These singularities stem from the change in the structure of expressions (19) for even values of c ($c = 2m$) and lie at the points $E_0 = \mathscr{E}_m$:

$$\mathscr{E}_m{}^2 = (^2/_3 + ^2/_3 m)(^7/_3 + ^4/_3 m), \quad m = 1, 2, \ldots .$$

The nature of the singularities can be seen quite easily by setting $c = 2m + c_1$, $c_1 \to 0$ in (19). For n we then find an expression

$$n(r) = \sum_{p,q} n_{p,q} r^{2(p+qm)} \left(1 + qc_1 \ln r + \frac{1}{2} q^2 c_1{}^2 \ln^2 r + \ldots\right)$$
$$= n_{0,0} + (n_{m,0} + n_{0,1}) r^{2m} + n_{0,1} c_1 r^{2m} \ln r + \ldots .$$

For the condition that the third term of this expression remain finite as $c_1 \to 0$ we find

$$n_{0,1} \propto 1/c_1 \propto 1/(E_0{}^2 - \mathscr{E}_m{}^2)$$

(the second term remains finite because the terms in the sum $n_{m,0} + n_{0,1}$ which increase without bound as $c_1 \to 0$, cancel out).

The self-similar solutions (which are representable by series in even powers of r as $r \to 0$) which were described by Zakharov and Shchur in Ref. 8 correspond to the zeros of the function $n_1(E_0)$. According to Ref. 8, these zeros are at the

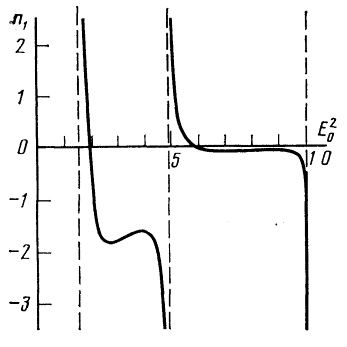

FIG. 2. Relationship between the parameters n_1 and E_0^2 imposed by the condition that the self-similar solutions must be regular.

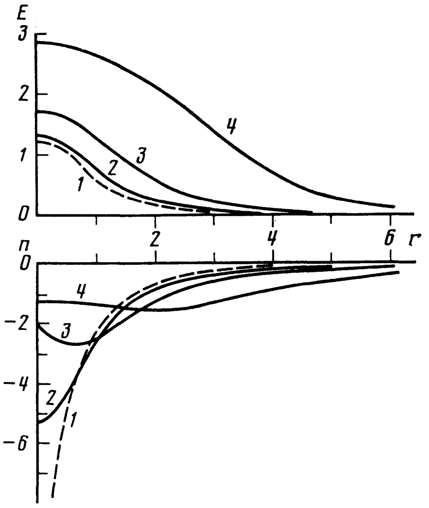

FIG. 3. Examples of self-similar solutions with various values of the field at the center of the cavity. 1—$E_0^2 = 14/9$; 2—a solution with $E_0^2 \approx 1913$ which is analytic at the center of the cavity; 3—$E_0^2 = 3 (c = 1)$, 4—$E_0^2 = 8$.

points $E_1^2 \approx 1.91$, $E_2^2 \approx 5.88$.... Figure 3 shows several typical examples of self-similar solutions of a general type. For all permissible values of the parameter E_0, the field E in these solutions depends monotonically on r. In the region $14/9 < E_0^2 < E_1^2$, the function (n,r) is also monotonic. In this region we have $n_1 > 0$, $c < 1$ and $dn/dr|_{r \to 0} \to \infty$; i.e., the "potential" $n(r)$ has a downward "ski-jump" at the origin. The size of this ski-jump increases as E_0^2 approaches the value $\mathscr{E}_0^2 = 14/9$. The solution which arises in the limit $E_0^2 \to 14/9$ the dashed line in Fig. 3) has a logarithmic singularity at the origin, $n(r)|_{r \to 0} \approx 14/11 \ln r +$ const. In the region $E_0^2 > E_1^2$ the function $n(r)$ has a single minimum. The latter moves away from the origin with increasing E_0^2. In the interval $E_1^2 < E_0^2 < 3$, the potential $n(r)$ still has a ski-jump, but now it is directed upward, by virtue of the relation $n_1 < 0$. The ski-jump disappears at $E_0^2 = 3$ (i.e., at $c = 1$). With a further increase in E_0^2, the singularity at the origin remains only for the derivatives of progressively higher order of the solution, and the "potential well" in n becomes progressively wider and flatter.

4. SELF-SIMILAR SOLUTIONS WITH HIGHER-INDEX BOUND STATES

A self-similar cavity is known to have an infinite number of bound states.[12] It has been assumed in the preceding sections of this paper that only the lowest state of this set is populated. Self-similar solutions for which higher-index states are populated also exist and must be taken into consideration in a description of turbulence. Although no examples of solutions of this sort have been found so far, they can be constructed, in several cases analytically. The structure of the set of all self-similar solutions with given occupation numbers of one or several bound states is also amenable to study. It can be shown that this structure is analogous to that which prevails during the filling of the ground state. The arguments which lead to this conclusion are largely the same as the arguments above, so we will not reproduce them here. We will restrict the discussion here to a simple example which gives an idea of the specific features of the self-similar solutions for which higher-index states are populated; for example, we will examine centrally symmetric solutions with a single populated level.

In the centrally symmetric case, Eq. (3) can be integrated once. Carrying out this integration, and introducing some new functions and variables,

$$\psi=E/E_0,\quad u=nr^2/E_0^2,\quad t=\ln(r/E_0),\tag{21}$$

we easily find the following system of equations from (2), (3):

$$\frac{1}{E_0^2}\left(\frac{d^2\psi}{dt^2}+\frac{d\psi}{dt}\right)=(u+e^{2t})\psi,\tag{22}$$

$$\frac{2}{3}\left(1+\frac{2}{3}\frac{d}{dt}\right)u=\left(1+\frac{d}{dt}\right)(\psi^2-1).\tag{23}$$

In terms of the new notation, the boundary conditions do not contain the parameter E_0:

$$\psi\to1,\quad u\to0\quad\text{as}\quad t\to-\infty,\qquad \psi\to0,\quad u\to-{}^3/_2\quad\text{as}\quad t\to+\infty.\tag{24}$$

A solution which corresponds to the filling of the ground state is described within the cavity in the approximation (5)–(7) by

$$u_*=-e^{2t},\quad \psi_*^2=1-{}^{14}/_{27}e^{2t}\tag{25}$$

and is also independent of E_0. As was mentioned above, the approximation (5)–(7) is good up to certain neighborhood of the boundary of the cavity, specifically, in the region

$$t_R-t\gg\delta/R,\quad t_R={}^1/_2\ln{}^{27}/_{14}.\tag{26}$$

In this region, the left side of (22) is small in comparison with each of the terms on the right, so that it becomes possible to calculate, by successive approximations, corrections to (25) which depend smoothly on t. In addition to these corrections, which contain a small factor on the order of the parameter E_0^{-1}, the exact solution differs from (25) by a correction which is exponentially small in region (26), for which the length scale of the variations is on the order of E_0^{-1}. It can be calculated by the WKB method; it decays exponentially as $t\to-\infty$. The coefficient of the exponential function is proportional to the parameter n_1, which was introduced in §3. If we treat n_1 as an adjustable parameter on which the family of solutions which are regular as $t\to-\infty$ depends, then a solution have a singularity like that described in §3 will correspond in a random way to each selected value of n_1. For solutions which depend smoothly on the coordinate within the cavity, the condition that this singularity be eliminated was already discussed above. It determines the curve $n_1(E_0)$. There are, however, other ways to eliminate the singularity, in which the solution executes several fast oscillations within the cavity and then becomes smooth again. (As long as the number of these oscillations is not very large, they can be represented as jumps from one branch of a smooth solution $\psi\approx\pm\psi_*$ to the other.) When we take this possibility into account, we see that condition (20) determines an entire set of $n_1^{(N)}(E_0)$ curves. Curve N corresponds to a function $\psi=\psi^{(N)}$, which has N zeros, i.e., to the filling of the Nth excited state in the cavity. In order to parametrize the family of solutions which are regular as $t\to-\infty$ and which have zeros, it is more convenient to switch from use of the coefficient n_1 to the use of the coordinate t_1 of the first zero of the functions $\psi(t_1$ is unambiguously related to n_1 for a given value of $E_0)$. When this parametrization is chosen, a countable set of curves $t_1^{(N)}(E_0)$ corresponds to solutions which are regular on the entire real axis.

To give a quantitative description of oscillatory solutions, it is natural to first make use of the fact that the length scale is small in comparison with the size of the cavity. Equations (22) and (23) form a third-order system, so it is sufficient to seek three approximate integrals of the fast motions in order to eliminate the shortest scale from the problem.[2] As can be seen from (23), the difference $\psi^2\_4/9u$ can be chosen as one of the weakly oscillating functions of the zeroth approximation. For use below, it is convenient to refine this choice in the following way:

$${}^4/_9(u+e^{2t})=\psi^2-\Phi.\tag{27}$$

After the function u is replaced by Φ, the system of equations (22), (23) becomes

$$\frac{1}{E_0^2}\left(\frac{d^2\psi}{dt^2}+\frac{d\psi}{dt}\right)=\frac{9}{4}(\psi^2-\Phi)\psi,\tag{28}$$

$$\frac{d\Phi}{dt}=1-\frac{3}{2}\Phi+\frac{1}{2}\psi^2-\frac{14}{9}e^{2t}.\tag{29}$$

If we ignore the dependence of Φ on t and the small term $(1/E_0^2)(d\psi/dt)$, Eq. (28) has an "energy" integral:

$$W=\frac{1}{2E_0^2}\left(\frac{d\psi}{dt}\right)^2-\frac{9}{16}(\psi^2-\Phi)^2.\tag{30}$$

Differentiating (30) with respect to t, and using the exact equations (28)–(30), we easily find an equation for the function W, which is obviously also weakly oscillating:

$$dW/dt=-2W-{}^9/_8(\Phi-\psi^2)(1-{}^1/_2\Phi-{}^1/_2\psi^2-{}^{14}/_9e^{2t}).\tag{31}$$

Using (30) and, after multiplication by ψ, Eq. (28), we can put (31) in a more convenient form, eliminating the term ψ^4 from its right side:

$$\frac{d}{dt}\left[W+\frac{1}{12E_0^2}\left(\frac{d\psi^2}{dt}+\psi^2\right)\right]=-\frac{5}{3}W-\frac{9}{8}(\Phi-\psi^2)\left(1-\frac{2}{3}\Phi-\frac{14}{9}e^{2t}\right).\tag{32}$$

As the third weakly oscillating function we should choose a large oscillation "frequency," $d\tau/dt\propto E_0$. If we want a determination of the "phase" τ which is suitable for all N, we must switch to functions which are even less oscillatory than W and Φ, which turns out to run into unnecessary complexities. We will thus offer some simpler although not universal determinations, corresponding to various values of N.

In the case $N=1$, in which the first excited state is populated, the solution is smooth within an exponentially small correction outside a narrow neighborhood of the single zero, $t_1=t_1^{(1)}$, of the function $\psi=\psi^{(1)}$ [but inside region (26)]. By virtue of (28), (30), this solution satisfies the relations

$$\psi^2=\Phi+O(E_0^{-2}),\quad W=\frac{1}{2E_0^2}\left(\frac{d\Phi^{1/2}}{dt}\right)^2+O(E_0^{-4}).\tag{33}$$

The quantities represented here by $O(E_0^{-k})$, which are on the order of e_0^{-k}, can be calculated without any particular difficulty, but we not need them for the discussion below. The variations of the functions Φ and W near the jump $|t-t_1^{(1)}|\leqslant E_0^{-1}$, are less than E_0^{-1} in order of magnitude, according to (29), (31). Making use of these facts, we find that integration of Eq. (32) across the jump leads to the relation

$$1-{}^2/_3\Phi(t_1^{(1)})-{}^{14}/_9 e^{2t_1^{(1)}}=O(E_0^{-1}), \tag{34}$$

while integration of Eq. (29) over the region in which the functions vary smoothly gives us

$$\Phi=\psi_*^2+O(E_0^{-1}). \tag{35}$$

Equations (34), (35), and (25) can be used for an approximate calculation of $t_1^{(1)}$:

$${}^{14}/_{27}e^{2t_1^{(1)}}={}^1/_7+O(E_0^{-1}). \tag{36}$$

The shape of the jump is found with the help of Eq. (30):

$$\psi=-\Phi^{1/2}\operatorname{th}\tau+O(E_0^{-1}),\quad \tau=\frac{3E_0}{2\sqrt{2}}\int_{t_1^{(1)}}^{t}\Phi^{1/2}(t')\,dt'. \tag{37}$$

Knowing the shape of the jump in the zeroth approximation, (37), we can integrate Eq. (29) more accurately, and we can calculate the function Φ in a first approximation; we can use the results to refine the position and shape of the jump, etc. There is no point in dwelling on this matter in more detail, since the simple equations (35)–(37) already give us a fairly detailed picture of the solution. The only important point is to emphasize the possibility of a systematic calculation of corrections which are small, on the order of the parameter E_0^{-1}. Near the jump, Eq. (37) should in fact be made somewhat cruder, by replacing the function $\Phi(t)$ with its value at the point $t_1^{(1)}$: $\Phi(t_1^{(1)})\approx 6/7$. It is useful to note that a jump in the field ψ (or E) corresponds to a soliton-like dip in a perturbation of the density n:

$$n+1\approx-7\operatorname{ch}^{-2}\left[\sqrt{\frac{7}{2}}(r-r_1^{(1)})\right],\quad r_1^{(1)}=E_0\exp(t_1^{(1)})\approx\frac{R}{\sqrt{7}}. \tag{38}$$

A solution with an index which is not very large, $N\ll E_0$, is, to lowest order, a set of N jumps similar to that described above. We denote by $t_1^{(N)}, t_2^{(N)},\dots,t_N^{(N)}$ the zeros of the function $\psi^N(t)$ in increasing order, and we denote by $s_1^{(N)},s_2^{(N)},\dots,s_{N-1}^{(N)}$ the extrema of this function which lie between these zeros. Under the condition $N\ll E_0$, all the jumps occur in a narrow neighborhood of the point $t_1^{(1)}$. In this neighborhood, the function $\Phi(t)$ does not have room to vary significantly, and it remains close to 6/7. The function $W(t)$ does not have room to increase significantly, and it remains small in comparison with unity. Because of this, the shape of the jumps remains the same as in the case $N=1$, in a zeroth approximation:

$$\psi^{(N)}(t)\approx(-1)^i\left(\frac{6}{7}\right)^{1/2}\operatorname{th}\left[\frac{3\sqrt{3}}{2\sqrt{7}}E_0(t-t_i^{(N)})\right],$$

$$t-s_{i-1}^{(N)}\gg E_0^{-1},\qquad s_i^{(N)}-t\gg E_0^{-1}. \tag{39}$$

Near each of the "stopping points" $s_i^{(N)}$ we can ignore the change in the function $W(t)$, and we can also integrate Eq. (30):

$$\Phi-\psi^2\approx\frac{4}{3}(W_i^{(N)})^{1/2}\operatorname{ch}\left[\frac{3\sqrt{3}}{\sqrt{7}}E_0(t-s_i^{(N)})\right],$$

$$W_i^{(N)}\equiv-W(s_i^{(N)}),\qquad t-t_i^{(N)}\gg E_0^{-1},\qquad t_{i+1}^{(N)}-t\gg E_0^{-1}. \tag{40}$$

A comparison of (39) and (40) in their common range of applicability makes it possible to express the distances between the adjacent zeros and the extrema of the function $\psi^{(N)}(t)$ in terms of the quantities $W_i^{(N)}$:

$$\Delta_i^{(N)}\equiv s_i^{(N)}-t_i^{(N)}\approx t_{i+1}^{(N)}-s_i^{(N)}\approx\frac{\sqrt{7}}{E_0 3\sqrt{3}}\ln\frac{6\sqrt{6}}{(7W_i^{(N)})^{1/2}}. \tag{41}$$

It can be seen from this expression that under the condition $W_i^{(N)}\ll 1$ the distances between adjacent jumps are significantly greater than the width of each jump; i.e., the jumps are indeed nearly isolated. To close the system of recurrence relations (41), it is sufficient to integrate Eq. (32) across a jump:

$$W_i^{(N)}-W_{i-1}^{(N)}\approx-2({}^3/_7)^{1/2}E_0^{-1}(t_i^{(N)}-t_1^{(1)}). \tag{42}$$

Using (42), we can eliminate the coordinates of the zeros of the function $\psi^{(N)}(t)$ from (41) and derive a system of equations containing only the quantities $W_i^{(N)}$:

$$W_{i+1}^{(N)}-2W_i^{(N)}+W_{i-1}^{(N)}\approx-\frac{2}{3E_0^2}\ln\frac{1}{W_i^{(N)}}. \tag{43}$$

The values of $W_0^{(N)}$ and $W_N^{(N)}$, which arise in (43) with $i=1$ and $i=N-1$, and which have not been determined previously, should be taken to be zero. The numerical factor has been omitted from within the logarithm on the right side of (43) because the derivation of (42) did not retain some small corrections on the order of the parameter $(\ln W_i^{(N)})^{-1}$. To within the same logarithmic accuracy, the solution of system (43) can be written

$$W_i^{(N)}\approx\frac{i(N-i)}{3E_0^2}\ln\frac{1}{W^{(N)}},\qquad W^{(N)}\approx\frac{N^2}{6E_0^2}\ln\frac{E_0}{N}, \tag{44}$$

where $W^{(N)}$ is the maximum value of $W_i^{(N)}$. As has been assumed, it is small in comparison with unity under the condition $N\ll E_0$. Knowing the quantities $W_i^{(N)}$, we can easily find the coordinates of the zeros of the function $\psi^{(N)}(t)$ from (42). At the accuracy level adopted above, these zeros are positioned symmetrically with respect to the point $t_1^{(1)}$, and they are separated from each other by distances on the order of $E_0^{-1}\ln(E_0/N)$. When small corrections on the order of the parameter $[\ln(E_0/N)]^{-1}$ are taken into account, the center of the distribution of zeros shifts leftward from the point $t_1^{(1)}$ by an amount of order $NE_0^{-1}\ll 1$, and the symmetry in the arrangement of zeros with respect to the center is slightly broken. A density perturbation which corresponds to solution N contains N soliton-like dips like that of (38), at the points $r_i^{(N)}=E_0 e^{t_i(N)}$:

$$n+1\approx-7\sum_{i=1}^{N}\operatorname{ch}^{-2}\left[\left(\frac{7}{2}\right)^{1/2}(r-r_i^{(N)})\right]. \tag{45}$$

Self-similar solutions with $1 \ll N \ll E_0$ can also be calculated by another method, which deserves mention here because in the centrally symmetric case it remains valid even for $N \gtrsim E_0$. The possibility of finding self-similar solutions without assuming that the ratio N/E_0 is small stems from the nearly exact coincidence of adjacent oscillations when the total number of these oscillations is large. Ignoring the difference between adjacent oscillations, i.e., assuming that the functions W and Φ are constants in (30), we can easily integrate this equation exactly. Under the condition

$$-{}^9/_{16}\Phi^2<W<0 \tag{46}$$

the solution would be periodic and would take the form

$$\psi=A\ \mathrm{sn}(\tau, k), \tag{47}$$

where sn is the elliptic sine,

$$A=\psi_-,\quad k=\psi_-/\psi_+,\quad \psi_\pm=[\Phi\pm{}^4/_3(-W)^{1/2}]^{1/2}, \tag{48}$$

$$d\tau/dt=3E_0\psi_-/k2\sqrt{2}. \tag{49}$$

For t-dependent functions $\Phi(t)$ and $W(t)$ which satisfy condition (46), relations (47), (48) should be thought of as equations for a transformation from an unknown function $\psi(t)$ to another unknown function $\tau(t)$. An equation for the function $\tau(t)$, which can easily be found with help of (29)–(31), (47), and (48), differs from (49) in having an oscillatory term on its right side which does not contain the large parameter E_0. An averaging method can be used for the system of three first-order equations for the three weakly oscillating functions τ, Φ, W (Ref. 14, for example). To lowest order in the parameters N^{-1} and E_0^{-1}, the equation for the average function τ is the same as (49), while the equations for the average functions Φ and W are

$$\frac{d\Phi}{dt}=1-\frac{3}{2}\Phi+\frac{\psi_-^2}{2k^2}\left[1-\frac{\mathrm{E}(k)}{\mathrm{K}(k)}\right]-\frac{14}{9}e^{2t}, \tag{50}$$

$$\frac{dW}{dt}=-\frac{5}{3}W-\frac{9}{8}\left\{\Phi-\frac{\psi_-^2}{k^2}\left[1-\frac{\mathrm{E}(k)}{\mathrm{K}(k)}\right]\right\}\times\left(1-\frac{2}{3}\Phi-\frac{14}{9}e^{2t}\right). \tag{51}$$

Here K and E are the complete elliptic integrals of the first and second kinds, and Φ, W, $\psi_\pm$, and k are functions which have been averaged over the period of the fast oscillations. These average functions are represented by the same symbols as the exact functions, since the average functions differ from the exact one only by corrections which are of no importance for the discussion below and which satisfy the same relations as before, (48). [In narrow neighborhoods of the ends of the oscillation-filled interval, the relative magnitude of the correction to the average function W is not small, and the validity of the averaging step is not obvious. Although these neighborhoods are not important to the discussion below (because of their narrowness), it is useful to note that a solution can also be calculated quite easily in these neighborhoods. The result demonstrates that the average equations are applicable even for describing several extreme oscillations. The reason lies in the very weak (logarithmic) dependence of the oscillation period on the quantity W as $W\to 0$.] At the left end of the oscillation-filled interval (which coincides to within a small correction with the point t_1), the solution of the average equations (50), (51) should join continuously with the smooth solution of the original system:

$$W\approx 0,\quad \Phi\approx\psi_*^2=1-{}^{14}/_{27}e^{2t}.$$

The family of solutions which arises when this matching is carried out depends on the single parameter

$$t_1<t_1^{(1)}\approx{}^1/_2\ln{}^{27}/_{98}$$

(remarkably, there is no E_0 dependence) and can easily be found by numerical integration. The results are shown in Fig. 4. For all possible values of t_1 the function $\Phi(t)$ is found to be a monotonically decreasing function, while the function $W(t)$ initially decreases, then increases, and manages to vanish before $\Phi(t)$ does. The point $t_N(t_1)$, where this occurs, is the right boundary of the oscillation filled interval. Over the region between $t_N(t_1)$ and the edge of the cavity, i.e., up to the point $t_R(t_1)$, at which the function $\Phi(t)$ vanishes, we again have a smooth solution of Eq. (29) with $\psi^2\approx\Phi$:

$$\Phi\approx 1-{}^{14}/_{27}e^{2t}-[1-{}^{14}/_{27}\exp(2t_N)-\Phi(t_N)]\exp(t_N-t). \tag{52}$$

For large negative values of the parameter t_1 ($-t_1\gg 1$), the solution of Eqs. (50) and (51) in the region $t-t_1\ll -t_1$ depends only on the difference $t-t_1$ to (to within small corrections). The asymptotic form of this universal solution in the region $t-t_1\gg 1$ (but $-t\gg 1$) is

$$\Phi\approx{}^2/_3,\quad \psi_-^2\approx Be^{-(t-t_1)}, \tag{53}$$

where the constant $B\sim 1$ is found numerically: $B\approx 0.91$. When t increases further, terms containing e^{2t} become important in Eqs. (50) and (51), and the solution deviates from the asymptotic solution (53):

$$\Phi\approx{}^2/_3(1-{}^2/_3e^{2t}),\quad \psi_-^2\approx Be^{-(t-t_1)}(1-{}^2/_3e^{2t})^{-1/2}. \tag{54}$$

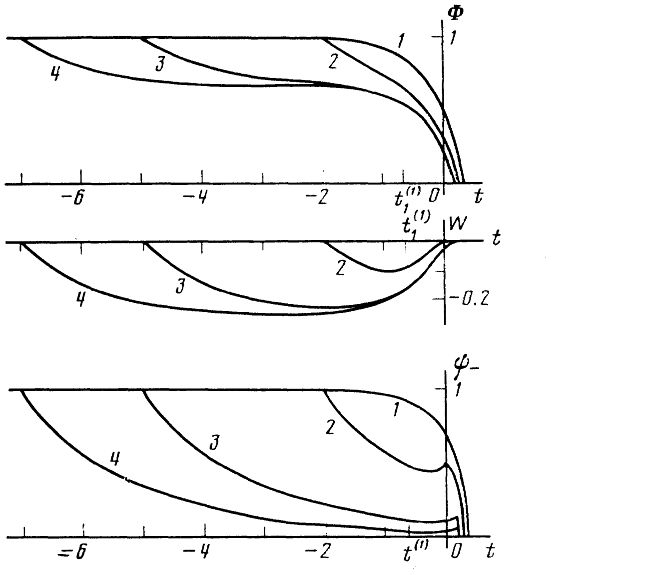

FIG. 4. The functions $\Phi(t)$, $W(t)$, and $\psi_-(t)$ for various values of the parameter t_1: $1-t_1=t_1^{(1)}=1/2\ln 27/98$; $2-t_1=-2$; $3-t_1=-5$; $4-t_1=-7$.

Expressions (54), derived in the linear approximation[3] in φ^2_-, are applicable as long as the condition $\psi^2_- \ll \Phi$, holds, i.e., in the region

$$t_\infty - t \gg e^{2t_1/3}, \quad t_\infty \equiv {}^1/_2 \ln {}^3/_2. \tag{55}$$

As the boundary of this region is approached, the nonlinearity of the oscillations becomes increasingly important; ψ^2_- then becomes comparable to Φ (W vanishes), and the solution becomes smooth again. Expression (54) can be used to estimate both the position of the right boundary of the t_N interval filled by oscillations and the value of $\Phi(t_N)$:

$$t_\infty - t_N \sim e^{2t_1/3}, \quad \Phi(t_N) \sim e^{2t_1/3}. \tag{56}$$

Since the part of the region of the smooth variation of the solution on the right is narrow, and since $\Phi(t_N)$ is small for large values of $-t_1$ we can simplify (52) in this limiting case:

$$\Phi(t) \approx \Phi(t_N) - {}^4/_3(t - t_N). \tag{57}$$

We can also estimate the position t_R of the edge of the cavity[4]:

$$t_R - t_N \sim e^{2t_1/3}. \tag{58}$$

It is pertinent to recall here that the lowest approximation in the parameter E_0^{-1} which we used above breaks down in a narrow neighborhood of the cavity boundary. The refinements required in this neighborhood are made in the same way as in §2, and they lead to the same Painlevé equation, (14), and the same estimate of the width of the neighborhood, δ. The point t_N can be calculated with the help of (50) and (51) if it lies outside the δ-neighborhood of the boundary of the cavity, i.e., for $t_R - t_N \gg \delta/E_0 \sim E_0^{-2/3}$, or, equivalently, for

$$e^{t_1} \gg E_0^{-1}. \tag{59}$$

In the opposite limit from (59), the distance $t_R - t_N$ which is formally found from (58) is less than the period of the last oscillation, and it becomes meaningless to distinguish between the points t_R and t_N. In the case $e^{t_1} \ll E_0^{-1}$ the oscillations remain linear up to the very edge of the cavity, and near it they are described by the ordinary Airy equation.

Note that the problem of finding self-similar solutions with $N \gg 1$ breaks up into two parts: constructing integral curves of Eqs. (50), (51) as a function of the parameter t_1 and "quantizing" the values of this parameter. The quantization condition amounts to fitting an integer number of oscillations in the interval (t_1, t_N) and can be found easily from (49), (47):

$$N = \frac{3E_0}{4\sqrt{2}} \int_{t_1}^{t_N(t_1)} dt \psi_+(t)/\mathrm{K}(k(t)). \tag{60}$$

It can be seen from (60) that the quantized values $t_1^{(N)}(E_0)$ of the parameter t_1 do not depend on both of the parameters E_0 and N separately but only their combination $\eta = N/E_0$. The functional dependence $t_1(\eta)$ is shown in Fig. 5. The asymptotic forms of this dependence can be calculated analytically. In the case $\eta \ll 1$ ($N \ll E_0$) the results are the same as those found by the other method (see the discussion above). In the case $\eta \gg 1$, the value of t_1 is negative and large, so that most of the oscillations can be assumed to be linear.

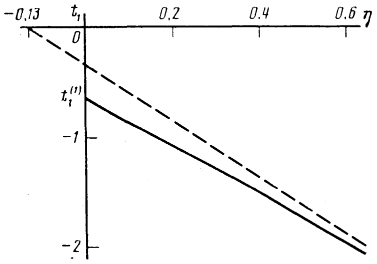

FIG. 5. Functional dependence of the parameter t_1 on $\eta = N/E_0$ found from the condition that the oscillatory self-similar solutions must be regular. The dashed line is the asymptote $t_1 = -(2/3)^{1/2}\pi(\eta + B_1)$, $B_1 \approx 0.13$.

The functional dependence $t_1(\eta)$ is also linear for $\eta \gg 1$:

$$t_1 \approx -({}^2/_3)^{1/2}\pi(\eta + B_1). \tag{61}$$

The term $B_1 \sim 1$ must be determined numerically because of the nonlinearity of the first $\Delta N \sim E_0$ oscillations. The result is $B_1 \sim 0.13$.

Up to this point, we have been discussing oscillatory self-similar solutions with $E_0 \gg 1$. Solutions with $E_0 \sim 1$ can be found numerically. Figure 6 illustrates the results with centrally symmetric solutions of Eqs. (2), (3) (these solutions are analytic at the center of the cavity) with the lowest possible field values at the center of the cavity, $E_0 = E_0(N)$, and various numbers N. These solutions were first rescaled according to

$$n(r) = |n(0)|\hat{n}(\hat{r}), \quad r = |n(0)|^{-1/2}\hat{r}. \tag{62}$$

After the scaling (62), the depth of the cavity at its center becomes unity, and the 'binding energy" of the populated state [which was assumed to be unity when the first of the basic equations was written in the form in (2)] is found to depend on N. Relatively large values of N correspond to relatively small binding energies, $\Omega_N \ll 1$, and to approximately equal values of the field at the center of the cavity:

$$E_0^2(N) = {}^{14}/_9(1 - \Omega_N)^{-1} \approx {}^{14}/_9(1 + \Omega_N). \tag{63}$$

By virtue of these properties, the equations and boundary conditions for the solutions with all possible $N \gg 1$ are found to be essentially identical, and the solutions themselves are

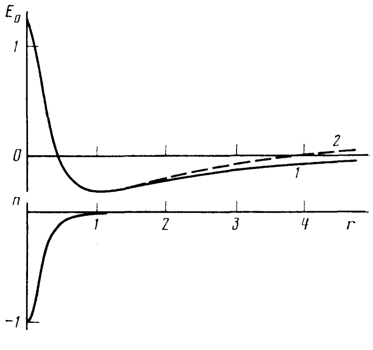

FIG. 6. Oscillatory self-similar solutions which are analytic as $v \to 0$ and which have the smallest field values at the center of the cavity, E_0, for various indices $(N + 1)$ of the filled states; 1—$N = 1$; 2—$N \geqslant 2$. The discrepancy between the curves of the functions $\hat{n}(\hat{r})$ with all possible $N > 1$ does not exceed the thickness of the line in this figure. The same is true of the functions $E(\hat{r})$ with $N \geqslant 2$ in the region $\hat{r} \leqslant 5$.

essentially the same except quite far from the center of the cavity. The difference between the "wave functions" $E^{(N)}(\hat{r})$ of the states with $N\gg 1$ is manifested only where the depth of the "potential well" $\hat{n}(\hat{r})$ becomes comparable to the binding energy Ω_N. In this region, the field $E^{(N)}(\hat{r})$ is still small and has essentially no effect on the potential $\hat{n}(\hat{r})$, which can be described quite accurately by the well-known asymptotic formula

$$\hat{n}\approx-\frac{3}{2}\frac{E_0^2}{\hat{r}^2}\approx-\frac{7}{3}\hat{r}^{-2}. \qquad (64)$$

Using (64), we can easily calculate the binding energy Ω_N:

$$\Omega_N\approx c_1\varkappa^N, \quad \varkappa=\exp(-4\sqrt{3}\pi/5)\approx 0.0128. \qquad (65)$$

The coefficient c_1, which depends on the behavior of the potential $\hat{n}(\hat{r})$ in the region $\hat{r}\sim 1$, must be found numerically. The result is $c_1\approx 0.151$. Since c_1 and $\varkappa$ are numerically small quantities, expression (65) remains highly accurate even in the case $N=1$.

5. CONCLUSION

The family of self-similar "scalar" collapse regimes turns out to be an extremely large one. It consists of a countable set of subfamilies corresponding to different ways in which the bound states available in the cavity are filled. Each subfamily is isomorphic to a certain class of regions of three-dimensional space — a class which contains in particular all convex regions with dimensions which are not too small. It is possible to discuss the structure of the set of self-similar regimes of scalar collapse and to analytically construct examples of representatives of this set because of the existence of regimes with high fields at the center of the cavity ($E_0\gg 1$. The attractive possibility of such a study of the self-similar regimes of the collapse of Langmuir waves is on closer inspection by no means obvious. At this point is not clear whether the difficulties which arise here are of a technical nature or are masking fundamental differences between the Zakharov equations and the scalar model. Yet another important question which requires further study, concerns the probabilities for the establishment of the various self-similar solutions in the cases in which the turbulence is excited by one method or another. At this point, we can offer no more than a few qualitative words about this topic. For example, it is clear that as E_0^2 increases the probability of obtaining such solution should decrease rapidly because of the instability which was discovered in Ref. 6. In this connection, the self-similar solutions constructed above with values of E_0^2 arbitrarily close to 14/9, i.e., with arbitrarily small exponents for the exponentially small powers for the power-law growth of unstable perturbations against the background of a collapsing cavity, deserve special attention. Remarkably, among the solutions which are close to the stability boundary there are some which are analytic at the center of the cavity and which are therefore essentially untouched by the smoothing effect of the acoustic term Δn in the equation for the perturbation of the density n (a term which was discarded in the self-similar limit).

[1)] To within the same error, the field $E(r)$ can be described by the common expression $E(r)\approx(28/27R\delta)^{1/2}f[(r^2-R^2)/2R\delta]$.

[2)] These integrals can then be refined by proceeding in the spirit of the method of accelerated convergence (Ref. 13, for example), specifically, by making the transition through successive replacements to progressively less oscillatory functions.

[3)] In this approximation the oscillations do not affect the shape of the cavity, so that (54) can also be derived by the ordinary WKB method for the linear Schrödinger equation.

[4] We can find a more accurate estimate by using Eq. (50): $t_R-t_\infty\sim e^{4t_i/3}$.

[1]A. A. Galeev, R. Z. Sagndeev, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **73**, 1352 (1977) [Sov. Phys. JETP **46**, 711 (1977)].

[2]G. Pelletier, Phys. Rev. Lett. **49**, 782 (1982).

[3]O. B. Budneva, V. E. Zakharov, and V. S. Synakh, Pis'ma Zh. Tekh. Fiz. **1**, 606 (1975) [Sov. Tech. Phys. Lett. **1**, 335 (1975)].

[4]V. E. Zakharov, A. F. Mastryukov, and V. S. Synakh, Fiz. Plazmy **1**, 614 (1975) [Sov. J. Plasma Phys. **1**, 614 (1975)].

[5]L. M. Degtyarev and V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 9 (1975) [JETP Lett. **21**, 4 (1975). L. M. Degtyarev, V. E. Zakharov, and L. I. Rudakov, Fiz. Plazmy **2**, 438 (1976) [Sov. J. Plasma Phys. **2**, 240 (1976)].

[6]V. M. Malkin, Zh. Eksp. Teor. Fiz. **87**, 433 (1984) [Sov. Phys. JETP **60**, 248 (1984)].

[7]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].

[8]V. E. Zakharov and L. H. Shchur, Zh. Eksp. Teor. Fiz. **81**, 2019 (1981) [Sov. Phys. JETP **54**, 1064 (1981)].

[9]V. M. Malkin and Yu. A. Tsidulko, Fiz. Plazmy **11**, 964 (1985) [Sov. J. Plasma Phys. **11**, 561 (1985)].

[10]V. V. Golubev, Lektsii po analiticheskoĭ teorii differentsial'nykh uravneniĭ (Lectures on the Analytic Theory of Differential Equations), Gostekhizdat, Moscow-Leningrad, 1950, p. 189.

[11]A. A. Samarskiĭ and V. B. Andreev, Raznostnye metody dlya éllipticheskikh uravneniĭ (Difference Methods for Elliptic Equations), Nauka, Moscow (1976).

[12]G. M. Fraĭman, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 557 (1979) [JETP Lett. **30**, 525 (1979)].

[13]N. N. Bogolyubov, Yu. A. Mitropol'skiĭ, and A. M. Samoĭlenko, Metod uskorennoĭ skhodimosti v nelineĭnoĭ mekhanike (Method of Accelerated Convergence in Nonlinear Mechanics), Naukova dumka, Kiev (1969).

[14]N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ, Nauka, Moscow, 1974, §25 (Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York).

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